Linearization theorems, Koopman operator and its application

Yueheng Lan
Department of Physics
Tsinghua University

May, 2013
Main contents

1 Introduction
   - Linear and nonlinear dynamics
   - Linearization in large
   - Examples

2 The Koopman operator
   - Its introduction
   - Koopman operator and partition of the phase space

3 Applications
   - The standard map
   - Application to fluid dynamics

4 Summary
1 Introduction
   - Linear and nonlinear dynamics
   - Linearization in large
   - Examples

2 The Koopman operator
   - Its introduction
   - Koopman operator and partition of the phase space

3 Applications
   - The standard map
   - Application to fluid dynamics

4 Summary
Main contents

1 Introduction
   - Linear and nonlinear dynamics
   - Linearization in large
   - Examples

2 The Koopman operator
   - Its introduction
   - Koopman operator and partition of the phase space

3 Applications
   - The standard map
   - Application to fluid dynamics

4 Summary
Outline

Main contents

1 Introduction
   - Linear and nonlinear dynamics
   - Linearization in large
   - Examples

2 The Koopman operator
   - Its introduction
   - Koopman operator and partition of the phase space

3 Applications
   - The standard map
   - Application to fluid dynamics

4 Summary
Main contents

1 Introduction
   - Linear and nonlinear dynamics
     - Linearization in large
     - Examples

2 The Koopman operator
   - Its introduction
   - Koopman operator and partition of the phase space

3 Applications
   - The standard map
   - Application to fluid dynamics

4 Summary
Dynamical systems and phase space

- State and dynamics
  \[
  \begin{align*}
  \dot{x}_1 &= f_1(x_1, x_2, \cdots, x_n) \\
  \dot{x}_2 &= f_2(x_1, x_2, \cdots, x_n) \\
  \vdots &= \vdots \\
  \dot{x}_n &= f_n(x_1, x_2, \cdots, x_n)
  \end{align*}
  \]

- The phase space - a geometric representation
- Vector field and trajectories
- Invariant set and organization of trajectories
Nonlinear dynamics vs complex systems

- Triumph of the nonlinear theory
  Local: bifurcation theory, normal form theory, linear stability analysis,...
  Global: Asymptotic analysis, topological methods, symbolic dynamics,...

- Troubles when treating complex systems
  (1) Huge number of interacting agents
  (2) Heterogeneity in spatiotemporal scales
  (3) Hierarchical structure and great many dynamic modes
  (4) Lack of exact mathematical description
  (5) Uncertainty in data or parameters (noise or ignorance)

- Solution: mean field approximation or statistical analysis?
Nonlinear dynamics vs complex systems

- Triumph of the nonlinear theory
  Local: bifurcation theory, normal form theory, linear stability analysis,...
  Global: Asymptotic analysis, topological methods, symbolic dynamics,...

- Troubles when treating complex systems
  1. Huge number of interacting agents
  2. Heterogeneity in spatiotemporal scales
  3. Hierarchical structure and great many dynamic modes
  4. Lack of exact mathematical description
  5. Uncertainty in data or parameters (noise or ignorance)

- Solution: mean field approximation or statistical analysis?
Nonlinear dynamics vs complex systems

- Triumph of the nonlinear theory
  Local: bifurcation theory, normal form theory, linear stability analysis,...
  Global: Asymptotic analysis, topological methods, symbolic dynamics,...

- Troubles when treating complex systems
  (1) Huge number of interacting agents
  (2) Heterogeneity in spatiotemporal scales
  (3) Hierarchical structure and great many dynamic modes
  (4) Lack of exact mathematical description
  (5) Uncertainty in data or parameters (noise or ignorance)

- Solution: mean field approximation or statistical analysis?
Structure of macromolecules
Introduction

The Koopman operator

Applications

Summary

Linear and nonlinear dynamics

linearization in large

Examples

Cell regulatory networks

Yueheng Lan

Linearization theorems, Koopman operator and its application
Linear systems and their solution

- General form: $\dot{x} = Ax$ with

\[
A = \begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{pmatrix}
\quad \text{and} \quad
x = \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}.
\]

- Idea: separate solutions into independent modes by assuming $x(t) = e^{\lambda t} v$.

