

FREDHOLM DETERMINANTS AND THE EVANS FUNCTION FOR DIFFERENCE EQUATIONS

DAVID CRAMER and YURI LATUSHKIN

Department of Mathematics, University of Missouri, Columbia, MO 65211, U.S.A.

E-mail: mathgr6@math.missouri.edu, yuri@math.missouri.edu

Abstract. We develop a difference equations analogue of recent results by F. Gesztesy, K. A. Makarov, and the second author relating the Evans function and Fredholm determinants of operators with semi-separable kernels.

1. Introduction. The purpose of this paper is to provide a difference equations version of some of the most recent results in [GM, GML, GML1] relating the Evans function and Fredholm determinants of operators with semi-separable kernels. Although our general strategy is close to that in [GM, GML] for the differential equations case, and for simplicity we consider less general assumptions than in [GML], the arguments and the results in the difference equations setting have some important differences. For a related work we cite [GGK, GKvS] and [BCK, KK]. For a detailed historical account and the bibliography we refer to [GML].

We consider an unperturbed difference equation $x_{j+1} = A_j x_j$ and its perturbation in the form $x_{j+1} = A_j^\times x_j$, $j \in \mathbb{Z}$, where $A_j^\times = A_j + B_j C_j$. Here and below, A_j, B_j , and C_j are $(d \times d)$ matrices with complex entries, and $\mathbf{x} = (x_j)_{j \in \mathbb{Z}}$ is a sequence of vectors $x_j \in \mathbb{C}^d$. Throughout, we assume that the matrices A_j and A_j^\times are invertible and that the unperturbed equation has an exponential dichotomy over \mathbb{Z} with the (unstable) dichotomy projection P . Let $\mathbf{U} = (U_j)_{j \in \mathbb{Z}}$ denote the fundamental matrix solution of the unperturbed equation normalized by $U_0 = I$, the identity matrix.

2000 *Mathematics Subject Classification*: Primary 34D09, 47B39; Secondary 39xx, 34D08.

Key words and phrases: Fredholm determinants, Evans function, Lyapunov exponents.

Research of the second author was supported in part by the NSF grants 0338743 and 0354339, by the CRDF grant UP1-2567-OD-03 and by the Research Board and the Research Council of the University of Missouri.

The authors thank F. Gesztesy for helpful discussions and K. A. Makarov for many valuable suggestions.

The paper is in final form and no version of it will be published elsewhere.

In the first part of the paper, following [GM], we give formulas for the (modified) Fredholm determinants of the difference operator $\mathcal{T} = (T_{jk})_{j,k \in \mathbb{Z}}$ on $\ell^2(\mathbb{Z}; \mathbb{C}^d)$ whose kernel is given by the formulas

$$(1.1) \quad T_{jk} = C_j U_j (I - P) U_{k+1}^{-1} B_k \text{ for } j > k \text{ and } T_{jk} = -C_j U_j P U_{k+1}^{-1} B_k \text{ for } j \leq k.$$

Note that the kernel of every difference operator with a semi-separable kernel admits a representation (1.1), see formulas (3.9) - (3.10) below. The choice of kernel (1.1) is related to the following elementary observation. Given the matrix sequence $\mathbf{A} = (A_j)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}; \mathbb{C}^{d \times d})$, define on $\ell^2(\mathbb{Z}; \mathbb{C}^d)$ an operator, $G_{\mathbf{A}}$, by $(G_{\mathbf{A}} \mathbf{x})_j = x_{j+1} - A_j x_j$ so that the inhomogeneous equations $x_{j+1} = A_j x_j + y_j, j \in \mathbb{Z}$, becomes $G_{\mathbf{A}} \mathbf{x} = \mathbf{y}$. Due to the exponential dichotomy [CL], the operator $G_{\mathbf{A}}$ is invertible in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$, and by a direct computation, its inverse is a difference operator, $\mathcal{K} = (K_{jk})_{j,k \in \mathbb{Z}}$, with kernel defined by

$$(1.2) \quad K_{jk} = U_j (I - P) U_{k+1}^{-1} \text{ for } j > k \text{ and } K_{jk} = -U_j P U_{k+1}^{-1} \text{ for } j \leq k.$$

If $\mathbf{A}^\times = (A_j^\times)_{j \in \mathbb{Z}}$ then the operator $G_{\mathbf{A}^\times} = G_{\mathbf{A}} - \text{diag}(B_j C_j)_{j \in \mathbb{Z}}$ can be represented as $G_{\mathbf{A}^\times} = G_{\mathbf{A}} (I - \mathcal{K} \text{diag}(B_j)_{j \in \mathbb{Z}} \text{diag}(C_j)_{j \in \mathbb{Z}})$, and $G_{\mathbf{A}^\times}$ is invertible if and only if the operator $I - \mathcal{T}$ is invertible; the kernel of $\mathcal{T} = \text{diag}(C_j)_{j \in \mathbb{Z}} \mathcal{K} \text{diag}(B_j)_{j \in \mathbb{Z}}$ is given by (1.1).

In the second part of the paper, following [GML, GML1], we construct appropriate matrix solutions of the perturbed difference equation whose determinant, \mathcal{E} , is called the *Evans determinant*. If the sequence $\mathbf{A}^\times = \mathbf{A}^\times(z)$ depends on a spectral parameter $z \in \mathbb{C}$, then the corresponding function $\mathcal{E} = \mathcal{E}(z)$ becomes the *Evans function*, a Wronskian type object widely used to detect unstable modes for operators obtained by linearizing nonlinear equations along special particular solutions such as travelling waves, see [AGJ] and recent reviews [JK, S] and the bibliographies therein. We stress that the Evans determinant, as defined in the current paper, is uniquely determined by the sequences \mathbf{A}^\times and \mathbf{A} . Moreover (and this is the central result of this paper), we derive a formula relating \mathcal{E} and the Fredholm determinant of $I - \mathcal{T}$ (for results in this spirit in the case of the Schrödinger differential operator see [KS, p. 861] and [KS1]). Finally, for the discrete Schrödinger operator, we show that the Evans function coincides with the Jost function, the classical object familiar from scattering theory, see e.g. [CS, Chap. XVII], [FT, Sec. III.2], [GH, Sec. 6], [T, Chap. 10], and [To, Chap. 3].

2. Notation and preliminaries. The set of $(d \times d)$ matrices with complex entries is denoted by $\mathbb{C}^{d \times d}$. Where possible, we abbreviate $\ell^2 = \ell^2(\mathbb{Z}; \mathbb{C}^d)$ or $\ell^2 = \ell^2(\mathbb{Z}; \mathbb{C}^{d \times d})$. We use boldface to denote sequences of vectors or matrices, e.g. $\mathbf{x} = (x_j)_{j \in \mathbb{Z}}, x_j \in \mathbb{C}^d$, or $\mathbf{a} = (a_j)_{j \in \mathbb{Z}}, a_j \in \mathbb{C}^{d \times d}$. We denote by $\sigma(\cdot)$ the spectrum of an operator, and by I (or sometimes $I_{d \times d}$) the identity operator. For a projection P on \mathbb{C}^d with $\dim \text{Im } P = d_1$ we often identify $I_{d_1 \times d_1}$ and P on $\text{Im } P$. The restriction of an operator A on a subspace (\cdot) is denoted by $A|_{(\cdot)}$. If A satisfies $A = AP$ then we denote $\|A\|_\bullet = \inf\{\|Ax\| : x = Px, \|x\| = 1\}$.

The sets of trace-class and Hilbert-Schmidt operators on a Hilbert space (\cdot) are denoted, respectively, by $\mathcal{B}_1 = \mathcal{B}_1(\cdot)$ and $\mathcal{B}_2 = \mathcal{B}_2(\cdot)$. Recall that $\ell^1 \subset \ell^2 \subset \ell^\infty$ and $\mathcal{B}_1(\cdot) \subset \mathcal{B}_2(\cdot)$. We will use the following properties of the (modified) Fredholm determinants, see, e.g. [GGK, Si] for more information:

$$(2.1) \quad \det(I - A) = \prod_{\lambda \in \sigma(A)} (1 - \lambda), \quad A \in \mathcal{B}_1,$$

$$(2.2) \quad \det_2(I - A) = \det[(I - A)e^A] = \prod_{\lambda \in \sigma(A)} (1 - \lambda)e^\lambda, \quad A \in \mathcal{B}_2,$$

$$(2.3) \quad \det_2(I - A) = \det(I - A)e^{\text{tr}A}, \quad A \in \mathcal{B}_1,$$

$$(2.4) \quad \det_2[(I - A)(I - B)] = \det_2(I - A) \det_2(I - B)e^{-\text{tr}(AB)}, \quad A, B \in \mathcal{B}_2.$$

Given matrix sequences $\mathbf{a}_j = (a_j)_{j \in \mathbb{Z}}$, $\mathbf{b}_j = (b_j)_{j \in \mathbb{Z}}$, and $\mathbf{d}_j = (d_j)_{j \in \mathbb{Z}}$, we define the upper triangular operator $V_{\mathbf{a}, \mathbf{b}}^+$, the lower triangular operator $V_{\mathbf{a}, \mathbf{b}}^-$, the diagonal operator $D_{\mathbf{d}}$, and the operator $V_{\mathbf{a}, \mathbf{b}}$ on $\ell^2(\mathbb{Z}; \mathbb{C}^d)$ as follows:

$$(2.5) \quad (V_{\mathbf{a}, \mathbf{b}}^+ \mathbf{x})_j = \sum_{k=j+1}^{\infty} a_j b_k x_k, \quad (V_{\mathbf{a}, \mathbf{b}}^- \mathbf{x})_j = \sum_{k=-\infty}^{j-1} a_j b_k x_k,$$

$$(2.6) \quad (D_{\mathbf{d}} \mathbf{x})_j = d_j x_j, \quad (V_{\mathbf{a}, \mathbf{b}} \mathbf{x})_j = \sum_{k=-\infty}^{\infty} a_j b_k x_k, \quad j \in \mathbb{Z}.$$

We summarize properties of these operators in the following elementary lemmas.

LEMMA 2.1. *Assume $\mathbf{a}, \mathbf{b}, \mathbf{d} \in \ell^2(\mathbb{Z}; \mathbb{C}^{d \times d})$. Then:*

$$(2.7) \quad \sigma(D_{\mathbf{d}}) = \{0\} \cup (\cup_{j \in \mathbb{Z}} \sigma(d_j));$$

$$(2.8) \quad V_{\mathbf{a}, \mathbf{b}}, V_{\mathbf{a}, \mathbf{b}}^\pm, D_{\mathbf{d}} \in \mathcal{B}_2(\ell^2(\mathbb{Z}; \mathbb{C}^d));$$

$$(2.9) \quad \sigma(V_{\mathbf{a}, \mathbf{b}}^+) = \sigma(V_{\mathbf{a}, \mathbf{b}}^-) = \{0\};$$

$$(2.10) \quad \det_2(I - V_{\mathbf{a}, \mathbf{b}}^+) = \det_2(I - V_{\mathbf{a}, \mathbf{b}}^-) = 1;$$

$$(2.11) \quad \det_2(I - D_{\mathbf{d}}) = \prod_{j \in \mathbb{Z}} \det(I_{d \times d} - d_j) e^{\text{tr} d_j}.$$

LEMMA 2.2. *Assume $\mathbf{a}, \mathbf{b}, \mathbf{d} \in \ell^1(\mathbb{Z}; \mathbb{C}^{d \times d})$. Then:*

$$(2.12) \quad V_{\mathbf{a}, \mathbf{b}}^\pm \text{ are compact operators on } \ell^\infty(\mathbb{Z}; \mathbb{C}^d) \text{ and } \sigma(V_{\mathbf{a}, \mathbf{b}}^\pm) = \{0\};$$

$$(2.13) \quad V_{\mathbf{a}, \mathbf{b}}, V_{\mathbf{a}, \mathbf{b}}^\pm, D_{\mathbf{d}} \in \mathcal{B}_1(\ell^2(\mathbb{Z}; \mathbb{C}^d));$$

$$(2.14) \quad \det(I - V_{\mathbf{a}, \mathbf{b}}^+) = \det(I - V_{\mathbf{a}, \mathbf{b}}^-) = 1;$$

$$(2.15) \quad \det(I - D_{\mathbf{d}}) = \prod_{j \in \mathbb{Z}} \det(I_{d \times d} - d_j).$$

Proof. If $\mathbf{d} \in \ell^2$ then $\|d_j\| \rightarrow 0$ as $|j| \rightarrow \infty$ and then, for any $r > 0$, there are only finitely many d_j 's having $\lambda \in \sigma(d_j)$ with $|\lambda| \geq r$. The formula $\sigma(D_{\mathbf{d}}) = \text{closure}(\cup_{j \in \mathbb{Z}} \sigma(d_j))$ now implies (2.7).

Note that $V_{\mathbf{a}, \mathbf{b}} = D_{\mathbf{d}} + V_{\mathbf{a}, \mathbf{b}}^+ + V_{\mathbf{a}, \mathbf{b}}^-$ with $\mathbf{d} = (a_j b_j)_{j \in \mathbb{Z}}$ and $V_{\mathbf{a}, \mathbf{b}}^- = (V_{\mathbf{b}^*, \mathbf{a}^*}^+)^*$ with $\mathbf{a}^* = (a_j^*)_{j \in \mathbb{Z}}$ and $\mathbf{b}^* = (b_j^*)_{j \in \mathbb{Z}}$. Since $\mathbf{a} \in \ell^\infty$ and thus $(a_j b_j)_{j \in \mathbb{Z}}$ is in ℓ^2 , resp. ℓ^1 , it is enough to prove (2.9) and (2.8), resp. (2.13), only for $V_{\mathbf{a}, \mathbf{b}}^+$ and $D_{\mathbf{d}}$. Using (2.7), we infer:

$$(2.16) \quad \begin{aligned} \|D_{\mathbf{d}}\|_{\mathcal{B}_1(\ell^2)} &= \text{tr}[(D_{\mathbf{d}}^* D_{\mathbf{d}})^{\frac{1}{2}}] = \text{tr}[D_{(\mathbf{d}^* \mathbf{d})^{\frac{1}{2}}}] = \sum_{j \in \mathbb{Z}} \text{tr}(d_j^* d_j)^{\frac{1}{2}} \leq d \sum_{j \in \mathbb{Z}} \|d_j\| = d \|\mathbf{d}\|_{\ell^1}, \\ \|D_{\mathbf{d}}\|_{\mathcal{B}_2(\ell^2)} &= \text{tr}(D_{\mathbf{d}^* \mathbf{d}}) = \sum_{j \in \mathbb{Z}} \text{tr}(d_j^* d_j) \leq d \sum_{j \in \mathbb{Z}} \|d_j\|^2 = d \|\mathbf{d}\|_{\ell^2}. \end{aligned}$$

Assuming $\mathbf{a}, \mathbf{b} \in \ell^1$ and writing $V_{\mathbf{a}, \mathbf{b}}^+ = \sum_{k=1}^{\infty} \mathcal{S}^{-k} D_{(a_{j-k} b_j)_{j \in \mathbb{Z}}}$, where $(\mathcal{S}\mathbf{x})_j = x_{j-1}$ is the shift operator, we have:

$$\begin{aligned} \|V_{\mathbf{a}, \mathbf{b}}^+\|_{\mathcal{B}_1(\ell^2)} &\leq \sum_{k=1}^{\infty} \|D_{(a_{j-k} b_j)_{j \in \mathbb{Z}}}\|_{\mathcal{B}_1(\ell^2)} = \sum_{k=1}^{\infty} \sum_{j \in \mathbb{Z}} \|a_{j-k} b_j\|_{\mathcal{B}_1(\mathbb{C}^d)} \\ &\leq d \sum_{j \in \mathbb{Z}} \sum_{k=1}^{\infty} \|b_j\| \|a_{j-k}\| \leq d \|\mathbf{a}\|_{\ell^1} \|\mathbf{b}\|_{\ell^1}. \end{aligned}$$

Assuming $\mathbf{a}, \mathbf{b} \in \ell^2$ and considering the basis $\mathbf{y}_{n,i} = (y_{n,i}(j))_{j \in \mathbb{Z}}$, $i = 1, \dots, d$, $n \in \mathbb{Z}$, in $\ell^2(\mathbb{Z}; \mathbb{C}^d)$ given by $y_{n,i}(j) = 0$ for $j \neq n$ and $y_{n,i}(n) = \mathbf{e}_i$, the standard ort in \mathbb{C}^d , we have:

$$\begin{aligned} \|V_{\mathbf{a}, \mathbf{b}}^+\|_{\mathcal{B}_2(\ell^2)}^2 &= \text{tr}[(V_{\mathbf{a}, \mathbf{b}}^+)^* V_{\mathbf{a}, \mathbf{b}}^+] = \sum_{n \in \mathbb{Z}} \sum_{i=1}^d \|V_{\mathbf{a}, \mathbf{b}}^+ \mathbf{y}_{n,i}\|_{\ell^2}^2 \\ &= \sum_{n \in \mathbb{Z}} \sum_{i=1}^d \sum_{j \in \mathbb{Z}} \left\| a_j \sum_{k=j+1}^{\infty} b_k y_{n,i}(k) \right\|^2 = \sum_{n \in \mathbb{Z}} \sum_{i=1}^d \sum_{j=n-1}^{\infty} \|a_j b_n \mathbf{e}_i\|^2 \\ &\leq d \|\mathbf{a}\|_{\ell^2}^2 \|\mathbf{b}\|_{\ell^2}^2. \end{aligned}$$

This proves (2.8) and (2.13). To prove (2.9), observe that $\sigma(V_{\mathbf{a}, \mathbf{b}}^+)$ consists of eigenvalues since the operator $V_{\mathbf{a}, \mathbf{b}}^+$ is compact by (2.8). Suppose there are $\lambda \neq 0$ and $\mathbf{0} \neq \mathbf{x} \in \ell^2(\mathbb{Z}; \mathbb{C}^d)$, so that $a_j \sum_{k=j+1}^{\infty} b_k x_k = \lambda x_j$. Then

$$(2.17) \quad \sum_{j=n+1}^{\infty} c_j y_j = \lambda y_n, \quad n \in \mathbb{Z},$$

for $y_n = \sum_{j=n+1}^{\infty} b_j x_j$ and $c_j = b_j a_j$. Using Cauchy-Schwarz, we have $y_k \rightarrow 0$ as $k \rightarrow \infty$. Using (2.17), we have

$$(2.18) \quad y_n = \prod_{j=1}^k (I + c_{n+k}/\lambda) y_{n+k}, \quad n \in \mathbb{Z}, \quad k = 1, 2, \dots$$

Since $(c_j) \in \ell^1$, the product $\prod_{j=1}^{\infty} (1 + \|c_{n+j}\|/\lambda)$ converges for each n . Using this in (2.18) and letting $k \rightarrow \infty$, we conclude that for each $n \in \mathbb{Z}$ one has $0 = y_n = \sum_{j=n+1}^{\infty} b_j x_j = b_{n+1} x_{n+1} + y_{n+1} = b_{n+1} x_{n+1}$. Since $b_n x_n = 0$, we conclude that $\lambda x_j = 0$ for all j , a contradiction. Formulas (2.10), (2.11), (2.14), and (2.15) now follow from (2.2) and (2.7). To prove (2.12), represent $V_{\mathbf{a}, \mathbf{b}}^+ = D_{\mathbf{a}} V_{\mathbf{1}, \mathbf{b}}^+$ where $\mathbf{1} = (I)_{j \in \mathbb{Z}}$. Since $\|a_j\| \rightarrow 0$ as $|j| \rightarrow \infty$ and $\|V_{\mathbf{1}, \mathbf{b}}^+\| \leq \|\mathbf{b}\|_{\ell^1}$, the operator $V_{\mathbf{a}, \mathbf{b}}^+$ is compact on $\ell^\infty(\mathbb{Z}; \mathbb{C}^d)$. The existence of nonzero $\mathbf{x} \in \ell^\infty$ and λ so that $V_{\mathbf{a}, \mathbf{b}}^+ \mathbf{x} = \lambda \mathbf{x}$ leads to a contradiction, as in (2.17) - (2.18) above, since $\|y_k\| \leq \|\mathbf{x}\|_{\ell^\infty} \sum_{j=k+1}^{\infty} \|b_j\| \rightarrow 0$ as $k \rightarrow \infty$ due to $\mathbf{b} \in \ell^1$, and $\mathbf{c} = (a_j b_j)_{j \in \mathbb{Z}} \in \ell^1$. ■

Let E denote the $(d \times d)$ matrix having 1's on the diagonal above the main diagonal, and with zero remaining entries.

LEMMA 2.3. *If $A = \lambda I + E$, $\lambda \in \mathbb{C}$, is a $(d \times d)$ Jordan block, then $\|A^j\| \leq c|j|^d |\lambda|^j$ for all $j \in \mathbb{Z}$ and some positive constant $c = c(d, \lambda)$.*

Proof. Since $E^d = 0$, we have $A^j = \lambda^j (I + \sum_{k=1}^{d-1} \binom{j}{k} (E/\lambda)^k)$ for $j > 0$. The polynomial growth with j of the binomial coefficients $\binom{j}{k}$ gives the result. For $j > 0$, we estimate

the norm of $A^{-j} = \lambda^{-j} \sum_{k_1=0}^{d-1} \sum_{k_2=k_1}^{d-1} \cdots \sum_{k_j=k_{j-1}}^{d-1} (-E/\lambda)^{k_j}$ by $c|\lambda|^{-j}S(0, j)$, where we denote for $k \in [0, d-1]$:

$$S(k, j) = \sum_{k_1=k}^{d-1} \sum_{k_2=k_1}^{d-1} \cdots \sum_{k_j=k_{j-1}}^{d-1} 1 = \sum_{k_1=k}^{d-1} S(k_1, j-1).$$

Using the formula $\sum_{k_1=1}^{d-k} k_1(k_1+1) \cdots (k_1+j-1) = (d-k) \cdots (d-k+j)/(j+1)$ and induction, one obtains $S(k, j) = (d-k)(d-k+1) \cdots (d-k+j-1)/j!$ and the lemma follows. ■

3. Fredholm determinants. Below, we will make use of the following *assumptions*:

$$(3.1) \quad A_j \text{ is invertible for each } j \in \mathbb{Z}, \text{ and } (A_j)_{j \in \mathbb{Z}}, (A_j^{-1})_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}; \mathbb{C}^{d \times d}),$$

$$(3.2) \quad \text{equation } x_{j+1} = A_j x_j, j \in \mathbb{Z}, \text{ has an exponential dichotomy } P \text{ on } \mathbb{Z},$$

$$(3.3) \quad (C_j U_j)_{j \in \mathbb{Z}}, (U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z}; \mathbb{C}^{d \times d}),$$

$$(3.4) \quad A_j^\times = A_j + B_j C_j \text{ is invertible for each } j \in \mathbb{Z}.$$

Assume (3.1). Then the fundamental matrix solution $\mathbf{U} = (U_j)_{j \in \mathbb{Z}}$ satisfying $U_{j+1} = A_j U_j$, $j \in \mathbb{Z}$, and $U_0 = I$ is given by the formulas

$$(3.5) \quad U_j = A_{j-1} \cdots A_0, \quad U_{-j} = A_{-j}^{-1} \cdots A_{-1}^{-1}, \quad j = 1, 2, \dots$$

Similarly, assume (3.4). Then any matrix solution $(U_j^\times)_{j \in \mathbb{Z}}$ satisfying $U_{j+1}^\times = A_j^\times U_j^\times$, $j \in \mathbb{Z}$, is given by the formulas

$$(3.6) \quad U_j^\times = A_{j-1}^\times \cdots A_0^\times U_0^\times, \quad U_{-j}^\times = (A_{-j}^\times)^{-1} \cdots (A_{-1}^\times)^{-1} U_0^\times, \quad j = 1, 2, \dots$$

Here, U_0^\times is an arbitrary (possibly singular!) matrix. Recall that \mathbf{U} has the exponential dichotomy over \mathbb{Z} with the (unstable) projection P provided the following inequalities hold for some $c \geq 1$ and $\alpha > 0$:

$$(3.7) \quad \|U_j(I-P)U_{k+1}^{-1}\| \leq ce^{-\alpha(j-k)} \text{ for } j > k, \quad \|U_j P U_{k+1}^{-1}\| \leq ce^{-\alpha(k-j)} \text{ for } j \leq k.$$

We let

$$(3.8) \quad d_1 = \dim \text{Im } P \text{ and } d_2 = \dim \text{Im}(I-P) \text{ so that } d_1 + d_2 = d.$$

Under assumptions (3.1) - (3.2) the operator \mathcal{T} in (1.1) is well-defined. Using notation (2.5) - (2.6), it can be written as $\mathcal{T} = V_{\mathbf{a}_1, \mathbf{b}_1}^- + D_{\mathbf{d}} + V_{\mathbf{a}_2, \mathbf{b}_2}^+$, where $\mathbf{a}_1 = (C_j U_j (I-P))_{j \in \mathbb{Z}}$, $\mathbf{b}_1 = ((I-P)U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}}$, $\mathbf{d} = (-C_j U_j P U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}}$, $\mathbf{a}_2 = (-C_j U_j P)_{j \in \mathbb{Z}}$, and $\mathbf{b}_2 = (P U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}}$. If assumption (3.3) holds then $\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2 \in \ell^2$ and, using Cauchy-Schwarz, $\mathbf{d} \in \ell^1 \subset \ell^2$, so that we have $\mathcal{T} \in \mathcal{B}_2(\ell^2)$ by (2.8), and thus $\det_2(I - \mathcal{T})$ is well-defined. Below, we will sometimes assume that $\mathcal{T} \in \mathcal{B}_1(\ell^2)$ so that $\det(I - \mathcal{T})$ is well-defined. Note that $\mathcal{T} \in \mathcal{B}_1(\ell^2)$ provided, say, (3.3) is replaced by the assumption $(C_j U_j)_{j \in \mathbb{Z}}, (U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}} \in \ell^1(\mathbb{Z}; \mathbb{C}^{d \times d})$. Indeed, under this latter assumption we have $\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2, \mathbf{d} \in \ell^1$; thus $\mathcal{T} \in \mathcal{B}_1(\ell^2)$ by (2.13).