- We then obtain an eigenvalue equation

\[
Av = \lambda v.
\]
Linear systems and their solution

- General form: $\dot{\mathbf{x}} = A\mathbf{x}$ with
  
  $$A = \begin{pmatrix}
  a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
  a_{2,1} & a_{2,2} & \cdots & \cdots \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n,1} & a_{n,2} & \cdots & a_{n,n}
  \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
  \end{pmatrix}.$$

- Idea: separate solutions into independent modes by assuming $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$.

- We then obtain an eigenvalue equation
  
  $$A\mathbf{v} = \lambda \mathbf{v}.$$
Linear systems and their solution

- General form: \( \dot{x} = Ax \) with
  
  \[
  A = \begin{pmatrix}
  a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
  a_{2,1} & a_{2,2} & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{n,1} & a_{n,2} & \cdots & a_{n,n}
  \end{pmatrix}
  \quad \text{and} \quad
  x = \begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
  \end{pmatrix}.
  \]

- Idea: separate solutions into independent modes by assuming \( x(t) = e^{\lambda t}v \).
- We then obtain an **eigenvalue equation**
  
  \[
  Av = \lambda v.
  \]
Linearization of nonlinear systems

- Consider the dynamics for $x \in D$ of $\mathbb{R}^n$,

$$\dot{x} = f(x) = Ax + v(x), \quad (1)$$

which induces a flow $\phi(x, t) : D \times \mathbb{R} \to D$

- Hartman-Grobman Theorem Let $f \in C^1(D)$. If $A$ is hyperbolic, $\exists h : U \to V$ with $U \subset D$, $0 \in U$ and $V \subset \mathbb{R}^n$, $0 \in V$ such that $\forall x_0 \in U$, $\exists I_0 \subset \mathbb{R}$ when $x_0 \in U$ and $t \in I_0$,

$$h \circ \phi(x_0, t) = e^{At} h(x_0); \quad (2)$$

i.e., $h$ maps trajectories of (1) near the origin to trajectories of $\dot{x} = Ax$ and preserves the time parametrization.

- Hartman’s theorem and Poincaré-Siegel theorem.

- Global linearization: weak nonlinearity or symmetry by lie group theory.
Consider the dynamics for $x \in D$ of $\mathbb{R}^n$,

$$\dot{x} = f(x) = Ax + v(x),$$

which induces a flow $\phi(x, t) : D \times \mathbb{R} \to D$

**Hartman-Grobman Theorem** Let $f \in C^1(D)$. If $A$ is hyperbolic, $\exists h : U \to V$ with $U \subset D, 0 \in U$ and $V \subset \mathbb{R}^n, 0 \in V$ such that $\forall x_0 \in U, \exists I_0 \subset \mathbb{R}$ when $x_0 \in U$ and $t \in I_0$,

$$h \circ \phi(x_0, t) = e^{At}h(x_0);$$

*i.e.*, $h$ maps trajectories of (1) near the origin to trajectories of $\dot{x} = Ax$ and preserves the time parametrization.

- Hartman’s theorem and Poincaré-Siegel theorem.
- Global linearization: weak nonlinearity or symmetry by lie group theory.
Linearization of nonlinear systems

- Consider the dynamics for \( x \in D \) of \( \mathbb{R}^n \),
  \[
  \dot{x} = f(x) = Ax + v(x),
  \]
  which induces a flow \( \phi(x, t) : D \times \mathbb{R} \rightarrow D \)

- **Hartman-Grobman Theorem** Let \( f \in C^1(D) \). If \( A \) is hyperbolic, \( \exists h : U \rightarrow V \) with \( U \subset D, 0 \in U \) and \( V \subset \mathbb{R}^n, 0 \in V \) such that \( \forall x_0 \in U, \exists I_0 \subset \mathbb{R} \) when \( x_0 \in U \) and \( t \in I_0 \),
  \[
  h \circ \phi(x_0, t) = e^{At}h(x_0); \]
  i.e., \( h \) maps trajectories of (1) near the origin to trajectories of \( \dot{x} = Ax \) and preserves the time parametrization.