We remark that every difference operator \mathcal{T} with a semi-separable kernel

$$T_{jk} = \mathbf{a}_2(j) \mathbf{b}_2(k) \text{ for } j > k \text{ and } T_{jk} = \mathbf{a}_1(j) \mathbf{b}_1(k) \text{ for } j \leq k,$$

with $\mathbf{a}_i = (\mathbf{a}_i(j))_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z}; \mathbb{C}^{d \times d_i})$ and $\mathbf{b}_i = (\mathbf{b}_i(j))_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z}; \mathbb{C}^{d_i \times d})$, $i = 1, 2$, and $d_1 + d_2 = d$, can be written in the form (1.1) by setting for a fixed $\alpha > 0$ and all $j \in \mathbb{Z}$:

$$(3.9) \quad P = \begin{bmatrix} 0 & 0 \\ 0 & I_{d_1 \times d_1} \end{bmatrix}, \quad U_j = \begin{bmatrix} e^{-\alpha j} I_{d_2 \times d_2} & 0 \\ 0 & e^{\alpha j} I_{d_1 \times d_1} \end{bmatrix},$$

$$(3.10) \quad C_j = [e^{\alpha j} \mathbf{a}_2(j) \quad e^{-\alpha j} \mathbf{a}_1(j)], \quad B_j = [e^{-\alpha(j+1)} \mathbf{b}_2(j) \quad e^{\alpha(j+1)} \mathbf{b}_1(j)]^\top,$$

where \top means transposition of the (1×2) block-row $[\cdot \quad \cdot]$.

We will use the following representations of the operator \mathcal{T} defined in (1.1):

$$(3.11) \quad (\mathcal{T}\mathbf{x})_j = \sum_{k=-\infty}^{j-1} C_j U_j (I - P) U_{k+1}^{-1} B_k x_k - \sum_{k=j}^{\infty} C_j U_j P U_{k+1}^{-1} B_k x_k$$

$$(3.11) \quad = \sum_{k=-\infty}^{j-1} C_j U_j U_{k+1}^{-1} B_k x_k - \sum_{k=-\infty}^{\infty} C_j U_j P U_{k+1}^{-1} B_k x_k$$

$$(3.12) \quad = - \sum_{k=j}^{\infty} C_j U_j U_{k+1}^{-1} B_k x_k + \sum_{k=-\infty}^{\infty} C_j U_j (I - P) U_{k+1}^{-1} B_k x_k, j \in \mathbb{Z}.$$

Accordingly, we define the following operators:

$$(3.13) \quad (H_- \mathbf{x})_j = \sum_{k=-\infty}^{j-1} C_j U_j U_{k+1}^{-1} B_k x_k, \quad (H_+ \mathbf{x})_j = - \sum_{k=j}^{\infty} C_j U_j U_{k+1}^{-1} B_k x_k,$$

$$(3.14) \quad (D\mathbf{x})_j = -C_j U_j U_{j+1}^{-1} B_j x_j, \quad (H_+^0 \mathbf{x})_j = - \sum_{k=j+1}^{\infty} C_j U_j U_{k+1}^{-1} B_k x_k,$$

$$(3.15) \quad (Qx)_j = C_j U_j P x, x \in \mathbb{C}^d, \quad R\mathbf{x} = -P \sum_{k=-\infty}^{\infty} U_{k+1}^{-1} B_k x_k \in \mathbb{C}^{d_1},$$

$$(3.16) \quad (Sx)_j = C_j U_j (I - P)x, x \in \mathbb{C}^d, \quad W\mathbf{x} = (I - P) \sum_{k=-\infty}^{\infty} U_{k+1}^{-1} B_k x_k \in \mathbb{C}^{d_2},$$

so that (3.11) - (3.12) in this notation become

$$(3.17) \quad \mathcal{T} = H_- + QR$$

$$(3.18) \quad = H_+ + SW = D + H_+^0 + SW.$$

We stress that the operators R and W have finite ranks d_1 and d_2 , respectively. Properties of the operators (3.13) - (3.16) are summarized in the following lemmas.

LEMMA 3.1. *Assume (3.1) - (3.3). Then $\sigma(H_-) = \{0\}$, $H_- \in \mathcal{B}_2(\ell^2)$, and $\det_2(I - H_-) = 1$. If, in addition to (3.1) - (3.3), we assume that $\mathcal{T} \in \mathcal{B}_1(\ell^2)$, then $H_- \in \mathcal{B}_1(\ell^2)$ and $\det(I - H_-) = 1$.*

LEMMA 3.2. *Assume (3.1) - (3.4). Then:*

$$(3.19) \quad D \in \mathcal{B}_1(\ell^2) \text{ and } \det(I - D) = \prod_{j \in \mathbb{Z}} (\det A_j^{-1} \det(A_j + B_j C_j)),$$

$$(3.20) \quad \text{the operator } (I - D) \text{ is invertible,}$$

$$(3.21) \quad \det_2(I - D) = \prod_{j \in \mathbb{Z}} (\det A_j^{-1} \det(A_j + B_j C_j)) \exp \sum_{j \in \mathbb{Z}} -\text{tr}(C_j U_j U_{j+1}^{-1} B_j),$$

$$(3.22) \quad \sigma((I - D)^{-1} H_+^0) = \{0\}, \quad (I - D)^{-1} H_+^0 \in \mathcal{B}_2(\ell^2), \quad \text{and} \\ \det_2(I - (I - D)^{-1} H_+^0) = 1.$$

If, in addition to (3.1) - (3.4), we assume that $\mathcal{T} \in \mathcal{B}_1(\ell^2)$ then

$$(3.23) \quad (I - D)^{-1} H_+^0 \in \mathcal{B}_1(\ell^2), \quad \text{and} \quad \det(I - (I - D)^{-1} H_+^0) = 1.$$

LEMMA 3.3. Assume (3.1) - (3.4). Then $H_+ \in \mathcal{B}_2(\ell^2)$, $\det_2(I - H_+) = \det_2(I - D)$, and the operator $I - H_+$ is invertible. If, in addition to (3.1) - (3.4), we assume that $\mathcal{T} \in \mathcal{B}_1(\ell^2)$ then $H_+ \in \mathcal{B}_1(\ell^2)$ and $\det(I - H_+) = \det(I - D)$.

Proof. Using notation (2.5), we remark that $H_- = V_{\mathbf{a}, \mathbf{b}}^-$ with $\mathbf{a} = (C_j U_j)_{j \in \mathbb{Z}}$ and $\mathbf{b} = (U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}}$. Since $\mathbf{a}, \mathbf{b} \in \ell^2$ by (3.3), the first three assertions in Lemma 3.1 follow, respectively, from (2.9), (2.8), and (2.10). If $\mathcal{T} \in \mathcal{B}_1(\ell^2)$ then $H_- \in \mathcal{B}_1(\ell^2)$ because $R \in \mathcal{B}_1(\ell^2)$ is of finite rank and (3.17) holds. We already know that $\sigma(H_-) = \{0\}$. Thus, $\det(I - H_-) = 1$ follows from (2.1), and Lemma 3.1 is proved.

To prove Lemma 3.2, note that $D = D_{\mathbf{d}}$, see (2.6), where $\mathbf{d} = (-C_j U_j U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}}$. Assumption (3.3) and Cauchy-Schwarz imply $\mathbf{d} \in \ell^1$. Then $D \in \mathcal{B}_1(\ell^2)$ by (2.13). Moreover, using (2.15) and the identity $U_{j+1} = A_j U_j$, $j \in \mathbb{Z}$, we verify (3.19) as follows:

$$\begin{aligned} \det(I - D) &= \prod_{j \in \mathbb{Z}} \det(I_{\mathbf{d} \times \mathbf{d}} + C_j U_j U_{j+1}^{-1} B_j) = \prod_{j \in \mathbb{Z}} \det(I + U_j U_{j+1}^{-1} B_j C_j) \\ &= \prod_{j \in \mathbb{Z}} \det(I + A_j^{-1} B_j C_j) = \prod_{j \in \mathbb{Z}} \det A_j^{-1} \det(A_j + B_j C_j). \end{aligned}$$

To prove (3.20), we use assumption (3.4) and the identity $U_{j+1} = A_j U_j$ to infer:

$$\begin{aligned} (I + C_j U_j U_{j+1}^{-1} B_j)^{-1} &= (I + C_j A_j^{-1} B_j)^{-1} = I - C_j (I + A_j^{-1} B_j C_j)^{-1} A_j^{-1} B_j \\ &= I - C_j (A_j + B_j C_j)^{-1} B_j = I - C_j U_j (U_{j+1}^{-1} (A_j + B_j C_j) U_j)^{-1} U_{j+1}^{-1} B_j \\ &= I - C_j U_j (I + U_{j+1}^{-1} B_j C_j U_j)^{-1} U_{j+1}^{-1} B_j. \end{aligned}$$

Note that $(C_j U_j)_{j \in \mathbb{Z}}, (U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}} \in \ell^2 \subset \ell^\infty(\mathbb{Z}; \mathbb{C}^{\mathbf{d} \times \mathbf{d}})$ and $\lim_{j \rightarrow \infty} \|U_{j+1}^{-1} B_j C_j U_j\| = 0$ by assumption (3.3). Thus (3.20) holds and, moreover, $(I - D)^{-1} = D_{\mathbf{d}}$, where

$$(3.24) \quad \mathbf{d} = ((I + C_j U_j U_{j+1}^{-1} B_j)^{-1})_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}; \mathbb{C}^{\mathbf{d} \times \mathbf{d}}).$$

Formula (3.21) follows from (3.19) and (2.3). Using notation (2.5) and (3.24) we remark that $(I - D)^{-1} H_+^0 = V_{\mathbf{a}, \mathbf{b}}^+$, where $\mathbf{a} = -(I + C_j U_j U_{j+1}^{-1} B_j)^{-1} C_j U_j)_{j \in \mathbb{Z}}$ and $\mathbf{b} = (U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}}$. By $\mathbf{d} \in \ell^\infty$ in (3.24) and assumption (3.3) we conclude $\mathbf{a}, \mathbf{b} \in \ell^2$, and then (3.22) follows from (2.9), (2.8), and (2.10). Finally, if $\mathcal{T} \in \mathcal{B}_1(\ell^2)$ then $H_+ = D + H_+^0 \in \mathcal{B}_1(\ell^2)$ by (3.18) since W is of finite rank. Since $D = D_{\mathbf{d}}$ with $\mathbf{d} = (-C_j U_j U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}} \in \ell^1$, see assumption (3.3), by (2.13) we have $D \in \mathcal{B}_1(\ell^2)$. Therefore $H_+^0 \in \mathcal{B}_1(\ell^2)$ and thus $(I - D)^{-1} H_+^0 \in \mathcal{B}_1(\ell^2)$ since $(I - D)^{-1}$ is a bounded operator. The last assertion in (3.23) now follows from $\sigma((I - D)^{-1} H_+^0) = \{0\}$ and (2.1). Assertions in Lemma 3.3 follow from the identity $I - H_+ = (I - D)(I - (I - D)^{-1} H_+^0)$. ■

Our first main result gives a formula for the (modified) Fredholm determinant of the (infinite-dimensional) operator $I - \mathcal{T}$ in terms of finite-dimensional determinants.