- **Hartman’s theorem and Poincaré-Siegel theorem.**

- **Global linearization:** weak nonlinearity or symmetry by lie group theory.
Consider the dynamics for $x \in D$ of $\mathbb{R}^n$,

$$\dot{x} = f(x) = Ax + v(x),$$  \hspace{1cm} (1)

which induces a flow $\phi(x, t) : D \times \mathbb{R} \rightarrow D$

**Hartman-Grobman Theorem** Let $f \in C^1(D)$. If $A$ is hyperbolic, $\exists h : U \rightarrow V$ with $U \subset D$, $0 \in U$ and $V \subset \mathbb{R}^n$, $0 \in V$ such that $\forall x_0 \in U$, $\exists I_0 \subset \mathbb{R}$ when $x_0 \in U$ and $t \in I_0$,

$$h \circ \phi(x_0, t) = e^{At}h(x_0);$$  \hspace{1cm} (2)

i.e., $h$ maps trajectories of (1) near the origin to trajectories of $\dot{x} = Ax$ and preserves the time parametrization.

**Hartman’s theorem and Poincaré-Siegel theorem.**

**Global linearization**: weak nonlinearity or symmetry by lie group theory.
Main contents

1 Introduction
   - Linear and nonlinear dynamics
   - Linearization in large
   - Examples

2 The Koopman operator
   - Its introduction
   - Koopman operator and partition of the phase space

3 Applications
   - The standard map
   - Application to fluid dynamics

4 Summary
Problems and general consideration

- Problems in the current linearization scheme:
  - Local theorems: too limited region of validity;
  - Global theorems: linear terms dominate or a complete solution of the equation is required.

- In essence, linearizability means a conjugacy a nonlinear system $\leftrightarrow$ a linear system

- Thus
  - (1) the linearizable region can contain only one equilibrium;
  - (2) chaotic trajectories cannot be linearized;
  - (3) At most one attractor or one repeller exists in the region.
Problems and general consideration

- Problems in the current linearization scheme:
  Local theorems: too limited region of validity;
  Global theorems: linear terms dominate or a complete solution of the equation is required.

- In essence, linearizability means a conjugacy:
  a nonlinear system $\leftrightarrow$ a linear system

- Thus
  (1) the linearizable region can contain only one equilibrium;
  (2) chaotic trajectories cannot be linearized;
  (3) At most one attractor or one repeller exists in the region.
Problems in the current linearization scheme:
Local theorems: too limited region of validity;
Global theorems: linear terms dominate or a complete solution of the equation is required.

In essence, linearizability means a conjugacy
a nonlinear system $\leftrightarrow$ a linear system

Thus
(1) the linearizable region can contain only one equilibrium;
(2) chaotic trajectories cannot be linearized;
(3) At most one attractor or one repeller exists in the region.
Autonomous flow linearization

- If all eigenvalues of $A$ have negative real parts. So, $x = 0$ is exponentially stable and let $\Omega$ be its basin of attraction. Then $\exists h(x) \in C^1(\Omega) : \Omega \to \mathbb{R}^n$, such that $y = a(x) = x + h(x)$ is a $C^1$ diffeomorphism with $Da(0) = I$ in $\Omega$ and satisfies $\dot{y} = Ay$.

- The map $h(x)$ could be obtained by solving

\[
\begin{align*}
\frac{dx}{dt} &= Ax + v(x), \\
\frac{dh}{dt} &= Ah - v(x),
\end{align*}
\]

where $h|_{\Sigma} = \tilde{h}|_{\Sigma}$. It is easy to see that $d(x + h)/dt = A \cdot (x + h)$ and the value $h(x)$ for $x \notin \Sigma$ is defined by the flow along the integral curve passing $x$. 

Yueheng Lan
Autonomous flow linearization

- If all eigenvalues of $A$ have negative real parts. So, $x = 0$ is exponentially stable and let $\Omega$ be its basin of attraction. Then $\exists h(x) \in C^1(\Omega) : \Omega \rightarrow \mathbb{R}^n$, such that $y = a(x) = x + h(x)$ is a $C^1$ diffeomorphism with $D a(0) = I$ in $\Omega$ and satisfies $\dot{y} = Ay$.

- The map $h(x)$ could be obtained by solving

$$\frac{dx}{dt} = Ax + v(x),$$
$$\frac{dh}{dt} = Ah - v(x),$$

where $h|_{\Sigma} = \tilde{h}|_{\Sigma}$. It is easy to see that $d(x + h)/dt = A \cdot (x + h)$ and the value $h(x)$ for $x \notin \Sigma$ is defined by the flow along the integral curve passing $x$. 

The generalization

- **Linearization of diffeomorphisms** Consider
  \[ x_{m+1} = f(x_m) = Ax_m + v(x_m) \] with \( x_m \in \mathbb{R}^n \), where \( v(x) \sim O(|x|^2) \) and \( A \) is an \( n \times n \) matrix with magnitude of all eigenvalues smaller than 1, then in the basin of attraction \( D \) of the origin \( x = 0 \), \( \exists y = a(x) = x + h(x) \) with \( Da(0) = I \) which transforms the original map \( f(x) \) to a linear one \( y_{m+1} = Ay_m \).

- Linearization around an attractive or repulsive periodic orbit.

- How to treat saddles? Applicable to flows on stable or unstable manifolds.
The generalization

- **Linearization of diffeomorphisms** Consider
  \[ x_{m+1} = f(x_m) = Ax_m + v(x_m) \] with \( x_m \in \mathbb{R}^n \), where
  \[ v(x) \sim O(|x|^2) \] and \( A \) is an \( n \times n \) matrix with magnitude of all eigenvalues smaller than 1, then in the basin of attraction \( D \) of the origin \( x = 0 \), \( \exists y = a(x) = x + h(x) \) with
  \[ Da(0) = I \] which transforms the original map \( f(x) \) to a linear one \( y_{m+1} = Ay_m \).

- Linearization around an attractive or repulsive periodic orbit.

- How to treat saddles? Applicable to flows on stable or unstable manifolds.
Linearization of diffeomorphisms Consider
\[ x_{m+1} = f(x_m) = Ax_m + v(x_m) \] with \( x_m \in \mathbb{R}^n \), where \( v(x) \sim O(|x|^2) \) and \( A \) is an \( n \times n \) matrix with magnitude of all eigenvalues smaller than 1, then in the basin of attraction \( D \) of the origin \( x = 0 \), \( \exists y = a(x) = x + h(x) \) with \( Da(0) = I \) which transforms the original map \( f(x) \) to a linear one \( y_{m+1} = Ay_m \).

- Linearization around an attractive or repulsive periodic orbit.
- How to treat saddles? Applicable to flows on stable or unstable manifolds.
Main contents

1 Introduction
   - Linear and nonlinear dynamics
   - Linearization in large
   - Examples

2 The Koopman operator
   - Its introduction
   - Koopman operator and partition of the phase space

3 Applications
   - The standard map
   - Application to fluid dynamics

4 Summary
Two examples

- Consider the 1-d equation $\dot{x} = x - x^3$. The transformation
  \[ x = b(y) = \frac{y}{\sqrt{1 + y^2}} \]
  results in $\dot{y} = y$, valid for $x \in [-1, 1]$.

- Consider the 2-d system $\dot{z}_1 = 2z_1, \dot{z}_2 = 4z_2 + z_1^2$. The transformation
  \[ z_1 = y_1, z_2 = y_2 + t(y_1, y_2)y_1^2 \]
  where $t(y_1, y_2) = \frac{1}{4} \ln y_1^2$ results in
  \[ \dot{y}_1 = 2y_1, \dot{y}_2 = 4y_2. \]

[Y. Lan and I. Mezic, Physica D. 242, 42(2013)]
Two examples

- Consider the 1-d equation $\dot{x} = x - x^3$. The transformation
  \[ x = b(y) = \frac{y}{\sqrt{1 + y^2}} \]
  results in $\dot{y} = y$, valid for $x \in [-1, 1]$.