THEOREM 3.4. *Assume (3.1), (3.2), (3.3), and (3.4). Then*

$$(3.25) \quad \det_2(I - \mathcal{T}) = \det_{\mathbb{C}^{d_1}}(I_{d_1 \times d_1} - R(I - H_-)^{-1}Q) \exp \sum_{j \in \mathbb{Z}} -\text{tr}(PU_{j+1}^{-1}B_j C_j U_j P)$$

$$(3.26) \quad \begin{aligned} &= \det_{\mathbb{C}^{d_2}}(I_{d_2 \times d_2} - W(I - H_+)^{-1}S) \\ &\quad \times \prod_{j \in \mathbb{Z}} \det A_j^{-1} \det(A_j + B_j C_j) \exp \sum_{j \in \mathbb{Z}} -\text{tr}(PU_{j+1}^{-1}B_j C_j U_j P). \end{aligned}$$

If, in addition to (3.1) - (3.4), we assume that $\mathcal{T} \in \mathcal{B}_1(\ell^2)$, then

$$(3.27) \quad \det(I - \mathcal{T}) = \det_{\mathbb{C}^{d_1}}(I_{d_1 \times d_1} - R(I - H_-)^{-1}Q)$$

$$(3.28) \quad = \det_{\mathbb{C}^{d_2}}(I_{d_2 \times d_2} - W(I - H_+)^{-1}S) \prod_{j \in \mathbb{Z}} \det A_j^{-1} \det(A_j + B_j C_j).$$

Proof. Using representation (3.17), recalling that $\det_2(I - H_-) = 1$ by Lemma 3.1, noting that $(I - H_-)^{-1}QR \in \mathcal{B}_1 \subset \mathcal{B}_2$ because R is of rank d_1 , and applying (2.4) and (2.3), we infer:

$$\begin{aligned} \det_2(I - \mathcal{T}) &= \det_2[(I - H_-)(I - (I - H_-)^{-1}QR)] \\ &= \det_2(I - H_-) \det_2(I - (I - H_-)^{-1}QR) \exp(-\text{tr}[H_-(I - H_-)^{-1}QR]) \\ &= \det(I - (I - H_-)^{-1}QR) \exp \text{tr}[(I - H_-)^{-1}QR] \exp(-\text{tr}[H_-(I - H_-)^{-1}QR]) \\ &= \det(I - R(I - H_-)^{-1}Q) \exp \text{tr}(QR) \\ &= \det_{\mathbb{C}^{d_1}}(I_{d_1 \times d_1} - R(I - H_-)^{-1}Q) \exp \text{tr}(RQ). \end{aligned}$$

Using (3.15), we have (3.25). To establish (3.26), we first apply representation (3.18), Lemma 3.3, and (2.4):

$$\begin{aligned} \det_2(I - \mathcal{T}) &= \det_2[(I - H_+)(I - (I - H_+)^{-1}SW)] \\ &= \det_2(I - H_+) \det_2(I - (I - H_+)^{-1}SW) \exp(-\text{tr}[H_+(I - H_+)^{-1}SW]) \\ &= \det_2(I - D) \det(I - (I - H_+)^{-1}SW) \exp(\text{tr}[(I - H_+)^{-1}SW - H_+(I - H_+)^{-1}SW]) \\ &= \det_2(I - D) \det(I - W(I - H_+)^{-1}S) \exp \text{tr}(SW) \\ &= \det_2(I - D) \det_{\mathbb{C}^{d_2}}(I_{d_2 \times d_2} - W(I - H_+)^{-1}S) \exp \text{tr}(WS), \end{aligned}$$

recalling that W is of rank d_2 , and thus $(I - H_+)^{-1}SW \in \mathcal{B}_1 \subset \mathcal{B}_2$, which allows us to use (2.3). Using (3.21) and (3.16) we therefore have:

$$\begin{aligned} \det_2(I - \mathcal{T}) &= \det_{\mathbb{C}^{d_2}}(I_{d_2 \times d_2} - W(I - H_+)^{-1}S) \prod_{j \in \mathbb{Z}} \det A_j^{-1} \det(A_j + B_j C_j) \\ &\quad \times \exp \sum_{j \in \mathbb{Z}} \text{tr}[-C_j U_j U_{j+1}^{-1} B_j + C_j U_j (I - P) U_{j+1}^{-1} B_j], \end{aligned}$$

which implies (3.26). Formula (3.27) follows from representation (3.17) and Lemma 3.1:

$$\begin{aligned} \det(I - \mathcal{T}) &= \det[(I - H_-)(I - (I - H_-)^{-1}QR)] \\ &= \det(I - H_-) \det(I - R(I - H_-)^{-1}Q) = \det_{\mathbb{C}^{d_1}}(I - R(I - H_-)^{-1}Q). \end{aligned}$$

Formula (3.28) follows from representation (3.18) and Lemmas 3.2 - 3.3:

$$\begin{aligned} \det(I - \mathcal{T}) &= \det[(I - H_+)(I - (I - H_+)^{-1}SW)] \\ &= \det(I - H_+) \det(I - (I - H_+)^{-1}SW) = \det(I - D) \det(I - W(I - H_+)^{-1}S), \end{aligned}$$

which concludes the proof. ■

Our next objective is to relate the finite-dimensional determinants in the right-hand side of (3.25) - (3.28) to the determinants of a particular matrix solution $(\mathcal{U}_j^\times)_{j \in \mathbb{Z}}$ satisfying $\mathcal{U}_{j+1}^\times = A_j^\times \mathcal{U}_j^\times$. We stress that this solution could be singular. For this, we consider the following matrix difference equations:

$$(3.29) \quad X_j^+ = C_j U_j (I - P) - \sum_{k=j}^{\infty} C_j U_j U_{k+1}^{-1} B_k X_k^+,$$

$$(3.30) \quad X_j^- = C_j U_j P + \sum_{k=-\infty}^{j-1} C_j U_j U_{k+1}^{-1} B_k X_k^-, \quad j \in \mathbb{Z}.$$

Using notation (3.13) - (3.16), equations (3.29) - (3.30) for $\mathbf{X}^\pm = (X_j^\pm)_{j \in \mathbb{Z}}$ can be rewritten as follows:

$$(3.31) \quad \mathbf{X}^+ x = Sx + H_+ \mathbf{X}^+ x, \quad \mathbf{X}^- x = Qx + H_- \mathbf{X}^- x, \quad x \in \mathbb{C}^d.$$

By assumption (3.3), we have $Sx, Qx \in \ell^2(\mathbb{Z}; \mathbb{C}^d)$. By Lemmas 3.1 and 3.3 operators $I - H_\pm$ are invertible. Thus, (3.29) - (3.30) have a unique pair of solutions $\mathbf{X}^\pm \in \ell^2(\mathbb{Z}; \mathbb{C}^{d \times d})$ given by

$$(3.32) \quad \mathbf{X}^+ = (I - H_+)^{-1}S \text{ and } \mathbf{X}^- = (I - H_-)^{-1}Q.$$

Since the solutions \mathbf{X}^\pm of (3.29) - (3.30) are unique, multiplying (3.29) by $(I - P)$ and (3.30) by P from the right, we also have: $X_j^+ = X_j^+(I - P)$ and $X_j^- = X_j^- P$, $j \in \mathbb{Z}$. Thus, we can treat matrices X_j^\pm as operators $X_j^+ : \text{Im}(I - P) \rightarrow \mathbb{C}^d$ and $X_j^- : \text{Im} P \rightarrow \mathbb{C}^d$. Using (3.32) and notations (3.15) - (3.16) we then have for X_j^\pm from (3.29) - (3.30):

$$(3.33) \quad \det_{\mathbb{C}^{d_1}}(I_{d_1 \times d_1} - R(I - H_-)^{-1}Q) = \det_{\mathbb{C}^{d_1}} \left(I_{d_1 \times d_1} + \sum_{k=-\infty}^{\infty} P U_{k+1}^{-1} B_k X_k^- P \right),$$

$$(3.34) \quad \det_{\mathbb{C}^{d_2}}(I_{d_2 \times d_2} - W(I - H_+)^{-1}S) \\ = \det_{\mathbb{C}^{d_2}} \left(I_{d_2 \times d_2} - \sum_{k=-\infty}^{\infty} (I - P) U_{k+1}^{-1} B_k X_k^+ (I - P) \right).$$

We remark that the series $\sum_{k=-\infty}^{\infty} U_{k+1}^{-1} B_k X_k^\pm$ converge absolutely by assumption (3.3), $\mathbf{X}^\pm \in \ell^2(\mathbb{Z}; \mathbb{C}^{d \times d})$, and Cauchy-Schwarz.

Using the direct sum decomposition $\mathbb{C}^d = \text{Im}(I - P) \oplus \text{Im} P$, consider a matrix sequence, $\mathcal{U}^\times = (\mathcal{U}_j^\times)_{j \in \mathbb{Z}}$, defined by $\mathcal{U}_j^\times = U_j V_j$ where $(U_j)_{j \in \mathbb{Z}}$ is the fundamental matrix solution of the unperturbed equation $x_{j+1} = A_j x_j$, $U_0 = I$, and the (2×2) block matrix V_j is defined as follows:

$$(3.35) \quad V_j = \begin{bmatrix} I - P - \sum_{k=j}^{\infty} (I - P)U_{k+1}^{-1}B_kX_k^+(I - P) & \sum_{k=-\infty}^{j-1} (I - P)U_{k+1}^{-1}B_kX_k^-P \\ - \sum_{k=j}^{\infty} PU_{k+1}^{-1}B_kX_k^+(I - P) & P + \sum_{k=-\infty}^{j-1} PU_{k+1}^{-1}B_kX_k^-P \end{bmatrix}$$

First, we claim that \mathcal{U}^\times is a solution of the matrix equation $\mathcal{U}_{j+1}^\times = A_j^\times \mathcal{U}_j^\times$, $j \in \mathbb{Z}$. Indeed, (3.35) and $U_{j+1} = A_j U_j$ imply:

$$\begin{aligned} \mathcal{U}_{j+1}^\times &= U_{j+1}V_{j+1} = U_{j+1}V_j + B_j [X_j^+(I - P) \quad X_j^-P] \\ &= A_j \mathcal{U}_j^\times + B_j [X_j^+(I - P) \quad X_j^-P], \end{aligned}$$

for the (1×2) block row $[X_j^+(I - P) \quad X_j^-P] : \text{Im}(I - P) \oplus \text{Im} P \rightarrow \mathbb{C}^d$. Using (3.29) - (3.30) and (3.35), we also have $[X_j^+(I - P) \quad X_j^-P] = C_j U_j V_j$, and the claim is proved. Second, we observe that there exist limits $V_{\pm\infty} = \lim_{j \rightarrow \pm\infty} U_j^{-1} \mathcal{U}_j^\times$, and the operators V_∞ and $V_{-\infty}$ are, respectively, upper- and lower-triangular matrices in the direct-sum decomposition $\mathbb{C}^d = \text{Im}(I - P) \oplus \text{Im} P$. Moreover, $\det_{\mathbb{C}^d} V_\infty$ and $\det_{\mathbb{C}^d} V_{-\infty}$ are equal, respectively, to the right hand sides of (3.33) and (3.34). Finally, since \mathbf{U} and \mathcal{U}^\times are solutions of the equations $U_{j+1} = A_j U_j$ and $\mathcal{U}_{j+1}^\times = A_j^\times \mathcal{U}_j^\times$, we can use formulas (3.5) - (3.6) to recalculate the right hand sides of (3.33) and (3.34) via $\det_{\mathbb{C}^d} \mathcal{U}_0^\times$ (recall that $\det U_0 = 1$ since $U_0 = I$). Using Theorem 3.4, we arrive to the following result.

THEOREM 3.5. *Assume (3.1) - (3.4), let X_j^\pm be the matrix solutions of (3.29) - (3.30), and define $\mathcal{U}^\times = (\mathcal{U}_j^\times)_{j \in \mathbb{Z}}$ as $\mathcal{U}_j^\times = U_j V_j$, where V_j are given by formula (3.35). Then*

$$\det_2(I - \mathcal{T}) = (\det_{\mathbb{C}^d} \mathcal{U}_0^\times) \prod_{j=0}^{\infty} \frac{\det(A_j + B_j C_j)}{\det A_j} \times \exp \sum_{j \in \mathbb{Z}} -\text{tr}(PU_{j+1}^{-1} B_j C_j U_j P).$$

If, in addition to (3.1) - (3.4), we assume that $\mathcal{T} \in \mathcal{B}_1(\ell^2)$, then

$$(3.36) \quad \det(I - \mathcal{T}) = (\det_{\mathbb{C}^d} \mathcal{U}_0^\times) \prod_{j=0}^{\infty} \frac{\det(A_j + B_j C_j)}{\det A_j}.$$

4. The Evans determinant. In this section we continue to assume that (3.1) - (3.4) hold. Because of (3.1), \mathbf{U} is exponentially bounded: $\sup_{j,k \in \mathbb{Z}} e^{-\alpha(k-j)} \|U_j U_{k+1}^{-1}\| < \infty$ for some $\alpha \in \mathbb{R}$. This allows us to introduce the following notions related to the Bohl (or, in other terminology, general Lyapunov) exponents. For the corresponding theory in differential equations case see [DK]; also, these notions are related to the so-called Sacker-Sell, or dynamical spectrum, see [SS] and the bibliography in [CL].