- Consider the 2-d system $\dot{z}_1 = 2z_1$, $\dot{z}_2 = 4z_2 + z_1^2$. The transformation
  \[ z_1 = y_1, \quad z_2 = y_2 + t(y_1, y_2)y_1^2 \]
  where $t(y_1, y_2) = \frac{1}{4} \ln y_1^2$ results in
  \[ \dot{y}_1 = 2y_1, \quad \dot{y}_2 = 4y_2. \]

[Y. Lan and I. Mezic, Physica D. 242, 42(2013)]
Main contents

1. Introduction
   - Linear and nonlinear dynamics
   - Linearization in large
   - Examples

2. The Koopman operator
   - Its introduction
   - Koopman operator and partition of the phase space

3. Applications
   - The standard map
   - Application to fluid dynamics

4. Summary
A nonlinear system can be linearized in the whole basin of attraction of an equilibrium or a periodic orbit. According to the Morse theory, the whole phase space can be viewed as a gradient system quotient the minimal transitive invariant sets. The phase space is a juxtaposition of linearizable patches.

- The theorems are existence ones and it is hard to identify exactly the linearization transformation.
- At present, theorems are only proved for equilibria and periodic orbits.
- Without explicit analytic expression, it is hard to deduce the linearization from experimental observation.
Remarks on the linearization theorem

- A nonlinear system can be linearized in the whole basin of attraction of an equilibrium or a periodic orbit. According to the Morse theory, the whole phase space can be viewed as a gradient system quotient the minimal transitive invariant sets. The phase space is a juxtaposition of linearizable patches.

- The theorems are existence ones and it is hard to identify exactly the linearization transformation.

- At present, theorems are only proved for equilibria and periodic orbits.

- Without explicit analytic expression, it is hard to deduce the linearization from experimental observation.
Remarks on the linearization theorem

- A nonlinear system can be linearized in the whole basin of attraction of an equilibrium or a periodic orbit. According to the Morse theory, the whole phase space can be viewed as a gradient system quotient the minimal transitive invariant sets. The phase space is a juxtaposition of linearizable patches.

- The theorems are existence ones and it is hard to identify exactly the linearization transformation.

- At present, theorems are only proved for equilibria and periodic orbits.

- Without explicit analytic expression, it is hard to deduce the linearization from experimental observation.
Remarks on the linearization theorem

- A nonlinear system can be linearized in the whole basin of attraction of an equilibrium or a periodic orbit. According to the Morse theory, the whole phase space can be viewed as a gradient system quotient the minimal transitive invariant sets. The phase space is a juxtaposition of linearizable patches.

- The theorems are existence ones and it is hard to identify exactly the linearization transformation.

- At present, theorems are only proved for equilibria and periodic orbits.

- Without explicit analytic expression, it is hard to deduce the linearization from experimental observation.
Koopman operator, a way out?

- A statistical point of view:
  Evolution of densities: the Perron-Frobenius Operator in analogy with the Schrödinger picture;
  Evolution of observables: the Koopman operator in analogy with the Heisenberg picture.

- Definition: for a map \( x_{n+1} = f(x_n) \) and a function \( g(x) \), the Koopman operator \( U \circ g(x) = g(f(x)) \)

- For a flow \( \phi(x, t) \) and a function \( g(x) \), a semigroup of Koopman operators could be defined as \( U_t \circ g(x) = g(\phi(x, t)) \).
Koopman operator, a way out?

- A statistical point of view:
  Evolution of densities: the Perron-Frobenius Operator in analogy with the Schrödinger picture;
  Evolution of observables: the Koopman operator in analogy with the Heisenberg picture.

- Definition: for a map $x_{n+1} = f(x_n)$ and a function $g(x)$, the Koopman operator $U \circ g(x) = g(f(x))$

- For a flow $\phi(x,t)$ and a function $g(x)$, a semigroup of Koopman operators could be defined as $U_t \circ g(x) = g(\phi(x,t))$. 
Koopman operator, a way out?

- A statistical point of view:
  Evolution of densities: the Perron-Frobenius Operator in analogy with the Schrödinger picture;
  Evolution of observables: the Koopman operator in analogy with the Heisenberg picture.

- Definition: for a map $x_{n+1} = f(x_n)$ and a function $g(x)$, the Koopman operator $U \circ g(x) = g(f(x))$

- For a flow $\phi(x, t)$ and a function $g(x)$, a semigroup of Koopman operators could be defined as $U_t \circ g(x) = g(\phi(x, t))$. 
It is a linear operator, which is unitary on a transitive invariant set.