If $J = \mathbb{Z}$ or $J = \mathbb{Z}^\pm$, and Q is a projection on \mathbb{C}^d so that $\sup_{k \in J} \|U_k Q U_k^{-1}\| < \infty$ then the upper and lower Bohl exponents, \varkappa_g and \varkappa'_g , are defined as follows:

$$(4.1) \quad \varkappa_g(Q; J) = \inf\{\alpha \in \mathbb{R} : \sup_{j \geq k \in J} e^{-\alpha(j-k)} \|U_j Q U_{k+1}^{-1}\| < \infty\},$$

$$(4.2) \quad \varkappa'_g(Q; J) = \sup\{\alpha \in \mathbb{R} : \sup_{j \leq k \in J} e^{-\alpha(j-k)} \|U_j Q U_{k+1}^{-1}\| < \infty\}.$$

For instance, if P is the dichotomy projection, cf. (3.7), then

$$(4.3) \quad \varkappa_g(I - P; \mathbb{Z}) < 0 < \varkappa'_g(P; \mathbb{Z}).$$

A system $\{Q_i\}_{i=1}^{d_0}$ of disjoint projections on \mathbb{C}^d , $1 \leq d_0 \leq d$, is called an *exponential splitting* of order d_0 if the following holds: $Q_1 + \dots + Q_{d_0} = I$, $\sup_{k \in J} \|U_k Q_i U_k^{-1}\| < \infty$, and the segments $[\varkappa'_g(Q_i; J), \varkappa_g(Q_i; J)]$ are all disjoint, $i = 1, \dots, d_0$. An exponential splitting is called the *finest* if there is no exponential splitting of order $d_0 + 1$.

Let $\{Q_i\}_{i=1}^{d_0}$ denote the finest exponential splitting over \mathbb{Z} for $\mathbf{U} = (U_j)_{j \in \mathbb{Z}}$. In what follows we assume that projections Q_i are numbered such that $\varkappa_g(Q_i; \mathbb{Z}) < \varkappa'_g(Q_{i+1}; \mathbb{Z})$; we set $Q_0 = 0$, $Q_{d_0+1} = I$. Since P is the dichotomy projection for \mathbf{U} , there exists an $n \in \{1, \dots, d_0\}$ such that $I - P = Q_1 + \dots + Q_n$ and $P = Q_{n+1} + \dots + Q_{d_0}$. Thus,

$$(4.4) \quad \varkappa'_g(Q_i; \mathbb{Z}) \leq \varkappa_g(Q_i; \mathbb{Z}) < 0 \quad \text{for } i = 1, \dots, n, \text{ and}$$

$$(4.5) \quad 0 < \varkappa'_g(Q_i; \mathbb{Z}) \leq \varkappa_g(Q_i; \mathbb{Z}) \quad \text{for } i = n + 1, \dots, d_0.$$

Clearly, $\{Q_i\}_{i=1}^{d_0}$ is also an exponential splitting for \mathbf{U} over \mathbb{Z}_+ and \mathbb{Z}_- . We denote:

$$(4.6) \quad \varkappa'_i = \varkappa'_g(Q_i; \mathbb{Z}), \quad \varkappa_i^\pm = \varkappa'_g(Q_i; \mathbb{Z}_\pm), \quad \varkappa_i = \varkappa_g(Q_i; \mathbb{Z}), \quad \varkappa_i^\pm = \varkappa_g(Q_i; \mathbb{Z}_\pm), \quad i = 1, \dots, d_0,$$

$$(4.7) \quad \varepsilon_0 = \frac{1}{2} \min\{-\varkappa_n, \varkappa'_{n+1}, \varkappa'_i - \varkappa_{i-1} : i = 1, \dots, n-1, n+1, \dots, d_0\}.$$

In this section we will use the following assumptions for the perturbation $(B_j C_j)_{j \in \mathbb{Z}}$: There exists a $\delta \in (0, \varepsilon_0)$ such that

$$(4.8) \quad \sum_{k=0}^{\infty} e^{(\varkappa_i^+ - \varkappa_i^+ + \delta)k} \|B_k C_k\| < \infty, \quad \text{for } i = 1, \dots, n,$$

$$(4.9) \quad \sum_{k=-\infty}^0 e^{-(\varkappa_i^- - \varkappa_i^- + \delta)k} \|B_k C_k\| < \infty, \quad \text{for } i = n + 1, \dots, d_0.$$

Our next objective is to construct matrix solutions of the perturbed difference equation $x_{j+1} = A_j^\times x_j$ that are asymptotic to the solutions $(U_j Q_i)_{j \in \mathbb{Z}}$ of the unperturbed equation $x_{j+1} = A_j x_j$. For some $N \in \mathbb{N}$ consider the following $(d \times d)$ matrix difference equations:

$$(4.10) \quad \begin{aligned} Y_j^{(i)} - U_j Q_i &= - \sum_{k=j}^{\infty} U_j (Q_i + \dots + Q_{d_0}) U_{k+1}^{-1} B_k C_k Y_k^{(i)} \\ &+ \sum_{k=N}^{j-1} U_j (I - (Q_i + \dots + Q_{d_0})) U_{k+1}^{-1} B_k C_k Y_k^{(i)}, \quad j > N, \quad i = 1, \dots, n, \end{aligned}$$

$$(4.11) \quad \begin{aligned} Y_j^{(i)} - U_j Q_i &= \sum_{k=-\infty}^{j-1} U_j (Q_1 + \dots + Q_i) U_{k+1}^{-1} B_k C_k Y_k^{(i)} \\ &- \sum_{k=j}^{-N} U_j (I - (Q_1 + \dots + Q_i)) U_{k+1}^{-1} B_k C_k Y_k^{(i)}, \quad j \leq -N, \quad i = n + 1, \dots, d_0. \end{aligned}$$

REMARK 4.1. If the sequence $\mathbf{Y}^{(i)} = (Y_j^{(i)})$, where $j > N$, resp. $j \leq -N$, is a solution of (4.10), resp. (4.11), then this sequence is a solution of the (perturbed) difference equation $Y_{j+1} = A_j^\times Y_j$ for $j > N$, resp. $j \leq -N$. If $\mathbf{U}^\times = (U_j^\times)_{j \in \mathbb{Z}}$, $U_0^\times = I$, denotes the fundamental matrix solution of the equation $U_{j+1}^\times = A_j^\times U_j^\times$ (this solution is non-singular

by assumption (3.4)), then the solutions $\mathbf{Y}^{(i)} = (Y_j^{(i)})$, $i = 1, \dots, n$, resp. $i = n+1, \dots, d_0$, originally given for $j > N$, resp. $j \leq -N$, could be extended to $\mathbb{Z}_+ = \{0, 1, \dots\}$, resp. $\mathbb{Z}_- = \{\dots, -1, 0\}$, by setting

$$(4.12) \quad \begin{aligned} Y_j^{(i)} &= U_j^\times (U_{N+1}^\times)^{-1} Y_{N+1}^{(i)}, \quad j = 0, \dots, N, \\ Y_j^{(i)} &= U_j^\times (U_{-N}^\times)^{-1} Y_{-N}^{(i)}, \quad j = -N+1, \dots, 0. \end{aligned}$$

LEMMA 4.2. *Assume (4.8), resp. (4.9). Then, for a sufficiently large $N = N(\delta)$, there exists a solution $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j>N}$, $i = 1, \dots, n$, of (4.10), resp. $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j \leq -N}$, $i = n+1, \dots, d_0$, of (4.11). Moreover, these solutions satisfy $Y_j^{(i)} = Y_j^{(i)} Q_i$ and have the following properties:*

$$(a) \quad \begin{aligned} \sup_{j>N} e^{-(\varkappa_i^+ + \frac{\delta}{2})j} \|Y_j^{(i)}\| &< \infty, \quad i = 1, \dots, n, \\ \sup_{j \leq -N} e^{-(\varkappa_i' - \frac{\delta}{2})j} \|Y_j^{(i)}\| &< \infty, \quad i = n+1, \dots, d_0; \end{aligned}$$

$$(b) \quad \begin{aligned} \inf_{j>N} e^{-(\varkappa_i^+ - \frac{\delta}{2})j} \|Y_j^{(i)}\|_\bullet &> 0, \quad i = 1, \dots, n, \\ \inf_{j \leq -N} e^{-(\varkappa_i^- + \frac{\delta}{2})j} \|Y_j^{(i)}\|_\bullet &> 0, \quad i = n+1, \dots, d_0; \end{aligned}$$

$$(c) \quad \begin{aligned} \sup_{j>N} e^{-(\varkappa_i^+ - \frac{\delta}{2})j} \|Y_j^{(i)} - U_j Q_i\| &< \infty, \quad i = 1, \dots, n, \\ \sup_{j \leq -N} e^{-(\varkappa_i^- + \frac{\delta}{2})j} \|Y_j^{(i)} - U_j Q_i\| &< \infty, \quad i = n+1, \dots, d_0. \end{aligned}$$

Proof. For $i = 1, \dots, n$ set $\alpha = \varkappa_i^+ - \delta/2$ so that $\alpha \in (\varkappa_{i-1}^+, \varkappa_i^+)$. Using (4.1) - (4.2) with $J = \mathbb{Z}_+$, find constants c_α and c'_α such that

$$(4.13) \quad \|U_j(I - (Q_i + \dots + Q_{d_0}))U_{k+1}^{-1}\| \leq c_\alpha e^{\alpha(j-k)}, \quad j \geq k \geq 0,$$

$$(4.14) \quad \|U_j(Q_i + \dots + Q_{d_0})U_{k+1}^{-1}\| \leq c'_\alpha e^{\alpha(j-k)}, \quad k \geq j \geq 0.$$

Using assumption (4.8), choose N so large that

$$(4.15) \quad q := \sum_{k=N}^{\infty} \max\{c_\alpha, c'_\alpha\} e^{(\varkappa_i^+ - \varkappa_i'^+ + \delta)k} \|B_k C_k\| < 1.$$

With this N , and letting $\beta = \varkappa_i^+ + \delta/2$, we define a Banach space, $\ell_{+, \beta}^\infty$, of $\mathbb{C}^{d \times d}$ -valued sequences $\mathbf{u} = (u_j)_{j>N}$ as follows: $\ell_{+, \beta}^\infty = \{\mathbf{u} : \|\mathbf{u}\|_{+, \beta} := \sup_{j>N} e^{-\beta j} \|u_j\| < \infty\}$. Let T denote the operator corresponding to the right-hand side of (4.10), so that this equation becomes $\mathbf{Y}^{(i)} - (U_j Q_i)_{j>N} = T\mathbf{Y}^{(i)}$. We claim that T is a contraction in $\ell_{+, \beta}^\infty$. Indeed, using (4.13) - (4.14), and then (4.15), we infer:

$$(4.16) \quad \begin{aligned} \|T\mathbf{u}\|_{+, \beta} &\leq \|\mathbf{u}\|_{+, \beta} \sup_{j>N} e^{-\beta j} \left(\sum_{k=j}^{\infty} c'_\alpha e^{\alpha(j-k)} \|B_k C_k\| e^{\beta k} + \sum_{k=N}^{j-1} c_\alpha e^{\alpha(j-k)} \|B_k C_k\| e^{\beta k} \right) \\ &\leq \|\mathbf{u}\|_{+, \beta} \max\{c_\alpha, c'_\alpha\} \sup_{j>N} \sum_{k=N}^{\infty} e^{(\alpha - \beta)(j-k)} \|B_k C_k\| < q \|\mathbf{u}\|_{+, \beta}, \end{aligned}$$

because $(\alpha - \beta)j = -(\varkappa_i^+ - \varkappa_i'^+ + \delta)j < 0$. Since $\sup_{j \geq 0} e^{-\beta j} \|U_j Q_i\| < \infty$ by (4.1) with $J = \mathbb{Z}_+$, $k = -1$, and $\beta > \varkappa_i^+$, we have $(U_j Q_i)_{j>N} \in \ell_{+, \beta}^\infty$. This proves the existence and

uniqueness of a solution $\mathbf{Y}^{(i)} \in \ell_{+, \beta}^\infty$ of (4.10), and thus (a). Also, multiplying (4.10) by Q_i from the right, and using the uniqueness, we have $Y_j^{(i)} = Y_j^{(i)} Q_i$. Similarly to estimate (4.16), we have

$$\|T\mathbf{u}\|_{+, \alpha} \leq \|\mathbf{u}\|_{+, \beta} \max\{c_\alpha, c'_\alpha\} \sum_{k=N}^{\infty} e^{(\beta-\alpha)k} \|B_k C_k\| = q \|\mathbf{u}\|_{+, \beta}.$$