The eigenvalues and eigenmodes are interesting objects.

For a 1-d linear map \( x_{n+1} = \lambda x_n \) and observable \( g(x) = x^n \),

\[
U \circ g(x) = (\lambda x)^n = \lambda^n x^n.
\]

For a 1-d equation \( \dot{x} = \lambda x \) and observable \( g(x) = x^n \)

\[
U_t \circ g(x) = (xe^{\lambda t})^n = e^{n\lambda t} x^n.
\]
• It is a linear operator, which is unitary on a transitive invariant set.
• The eigenvalues and eigenmodes are interesting objects.
• For a 1-d linear map $x_{n+1} = \lambda x_n$ and observable $g(x) = x^n$,
  $$U \circ g(x) = (\lambda x)^n = \lambda^n x^n.$$ 
• For a 1-d equation $\dot{x} = \lambda x$ and observable $g(x) = x^n$
  $$U_t \circ g(x) = (xe^{\lambda t})^n = e^{n\lambda t}x^n.$$
Nontrivial examples

- Note that $h(x)$ in the Hartman-Grobman’s theorem satisfies
  \[ U_t h(x) = h \circ \phi(x, t) = e^{At} h(x) \]

- Suppose
  \[ V^{-1} AV = \Lambda, \]
  then we get
  \[ V^{-1} h \circ \phi(x, t) = V^{-1} e^{At} h(x), \]
  and so $k = V^{-1} h$ satisfies
  \[ k \circ \phi(x_0, t) = e^{\Lambda t} k(x_0) \]
  i.e. each component function of $k$ is an eigenfunction of $U_t$. 
Nontrivial examples

- Note that \( h(x) \) in the Hartman-Grobman’s theorem satisfies
  \[
  U_t h(x) = h \circ \phi(x, t) = e^{At} h(x)
  \]

- Suppose
  \[
  V^{-1}AV = \Lambda,
  \]
  then we get
  \[
  V^{-1}h \circ \phi(x, t) = V^{-1}e^{At} h(x),
  \]
  and so \( k = V^{-1}h \) satisfies
  \[
  k \circ \phi(x_0, t) = e^{At} k(x_0)
  \]
  i.e. each component function of \( k \) is an eigenfunction of \( U_t \).
Main contents

1 Introduction
   - Linear and nonlinear dynamics
   - linearization in large
   - Examples

2 The Koopman operator
   - Its introduction
   - Koopman operator and partition of the phase space

3 Applications
   - The standard map
   - Application to fluid dynamics

4 Summary
Construction of eigenmodes along trajectories

- For the map \( x_{n+1} = T(x_n) \) and function \( g(x) \), consider

\[
g^*(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} g(T^j x),
\]

which is an eigenfunction of the Koopman operator with eigenvalue 1.

- Furthermore, the construction

\[
g^\omega(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{i2\pi j \omega} g(T^j x)
\]

defines an eigenfunction of the Koopman operator with eigenvalue \( e^{-i2\pi \omega} \).
Construction of eigenmodes along trajectories

- For the map $x_{n+1} = T(x_n)$ and function $g(x)$, consider

$$g^*(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} g(T^j x),$$

which is an eigenfunction of the Koopman operator with eigenvalue 1.

- Furthermore, the construction

$$g^\omega(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{i2\pi j \omega} g(T^j x)$$

defines an eigenfunction of the Koopman operator with eigenvalue $e^{-i2\pi \omega}$. 
Spectral decomposition of evolution equations

- For an evolution equation in an infinite-dimensional Hilbert space \( v(x)^{n+1} = N(v(x)^n, p) \) and if the attractor \( M \) is of finite dimension with the evolution \( m^{n+1} = T(m^n) \).

For an observable \( g(x, m) \), we have

\[
Ug(x, m) = U_sg(x, m) + U_r g(x, m)
\]

\[
= g^*(x) + \sum_{j=1}^{k} \lambda_j f_j(m) g_j(x) + \int_{0}^{1} e^{i2\pi \alpha} dE(\alpha) g(x, m).
\]

- \( U_s \): the singular part of the operator corresponding to the discrete part of the spectrum, viewed as a deterministic part.
- \( U_r \): the regular part of the operator corresponding to the continuous part of the spectrum, viewed as a stochastic part.

[I. Mezic, Nonlinear Dynamics 41, 309(2005)]
Spectral decomposition of evolution equations

- For an evolution equation in an infinite-dimensional Hilbert space $v(x)^{n+1} = N(v(x)^n, p)$ and if the attractor $M$ is of finite dimension with the evolution $m^{n+1} = T(m^n)$.

For an observable $g(x, m)$, we have

$$Ug(x, m) = Usg(x, m) + Ur g(x, m)$$

$$= g^*(x) + \sum_{j=1}^{k} \lambda_j f_j(m)g_j(x) + \int_0^1 e^{i2\pi\alpha}dE(\alpha)g(x, m).$$

- $Us$: the singular part of the operator corresponding to the discrete part of the spectrum, viewed as a deterministic part.
- $Ur$: the regular part of the operator corresponding to the continuous part of the spectrum, viewed as a stochastic part.

[I. Mezic, Nonlinear Dynamics 41, 309(2005)]
1 Introduction
   • Linear and nonlinear dynamics
   • Linearization in large
   • Examples

2 The Koopman operator
   • Its introduction
   • Koopman operator and partition of the phase space

3 Applications
   • The standard map
   • Application to fluid dynamics

4 Summary
The Chirikov standard map is

\[ x_1^* = x_1 + 2\pi \epsilon \sin(x_2) (\text{mod} 2\pi) \]

\[ x_2^* = x_1^* + x_2 (\text{mod} 2\pi) \]

The embedding of dynamics into space of three observables.

[M. Budisic and I. Mezic, 48th IEEE Conference on Decision and Control]
Several eigenmodes of the Koopman operator

Eigenvalues for $\lambda_1, \lambda_2, \lambda_7, \lambda_{17}$ at $\epsilon = 0.133$. 
Main contents

1 Introduction
   - Linear and nonlinear dynamics
   - Linearization in large
   - Examples

2 The Koopman operator
   - Its introduction
   - Koopman operator and partition of the phase space

3 Applications
   - The standard map
   - Application to fluid dynamics

4 Summary
Fluid dynamics

- The Navier-Stokes equation

\[ v_t + v \cdot \nabla v = -\frac{\nabla p}{\rho} + \nu \nabla^2 v \]

with \( \nabla \cdot v = 0 \) describes incompressible Newtonian fluids.

- With different Reynold’s number \( Re = Lv/\nu \), the system experience a series of bifurcation: laminar \( \rightarrow \) periodic \( \rightarrow \) turbulent

- Turbulence is a spatiotemporal chaos with enormous space-time structures and scales.

- Jet in cross flow: turbulent but with large eddies. Could we describe it with the Koopman operator approach?
The Navier-Stokes equation

\[ \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v} \]

with \( \nabla \cdot \mathbf{v} = 0 \) describes incompressible Newtonian fluids.

With different Reynold’s number \( Re = Lv/\nu \), the system experience a series of bifurcation: laminar \( \rightarrow \) periodic \( \rightarrow \) turbulent

Turbulence is a spatiotemporal chaos with enormous space-time structures and scales.

Jet in cross flow: turbulent but with large eddies. Could we describe it with the Koopman operator approach?
• The Navier-Stokes equation

\[ \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v} \]

with \( \nabla \cdot \mathbf{v} = 0 \) describes incompressible Newtonian fluids.

• With different Reynold’s number \( Re = \frac{Lv}{\nu} \), the system experience a series of bifurcation: laminar \( \rightarrow \) periodic \( \rightarrow \) turbulent

• Turbulence is a spatiotemporal chaos with enormous space-time structures and scales.

• Jet in cross flow: turbulent but with large eddies. Could we describe it with the Koopman operator approach?
Fluid dynamics

- The Navier-Stokes equation

\[ \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v} \]

with \( \nabla \cdot \mathbf{v} = 0 \) describes incompressible Newtonian fluids.

- With different Reynold’s number \( Re = L\nu/\nu \), the system experience a series of bifurcation:
laminar \( \rightarrow \) periodic \( \rightarrow \) turbulent

- Turbulence is a spatiotemporal chaos with enormous space-time structures and scales.