Since $\mathbf{Y}^{(i)} - (U_j Q_i)_{j>N} = T\mathbf{Y}^{(i)}$ and $\mathbf{Y}^{(i)} \in \ell_{+, \beta}^\infty$, we have $\mathbf{Y}^{(i)} - (U_j Q_i)_{j>N} \in \ell_{+, \alpha}^\infty$ and assertion (c) in the lemma follows. To prove assertion (b), we estimate

$$(4.17) \quad \begin{aligned} \|Y_j^{(i)}\|_\bullet &= \|Y_j^{(i)} - U_j Q_i + U_j Q_i\|_\bullet \geq \|U_j Q_i\|_\bullet - \|Y_j^{(i)} - U_j Q_i\| \\ &\geq \|U_j Q_i\|_\bullet - c_1 e^{\alpha j} \end{aligned}$$

with some positive constant c_1 from (c). If $\|x\| = 1$ and $Q_i x = x$ then, using (4.2) for $J = \mathbb{Z}_+$, and that $\alpha + \delta/4 = \varkappa_j^+ - \delta/4 < \varkappa_j^+$, we infer for any $k > 0$:

$$\begin{aligned} 1 &= \|Q_i x\| \leq \|Q_i U_{k+1}^{-1}\| \cdot \|U_{k+1}^{-1} Q_i x\| = (e^{(\alpha+\delta/4)k} \|Q_i U_{k+1}^{-1}\|) (e^{-(\alpha+\delta/4)k} \|U_{k+1}^{-1} Q_i x\|) \\ &\leq \left(\sup_{0 \leq j \leq k} e^{-(\alpha+\delta/4)(j-k)} \|U_j Q_i U_{k+1}^{-1}\| \right) (e^{-(\alpha+\delta/4)k} \|U_{k+1}^{-1} Q_i x\|) \\ &\leq c e^{-(\alpha+\delta/4)k} \|U_{k+1}^{-1} Q_i x\|. \end{aligned}$$

This implies that $\|U_j Q_i\|_\bullet \geq c_2 e^{(\alpha+\delta/4)j}$ for all $j \in \mathbb{Z}_+$ and some $c_2 > 0$. Using (4.17), we have $e^{-\alpha j} \|Y_j^{(i)}\|_\bullet \geq c_2 e^{(\delta/4)j} - c_1 \geq 1$ starting from some sufficiently large $j_0 > N$. Since $\mathbf{Y}^{(i)} = \mathbf{Y}^{(i)} Q_i$ is a solution of the equation $Y_{j+1}^{(i)} = A_j^\times Y_j^{(i)}$, the last assertion and assumption (3.4) imply that $\|Y_j^{(i)}\|_\bullet > 0$ for each $j \in [N, j_0]$. This proves (b), and the proof of the lemma for $i = 1, \dots, n$ is complete. The proof for $i = n+1, \dots, d_0$ is similar. ■

REMARK 4.3. By Lemma 4.2 and Remark 4.1 we thus have matrix solutions $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j \geq 0}$, $i = 1, \dots, n$, resp. $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j \leq 0}$, $i = n+1, \dots, d_0$, of the perturbed equation $x_{j+1} = A_j^\times x_j$ on \mathbb{Z}_+ , resp. \mathbb{Z}_- . Define $\mathbf{Y}^+ = \mathbf{Y}^{(1)} + \dots + \mathbf{Y}^{(n)}$ and $\mathbf{Y}^- = \mathbf{Y}^{(n+1)} + \dots + \mathbf{Y}^{(d_0)}$. Using the property $Y_j^{(i)} Q_i = Y_j^{(i)}$, we may view Y_j^+ , resp. Y_j^- , as operators acting from $\text{Im}(I - P) = \text{Im} Q_1 \oplus \dots \oplus \text{Im} Q_n$, resp. $\text{Im} P = \text{Im} Q_{n+1} \oplus \dots \oplus \text{Im} Q_{d_0}$, to \mathbb{C}^d and thus write $\mathbf{Y}^+ = [\mathbf{Y}^{(1)} \dots \mathbf{Y}^{(n)}]$, resp. $\mathbf{Y}^- = [\mathbf{Y}^{(n+1)} \dots \mathbf{Y}^{(d_0)}]$ either as $(1 \times n)$ -, resp. $(1 \times (d_0 - n))$ block rows, or, cf. (3.8), as $(d \times d_2)$ -, resp. $(d \times d_1)$ -matrices.

Assertions (a) and (b) of Lemma 4.2 and Remark 4.1 show that the solution $\mathbf{Y}^{(i)}$ of the perturbed equation $x_{j+1} = A_j^\times x_j$, $j \in \mathbb{Z}_+$, resp. \mathbb{Z}_- , corresponds to the segment $[\varkappa_i^+, \varkappa_i^+]$, resp. $[\varkappa_i^-, \varkappa_i^-]$, in the Bohl spectrum for $i = 1, \dots, n$, resp. $i = n+1, \dots, d_0$. In particular, for $i = 1, \dots, n$ one has $\|Y_j^{(i)}\| \rightarrow 0$ as $j \rightarrow \infty$, and for $i = n+1, \dots, d_0$ one has $\|Y_j^{(i)}\| \rightarrow 0$ as $j \rightarrow -\infty$, see (4.4) - (4.5) and recall that $\varkappa_i \leq \varkappa_i^\pm \leq \varkappa_i^\pm \leq \varkappa_i$ by (4.1) - (4.2). Assertion (c) shows that the solutions $\mathbf{Y}^{(i)}$ of the perturbed equation are asymptotic to the solutions $(U_j Q_i)_{j \in \mathbb{Z}}$ of the unperturbed equation $x_{j+1} = A_j x_j$, $j \in \mathbb{Z}$, as $j \rightarrow \infty$ for $i = 1, \dots, n$, resp. $j \rightarrow -\infty$ for $i = n+1, \dots, d_0$. This motivates the following definition.

DEFINITION 4.4. If $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j \geq 0}$, $i = 1, \dots, n$ and $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j \leq 0}$, $i = n+1, \dots, d_0$ are the matrix solutions of the perturbation $x_{j+1} = A_j^\times x_j$, $A_j^\times = A_j + B_j C_j$, of the

difference equation $x_{j+1} = A_j x_j$, then the *Evans determinant*, \mathcal{E} , is defined as follows:

$$(4.18) \quad \mathcal{E} = \det[Y_0^+ + Y_0^-], \text{ where } Y_0^+ = Y_0^{(1)} + \cdots + Y_0^{(n)} \text{ and } Y_0^- = Y_0^{(n+1)} + \cdots + Y_0^{(d_0)}.$$

The terminology is related to the so-called Evans function, a powerful tool frequently used for detecting isolated eigenvalues of differential (and difference) operators that appear after linearizing nonlinear equations about such special solutions as travelling waves, see [AGJ, JK, S] and [BCK, KK], and the bibliographies therein. The Evans function, $\mathcal{D}(z)$, is usually defined, cf. [S], in the situation when the coefficients of the unperturbed and perturbed equations depend (analytically) on a (spectral) parameter z . Thus, in our terminology, the values of the Evans function for fixed z 's are called the Evans determinants. In Proposition 4.5 we will show that the definition of the Evans determinant \mathcal{E} given in (4.18) coincides with the definition of the Evans determinant \mathcal{D} standardly accepted in the literature on the Evans function, cf. [JK, S]; moreover, the Evans determinant \mathcal{E} gives a canonical choice among the standard Evans determinants \mathcal{D} whose definition in [JK, S] is not unique.

Out of several available (equivalent) standard definitions of \mathcal{D} we chose the definition using the exponential dichotomies on \mathbb{Z}_+ and \mathbb{Z}_- , cf. [S, Def. 4.1] for the differential equations case. Recall the definition from [S]. Assume that the perturbed equation $x_{j+1} = A_j^\times x_j$ has exponential dichotomies P_+ and P_- on \mathbb{Z}_+ and \mathbb{Z}_- , respectively, so that for its solution $\mathbf{x} = (x_j)$ one has: $\|x_j\| \rightarrow 0$ as $j \rightarrow \infty$ if and only if $x_0 \in \text{Im}(I - P_+)$; $\|x_j\| \rightarrow \infty$ as $j \rightarrow \infty$ if and only if x_0 has a nonzero component in $\text{Im} P_+$; $\|x_j\| \rightarrow 0$ as $j \rightarrow -\infty$ if and only if $x_0 \in \text{Im} P_-$; and $\|x_j\| \rightarrow \infty$ as $j \rightarrow -\infty$ if and only if x_0 has a nonzero component in $\text{Im}(I - P_-)$. In addition, following [S], assume that $\dim \text{Im}(I - P_+) = \dim \text{Im}(I - P_-)$ and denote the common value of these dimensions d' . Choose ordered bases $u_1, \dots, u_{d'}$ and $u_{d'+1}, \dots, u_d$ of $\text{Im}(I - P_+)$ and $\text{Im} P_-$, respectively, and define

$$(4.19) \quad \mathcal{D} = \det_{\mathbb{C}^d}[u_1 \cdots u_d],$$

the Evans determinant. Note that \mathcal{D} depends on the choice of the basis vectors u_i ; however, if $\tilde{\mathcal{D}}$ is the determinant corresponding to another choice of \tilde{u}_i , then $\mathcal{D} = c\tilde{\mathcal{D}}$ for a nonzero c ; if $\mathbf{A}^\times = \mathbf{A}^\times(z)$ and $u_i = u_i(z)$ and $\tilde{u}_i = \tilde{u}_i(z)$ depend on z analytically then one can show [S] that $c = c(z)$ is a nonvanishing analytic multiplier.

PROPOSITION 4.5. *Assume (3.1), (3.2), (3.4), and (4.8) - (4.9). Then the Evans determinant \mathcal{E} , as defined in (4.18), coincides with \mathcal{D} , as defined in (4.19), where $u_1, \dots, u_{d'}$ and $u_{d'+1}, \dots, u_d$ are the columns of the matrices Y_0^+ and Y_0^- , respectively.*

Proof. First, we claim that

$$(4.20) \quad \dim \text{Im}(I - P) = \dim \text{Im}(I - P_+) \text{ and } \dim \text{Im} P = \dim \text{Im} P_-$$

for the dichotomy projection P over \mathbb{Z} of the unperturbed equation $x_{j+1} = A_j x_j$ and the dichotomy projections P_\pm over \mathbb{Z}_\pm of the perturbed equation $y_{j+1} = A_j^\times y_j$. To prove the first equality in (4.20) (the second is proved similarly), we recall that the perturbation $B_j C_j = A_j^\times - A_j$, $j \in \mathbb{Z}$, is assumed to satisfy $(B_j C_j)_{j \in \mathbb{Z}_+} \in \ell^1(\mathbb{Z}_+; \mathbb{C}^{d \times d})$, cf. (4.8). Under this assumption, following the proof of [Co, Prop. 4.3], one can see that the (bounded on \mathbb{Z}_+) solutions $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_+}$ of the equation $x_{j+1} = A_j x_j$ and $\mathbf{y} = (y_j)_{j \in \mathbb{Z}}$ of the

equation $y_{j+1} = A_j^\times y_j$ are in one-to-one correspondence. Indeed, using (3.7), choose N so large that the operator T defined by

$$(T\mathbf{y})_j = -\sum_{k=j}^{\infty} U_j P U_{k+1}^{-1} B_k C_k y_k + \sum_{k=N}^{j-1} U_j (I - P) U_{k+1}^{-1} B_k C_k y_k, \quad j \in \mathbb{Z}_+,$$

is a contraction on $\ell^\infty(\mathbb{Z}_+; \mathbb{C}^{d \times d})$, cf. the proof of Lemma 4.2. Then the above-mentioned correspondence is given by the formula $\mathbf{y} = \mathbf{x} + T\mathbf{y}$. Since the solutions \mathbf{x} and \mathbf{y} are bounded on \mathbb{Z}_+ if and only if $x_0 \in \text{Im}(I - P)$, resp. $y_0 \in \text{Im}(I - P_+)$, and using notation (3.8), the equality $d_2 = d'$ in (4.20) follows.

Next, we claim that for the solutions $\mathbf{Y}^+ = [\mathbf{Y}^{(1)} \dots \mathbf{Y}^{(n)}]$ and $\mathbf{Y}^- = [\mathbf{Y}^{(n+1)} \dots \mathbf{Y}^{(d_0)}]$, see Remark 4.3, of the equation $y_{j+1} = A_j^\times y_j$, obtained in Lemma 4.2 and Remark 4.1, one has the following equalities:

$$(4.21) \quad \text{rank } Y_0^+ = \dim \text{Im}(I - P_+) \text{ and } \text{rank } Y_0^- = \dim \text{Im } P_-.$$

As soon as (4.21) is proved, the equality $\mathcal{E} = \mathcal{D}$ follows. Indeed, recalling (4.4) - (4.5), and the inequality $\delta < \varepsilon_0$ for ε_0 defined in (4.7), we use Lemma 4.2(a) to observe that $\|Y_j^{(i)}\| \rightarrow 0$ as $j \rightarrow \infty$, resp. $j \rightarrow -\infty$, for $i = 1, \dots, n$, resp. $i = n + 1 \dots, d_0$. Therefore, recalling (4.20), the definitions of the exponential dichotomies P_\pm , see (3.7), and Remark 4.3, we observe that the columns u_1, \dots, u_{d_2} and u_{d_2+1}, \dots, u_d of the matrices Y_0^+ and, respectively, Y_0^- belong to the subspace $\text{Im}(I - P_+)$, respectively, to the subspace $\text{Im } P_-$. By (4.21), these columns form bases in the respective subspaces, and thus $\mathcal{E} = \mathcal{D}$.

To prove the first equality in (4.21) (the second is proved similarly), we will use (4.20) and will show that $\text{rank } Y_0^+ = \dim \text{Im}(I - P)$. For this, it is enough to check that $\text{rank } Y_j^+ = \dim \text{Im}(I - P)$ for sufficiently large j because $\text{rank } Y_j^+ = \text{rank } Y_0^+$ using $Y_j^+ = U_j^\times Y_0^+$ for the invertible by (3.4) matrices U_j^\times forming the fundamental matrix solution \mathbf{U}^\times of the equation $y_{j+1} = A_j^\times y_j$. Finally, since matrices U_j are also invertible, and $I - P = Q_1 + \dots + Q_n$, we have:

$$\dim \text{Im}(I - P) = \text{rank}[Q_1 \dots Q_n] = \text{rank}[U_j Q_1 \dots U_j Q_n] = \text{rank}[Y_j^{(1)} \dots Y_j^{(n)}] = \text{rank } Y_j^+,$$

where the relation $\|Y_j^{(i)} - U_j Q_i\| \rightarrow 0$ as $j \rightarrow \infty$, $i = 1, \dots, n$, from Lemma 4.2 (c) has been used to justify the third equality. Thus, (4.21) follows, and $\mathcal{E} = \mathcal{D}$ is proved. ■

One advantage of Definition 4.4 adapted in the current paper is that the Evans determinant given in (4.18) is uniquely defined by \mathbf{A} and \mathbf{A}^\times (indeed, the finest exponential splitting $\{Q_i\}_{i=1}^{d_0}$ is uniquely defined; although the solutions $\mathbf{Y}^{(i)}$ depend on N , see Lemma 4.2, the determinant (4.18) is proved below to be N -independent, see (4.22)). In other words, the columns of the matrices Y_0^+ and Y_0^- give the canonical choice of the bases in $\text{Im}(I - P_+)$ and $\text{Im } P_-$ needed in (4.19). Also, if $\mathbf{A} = \mathbf{A}(z)$ and $\mathbf{A}^\times = \mathbf{A}^\times(z)$ depend on z analytically, then the analyticity of the Evans function $\mathcal{E} = \mathcal{E}(z)$, where the Evans determinant $\mathcal{E}(z)$ is defined for each fixed z using $\mathbf{A}(z)$ and $\mathbf{A}^\times(z)$ as indicated in (4.18), follows automatically from the analyticity of the corresponding Fredholm determinant whose connection to the Evans determinant is given next.

THEOREM 4.6. *Assume that $(B_j)_{j \in \mathbb{Z}}$, $(C_j)_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z}; \mathbb{C}^{d \times d})$, and (3.1), (3.2), (3.4), (4.8), (4.9) hold. Then the Fredholm determinant for the operator $\mathcal{T} \in \mathcal{B}_1(\ell^2(\mathbb{Z}; \mathbb{C}^{d \times d}))$,*

defined in (1.1), and the Evans determinant (4.18) are related as follows:

$$(4.22) \quad \det(I - T) = \mathcal{E} \prod_{j=0}^{\infty} \frac{\det(A_j + B_j C_j)}{\det A_j}.$$

REMARK 4.7. In this paper we study the perturbed equation $x_{j+1} = (A_j + B_j C_j)x_j$ with the perturbation term $R_j = B_j C_j$ having a *predefined* factorization. Given a perturbed equation $x_{j+1} = (A_j + R_j)x_j$, we can *define* the factorization $R_j = B_j C_j$ by using the polar decomposition $R_j = V_{R_j}|R_j|$ and setting $B_j = V_{R_j}|R_j|^{\frac{1}{2}}$ and $C_j = |R_j|^{\frac{1}{2}}$. If (4.8) - (4.9) are assumed for $R_j = B_j C_j$ then $(R_j)_{j \in \mathbb{Z}} \in \ell^1$ and $(B_j)_{j \in \mathbb{Z}}, (C_j)_{j \in \mathbb{Z}} \in \ell^2$.

Proof. First, we remark that $\mathcal{T} \in \mathcal{B}_1(\ell^2)$ provided $\mathbf{b} = (B_j)_{j \in \mathbb{Z}}$ and $\mathbf{c} = (C_j)_{j \in \mathbb{Z}}$ belong to ℓ^2 . Indeed, let $\mathcal{K}, (\mathcal{K}\mathbf{x})_j = \sum_{k \in \mathbb{Z}} K_{jk} x_k$, denote the (bounded on ℓ^2) operator with the kernel given by (1.2) and satisfying $\|K_{jk}\| \leq ce^{-\alpha|k-j|}$, see (3.7). Then $\mathcal{T} = D_{\mathbf{c}} \mathcal{K} D_{\mathbf{b}}$ using notation (2.6). By (2.8) we have $D_{\mathbf{c}}, D_{\mathbf{b}} \in \mathcal{B}_2(\ell^2)$ and thus $\mathcal{T} \in \mathcal{B}_1$, see, e.g. [GGK, Lem. IV.7.2]. Next, we remark that the product in the right-hand side of (4.22) converges absolutely provided (3.1) and $\mathbf{b}, \mathbf{c} \in \ell^2$ hold:

$$\prod_{j=0}^{\infty} \left| \frac{\det(A_j + B_j C_j)}{\det A_j} \right| = \prod_{j=0}^{\infty} |\det(I + A_j^{-1} B_j C_j)| \leq \prod_{j=0}^{\infty} (1 + d\|(A_j^{-1})_{j \in \mathbb{Z}}\|_{\ell^\infty} \|B_j C_j\|) < \infty$$

because $(B_j C_j)_{j \in \mathbb{Z}} \in \ell^1$.

Further, we *claim* that it is enough to prove (4.22) for finitely supported \mathbf{b} and \mathbf{c} . For $M \in \mathbb{N}$ let χ_M denote the characteristic function of $[-M, M] \cap \mathbb{Z}$, and set $\mathbf{b}^{(M)} = (\chi_M(j) B_j)_{j \in \mathbb{Z}}, \mathbf{c}^{(M)} = (\chi_M(j) C_j)_{j \in \mathbb{Z}}, \mathcal{T}^{(M)} = D_{\mathbf{c}^{(M)}} \mathcal{K} D_{\mathbf{b}^{(M)}}$. By (2.16), $D_{\mathbf{c}^{(M)}} \rightarrow D_{\mathbf{c}}$ and $D_{\mathbf{b}^{(M)}} \rightarrow D_{\mathbf{b}}$ in $\mathcal{B}_2(\ell^2)$ -norm as $M \rightarrow \infty$. Using e.g. [GGK, Lem. IV.7.2], we conclude that $\mathcal{T}^{(M)} \rightarrow \mathcal{T}$ in $\mathcal{B}_1(\ell^2)$ -norm and hence $\det(I - \mathcal{T}^{(M)}) \rightarrow \det(I - \mathcal{T})$ as $M \rightarrow \infty$, see e.g. [GGK, (IV.5.14)]. Let $\mathbf{d}_+ = (A_j^{-1} B_j C_j)_{j \geq 0}$ and $\mathbf{d}_+^{(M)} = (\chi_M(j) A_j^{-1} B_j C_j)_{j \geq 0}$. Then the product in the right-hand side of (4.22) is equal to $\det(I + D_{\mathbf{d}_+}) = \lim_{M \rightarrow \infty} \det(I + D_{\mathbf{d}_+^{(M)}})$ because $D_{\mathbf{d}_+^{(M)}} \rightarrow D_{\mathbf{d}_+}$ in $\mathcal{B}_1(\ell^2(\mathbb{Z}_+; \mathbb{C}^d))$ -norm, as above. Thus, to prove the claim it remains to show that

$$(4.23) \quad \mathcal{E} = \lim_{M \rightarrow \infty} \mathcal{E}^{(M)}$$

where $\mathcal{E}^{(M)}$ is defined as in (4.18) using the solutions $\mathbf{Y}^{(i, M)}$ of equations (4.10) - (4.11) with B_j and C_j replaced by $B_j^{(M)} = \chi_M(j) B_j$ and $C_j^{(M)} = \chi_M(j) C_j$. Note that $q^{(M)} \leq q$, cf. (4.15). Thus, N in Lemma 4.2 could be fixed independent of M . Note that for $i = 1, \dots, n$ (and similarly for $i = n+1, \dots, d_0$) we have $\mathbf{Y}^{(i)} = (I - T)^{-1} (U_j Q_i)_{j > N}$ and $\mathbf{Y}^{(i, M)} = (I - T^{(M)})^{-1} (U_j Q_i)_{j > N}$ for the operators T and $T^{(M)}$ defined by the right-hand side of (4.10), and that $\|T\| < q < 1$ and $\|T^{(M)}\| < q$ uniformly for $M \in \mathbb{N}$, for the operator norm in $l_{+, \beta}^\infty$, cf. (4.16). Also, $U_j^{\times, M} \rightarrow U_j^\times$ as $M \rightarrow \infty$ uniformly for $|j| \leq N$, cf. Remark 4.1. Thus, (4.23) is proved as soon as we show that $\|T - T^{(M)}\| \rightarrow 0$ as $M \rightarrow \infty$. But $T - T^{(M)}$ is again given by the same expression as T , see (4.10), except the product $B_k C_k$ should be replaced by $B_k C_k - B_k^{(M)} C_k^{(M)} = (1 - \chi_M(k)) B_k C_k$. We can now use (4.16) to estimate for $i = 1, \dots, n$ the operator norm of $T - T^{(M)}$ in $l_{+, \beta}^\infty$:

$$\begin{aligned}
(4.24) \quad \|T - T^{(M)}\| &\leq \max\{c_\alpha, c'_\alpha\} \sum_{k=N}^{\infty} (1 - \chi_M(k)) e^{(\varkappa_i^+ - \varkappa_i'^+ + \delta)k} \|B_k C_k\| \\
&= \max\{c_\alpha, c'_\alpha\} \sum_{k=M+1}^{\infty} e^{(\varkappa_i^+ - \varkappa_i'^+ + \delta)k} \|B_k C_k\| \rightarrow 0 \text{ as } M \rightarrow \infty,
\end{aligned}$$

using (4.8). A similar argument works for $i = n + 1, \dots, d_0$. Thus, (4.23) holds, and the claim above is verified.

From now on we therefore assume that $(B_j)_{j \in \mathbb{Z}}$ and $(C_j)_{j \in \mathbb{Z}}$ are finitely supported. Note that then assumption (3.3) holds, and thus (3.36) holds. So, to establish (4.22) we need to prove that $\mathcal{E} = \det \mathcal{U}_0^\times$, where $\mathcal{U}_0^\times = V_0$ is given by (3.35) in terms of the solutions \mathbf{X}^\pm of difference equations (3.29) - (3.30).