- Jet in cross flow: turbulent but with large eddies. Could we describe it with the Koopman operator approach?
Jet in cross flow

The Arnoldi algorithm

Consider a linear dynamical system $x_{k+1} = Ax_k$ and construct the matrix

$$K = [x_0, x_1, \cdots, x_{m-1}] = [x_0, Ax_0, \cdots, A^{m-1}x_0].$$

If the $m$th iterate $x_m = Ax_{m-1} = \sum_{k=0}^{m-1} c_k x_k + r$, the we can write $AK \approx KC$, where

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & c_0 \\ 1 & 0 & 0 & \cdots & c_1 \\ 0 & 1 & 0 & \cdots & c_2 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & c_0 \end{pmatrix}.$$

If $Ca = \lambda a$, then the value $\lambda$ and the vector $v = Ka$ are approximate eigenvalue and eigenvector of the original matrix $A$. 

Yueheng Lan

Linearization theorems, Koopman operator and its applications
Consider a linear dynamical system $x_{k+1} = Ax_k$ and construct the matrix

$$K = [x_0, x_1, \cdots, x_{m-1}] = [x_0, Ax_0, \cdots, A^{m-1}x_0].$$

If the $m$th iterate $x_m = Ax_{m-1} = \sum_{k=0}^{m-1} c_k x_k + r$, then we can write $AK \approx KC$, where

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & c_0 \\ 1 & 0 & 0 & \cdots & c_1 \\ 0 & 1 & 0 & \cdots & c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & c_0 \end{pmatrix}. $$

If $Ca = \lambda a$, then the value $\lambda$ and the vector $v = Ka$ are approximate eigenvalue and eigenvector of the original matrix $A$. 
Two structure functions

The two eigenmodes
Summary

- Linearization is possible in the basin of attraction of a hyperbolic set.
- Koopman operator provides a way to identify the linearization transformation.
- On the transitive invariant set, the spectrum of the Koopman operator is on the unit circle in the complex plane.
- The eigenmodes could be constructed through numerical computation, revealing the most important dynamics.
- Generalizations and challenges:
  1. Can deal with stochastic systems.
  2. How to deal with uncertainty embedded in complex systems.
  3. How to construct eigenmodes from information pieces.
Summary

- Linearization is possible in the basin of attraction of a hyperbolic set.
- Koopman operator provides a way to identify the linearization transformation.
- On the transitive invariant set, the spectrum of the Koopman operator is on the unit circle in the complex plane.
- The eigenmodes could be constructed through numerical computation, revealing the most important dynamics.
- Generalizations and challenges:
  1. Can deal with stochastic systems.
  2. How to deal with uncertainty embedded in complex systems.
  3. How to construct eigenmodes from information pieces.
Summary

- Linearization is possible in the basin of attraction of a hyperbolic set.
- Koopman operator provides a way to identify the linearization transformation.
- On the transitive invariant set, the spectrum of the Koopman operator is on the unit circle in the complex plane.
- The eigenmodes could be constructed through numerical computation, revealing the most important dynamics.
- Generalizations and challenges:
  1. Can deal with stochastic systems.
  2. How to deal with uncertainty embedded in complex systems.
  3. How to construct eigenmodes from information pieces.
Summary

- Linearization is possible in the basin of attraction of a hyperbolic set.
- Koopman operator provides a way to identify the linearization transformation.
- On the transitive invariant set, the spectrum of the Koopman operator is on the unit circle in the complex plane.
- The eigenmodes could be constructed through numerical computation, revealing the most important dynamics.
- Generalizations and challenges:
  1. Can deal with stochastic systems.
  2. How to deal with uncertainty embedded in complex systems.
  3. How to construct eigenmodes from information pieces.
Summary

- Linearization is possible in the basin of attraction of a hyperbolic set.
- Koopman operator provides a way to identify the linearization transformation.
- On the transitive invariant set, the spectrum of the Koopman operator is on the unit circle in the complex plane.
- The eigenmodes could be constructed through numerical computation, revealing the most important dynamics.
- Generalizations and challenges:
  1. Can deal with stochastic systems.
  2. How to deal with uncertainty embedded in complex systems.
  3. How to construct eigenmodes from information pieces.