Consider the following matrix difference equations:

$$(4.25) \quad Z_j^+ - (I - P) = - \sum_{k=j}^{\infty} U_{k+1}^{-1} B_k C_k U_k Z_k^+,$$

$$(4.26) \quad Z_j^- - P = \sum_{k=-\infty}^{j-1} U_{k+1}^{-1} B_k C_k U_k Z_k^-,$$

and introduce on $\ell^\infty(\mathbb{Z}; \mathbb{C}^{d \times d})$ operators \tilde{H}_+ and \tilde{H}_- corresponding to the right-hand sides of (4.25) - (4.26) so that these equations become $(I - \tilde{H}_+) \mathbf{Z}^+ = (I - P)_{j \in \mathbb{Z}}$ and $(I - \tilde{H}_-) \mathbf{Z}^- = (P)_{j \in \mathbb{Z}}$. We claim that the operators $I - \tilde{H}_\pm$ are invertible on $\ell^\infty(\mathbb{Z}; \mathbb{C}^{d \times d})$. Indeed, using notation (2.5) - (2.6) we have $\tilde{H}_+ = -D_{\mathbf{d}} - V_{\mathbf{a}, \mathbf{b}}^+$ and $\tilde{H}_- = V_{\mathbf{a}, \mathbf{b}}^-$ with $\mathbf{a} = (U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}} \in \ell^1$, $\mathbf{b} = (C_j U_j)_{j \in \mathbb{Z}} \in \ell^1$, and $\mathbf{d} = (U_{j+1}^{-1} B_j C_j U_j)_{j \in \mathbb{Z}} \in \ell^1$. The operator $I + D_{\mathbf{d}}$ is invertible in $\ell^\infty(\mathbb{Z}; \mathbb{C}^{d \times d})$ since the operator $I + D_{\mathbf{d}'}$, $\mathbf{d}' = (B_j C_j U_j U_{j+1}^{-1})_{j \in \mathbb{Z}}$ is invertible in $\ell^\infty(\mathbb{Z}; \mathbb{C}^{d \times d})$ by assumptions (3.1) and (3.4) and the identities $I + B_j C_j U_j U_{j+1}^{-1} = A_j^\times A_j^{-1}$, $j \in \mathbb{Z}$ (recall that $A_j^\times = A_j$ for sufficiently large $|j|$ because $(B_j)_{j \in \mathbb{Z}}$ and $(C_j)_{j \in \mathbb{Z}}$ are finitely supported). Moreover, $(I + D_{\mathbf{d}})^{-1} = D_{\mathbf{d}''}$ with some $\mathbf{d}'' = (d_j'')_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}; \mathbb{C}^{d \times d})$. Since

$$I - \tilde{H}_+ = (I + D_{\mathbf{d}})(I + (I + D_{\mathbf{d}})^{-1} V_{\mathbf{a}, \mathbf{b}}^+) = (I + D_{\mathbf{d}})(I + V_{\mathbf{a}', \mathbf{b}}^+)$$

with $\mathbf{a}' = (d_j'' U_{j+1}^{-1} B_j)_{j \in \mathbb{Z}} \in \ell^1$, the operators $I - \tilde{H}_+$ and $I - \tilde{H}_-$ are invertible by (2.12), as claimed. Therefore, equations (4.25) - (4.26) have a unique pair of solutions $\mathbf{Z}^\pm \in \ell^\infty(\mathbb{Z}; \mathbb{C}^{d \times d})$ given by

$$(4.27) \quad \mathbf{Z}^+ = (I - \tilde{H}_+)^{-1} (I - P)_{j \in \mathbb{Z}} \text{ and } \mathbf{Z}_- = (I - \tilde{H}_-)^{-1} (P)_{j \in \mathbb{Z}}.$$

In particular $Z_j^+ (I - P) = Z_j^+$ and $Z_j^- P = Z_j^-$, $j \in \mathbb{Z}$. Also, using (4.25) - (4.26), it is easy to see that $(U_j Z_j^\pm)_{j \in \mathbb{Z}}$ are solutions of the equation $x_{j+1} = A_j^\times x_j$, $j \in \mathbb{Z}$. Thus, if $(U_j^\times)_{j \in \mathbb{Z}}$ is the fundamental matrix solution of the equation $x_{j+1} = A_j^\times x_j$, then the following relations hold:

$$(4.28) \quad Z_0^+ = (U_{N+1}^\times)^{-1} (U_{N+1} Z_{N+1}^+) \text{ and } Z_0^- = (U_{-N}^\times)^{-1} (U_{-N} Z_{-N}^-)$$

(recall that $U_0 = U_0^\times = I$, and N here is chosen as in Lemma 4.2). On the other hand, multiplying (4.25) - (4.26) by $C_j U_j$ from the left, letting $X_j^\pm = C_j U_j Z_j^\pm$ and recalling that $(C_j U_j)_{j \in \mathbb{Z}}$ is finitely supported, we conclude that $\mathbf{X}^\pm = (X_j)_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z}; \mathbb{C}^{d \times d})$ is the

unique pair of solutions of equations (3.29) - (3.30). Using the direct sum decomposition $\mathbb{C}^d = \text{Im}(I - P) \oplus \text{Im} P$, and writing $Z_j^+ + Z_j^-$, $j \in \mathbb{Z}$, as a (2×2) -block matrix, we observe that $Z_j^+ + Z_j^- = V_j$, where V_j is given in (3.35) in terms of \mathbf{X}^\pm . So, to complete the proof of the theorem we need to show that $\mathcal{E} = \det[Z_0^+ + Z_0^-]$.

Since $(U_{j+1}^{-1}B_j)_{j \in \mathbb{Z}}$ and $(C_j U_j)_{j \in \mathbb{Z}}$ are finitely supported, equations (4.10) - (4.11) can be equivalently rewritten as follows:

$$(4.29) \quad Y_j^{(i)} = U_j Q_i - \sum_{k=j}^{\infty} U_j U_{k+1}^{-1} B_k C_k Y_k^{(i)} \\ + \sum_{k=N}^{\infty} U_j (I - (Q_i + \cdots + Q_{d_0})) U_{k+1}^{-1} B_k C_k Y_k^{(i)}, \quad j > N,$$

$$(4.30) \quad Y_j^{(i)} = U_j Q_i + \sum_{k=-\infty}^{j-1} U_j U_{k+1}^{-1} B_k C_k Y_k^{(i)} \\ - \sum_{k=-\infty}^{-N} U_j (I - (Q_1 + \cdots + Q_i)) U_{k+1}^{-1} B_k C_k Y_k^{(i)}, \quad j \leq -N,$$

Recall that $Y_j^{(i)} Q_i = Y_j^{(i)}$ by Lemma 4.2, and that $I - P = Q_1 + \cdots + Q_n$, $P = Q_{n+1} + \cdots + Q_{d_0}$, so that $(I - P)Q_i = Q_i$ for $i = 1, \dots, n$ and $PQ_i = Q_i$ for $i = n+1, \dots, d_0$. Using notation $Y_j^+ = \sum_{i=1}^n Y_j^{(i)}$ and $Y_j^- = \sum_{i=n+1}^{d_0} Y_j^{(i)}$, we derive from (4.29) - (4.30):

$$(4.31) \quad Y_j^+ = U_j (I - P) - \sum_{k=j}^{\infty} U_j U_{k+1}^{-1} B_k C_k Y_k^+ + U_j T_+, \quad j > N,$$

$$(4.32) \quad Y_j^- = U_j P + \sum_{k=-\infty}^{j-1} U_j U_{k+1}^{-1} B_k C_k Y_k^- + U_j T_-, \quad j \leq -N,$$

where we have denoted:

$$(4.33) \quad T_+ = \sum_{i=1}^n (I - (Q_i + \cdots + Q_{d_0})) \eta_i Q_i, \quad \eta_i = \sum_{k=N}^{\infty} U_{k+1}^{-1} B_k C_k Y_k^{(i)},$$

$$(4.34) \quad T_- = \sum_{i=n+1}^{d_0} (I - (Q_1 + \cdots + Q_i)) \eta_i Q_i, \quad \eta_i = \sum_{k=-\infty}^{-N} U_{k+1}^{-1} B_k C_k Y_k^{(i)}.$$

We remark that, in the direct sum decomposition $\mathbb{C}^d = \text{Im}(I - P) \oplus \text{Im} P$,

$$(4.35) \quad T_+ = (I - P)T_+(I - P), \quad T_- = PT_-P, \quad \text{and} \quad T_+ + T_- = T_+ \oplus T_-.$$

Using the operators \tilde{H}_\pm , we can also rewrite (4.31) - (4.32) as follows:

$$(I - \tilde{H}_+)(U_j^{-1}Y_j^+)_{j > N} = ((I - P)(I - P + T_+))_{j > N}, \\ (I - \tilde{H}_-)(U_j^{-1}Y_j^-)_{j \leq -N} = (P(P + T_-))_{j \leq -N}.$$

Comparison with (4.27) yields:

$$Y_j^+ = U_j Z_j^+ (I - P + T_+), \quad j > N, \quad \text{and} \quad Y_j^- = U_j Z_j^- (P + T_-), \quad j \leq -N.$$

Formulas (4.12) and (4.28) with the fundamental matrix solution $(U_j^\times)_{j \in \mathbb{Z}}$ of $x_{j+1} = A_j^\times x_j$ then imply:

$$\begin{aligned} Y_0^+ + Y_0^- &= (U_{N+1}^\times)^{-1} Y_{N+1}^+ + (U_{-N}^\times)^{-1} Y_{-N}^- \\ &= (U_{N+1}^\times)^{-1} (U_{N+1} Z_{N+1}^+ (I - P + T_+)) + (U_{-N}^\times)^{-1} (U_{-N} Z_{-N}^- (P + T_-)) \\ &= Z_0^+ (I - P + T_+) + Z_0^- (P + T_-) = (Z_0^+ + Z_0^-) (I + T_+ + T_-). \end{aligned}$$

In the last equality we also used the fact that $Z_0^+ = Z_0^+ (I - P)$ and $Z_0^- = Z_0^- P$, and assertions (4.35). Since $\mathcal{E} = \det[Y_0^+ + Y_0^-]$ by (4.18), to finish the proof of the identity $\mathcal{E} = \det[Z_0^+ + Z_0^-]$ (and thus the proof of the theorem), it suffices to prove that

$$(4.36) \quad \det[I + T_+ + T_-] = 1.$$

For this, cf. (4.35), we note that the operators $T_+ = (I - P)T_+(I - P)$ on $\text{Im}(I - P)$ and, resp. $T_- = PT_-P$ on $\text{Im} P$ are represented by lower-, resp. upper-triangular matrices with zero main diagonals in the direct sum decomposition $\text{Im}(I - P) = \text{Im} Q_1 \oplus \cdots \oplus \text{Im} Q_n$, resp. $\text{Im} P = \text{Im} Q_{n+1} \oplus \cdots \oplus \text{Im} Q_{d_0}$ (cf. (4.33) - (4.34)). Thus, the proof of (4.36) and the theorem is completed. ■

5. Constant coefficients. In this section we specialize to the case where $A_j \equiv A$, $j \in \mathbb{Z}$, with a given matrix $A \in \mathbb{C}^{d \times d}$ satisfying the assumptions

$$(5.1) \quad 0 \notin \sigma(A) \quad \text{and} \quad \sigma(A) \cap \{z \in \mathbb{C} : |z| = 1\} = \emptyset,$$

cf. assumptions (3.1) - (3.2). We will continue to assume that $A_j^\times = A + B_j C_j$ is invertible for each $j \in \mathbb{Z}$, cf. (3.4). Note that $U_j = A^j$, $j \in \mathbb{Z}$. We will show that in this constant coefficients case assumptions (4.8) - (4.9) of the exponential decay of the perturbation could be replaced with a weaker assumption of a polynomial decay. Specifically, let m denote the maximal size of Jordan blocks in the Jordan canonical form of the $(d \times d)$ -matrix A . (Then $m = 0$ provided all eigenvalues of A are semi-simple, that is, when all Jordan blocks are diagonal.) The following assumption will replace (4.8) - (4.9):

$$(5.2) \quad \sum_{k=-\infty}^{\infty} |k|^{2m} \|B_k C_k\| < \infty.$$

Let us decompose $\sigma(A) = \cup_{i=1}^{d_0} \sigma_i$ where σ_i 's belong to concentric circles whose radii are denoted by e^{\varkappa_i} so that $\sigma_i = \{\lambda \in \sigma(A) : |\lambda| = e^{\varkappa_i}\}$, and enumerate the numbers \varkappa_i so that

$$(5.3) \quad \varkappa_1 < \cdots < \varkappa_n < 0 < \varkappa_{n+1} < \varkappa_{d_0}$$

for some $n \in \{1, \dots, d_0\}$. Let Q_i denote the spectral projection for A such that $\sigma(A|_{\text{Im} Q_i}) = \sigma_i$. Note that $\varkappa_i = \varkappa_g(Q_i; \mathbb{Z}) = \varkappa'_g(Q_i; \mathbb{Z}) = \varkappa_i'^{\pm} = \varkappa_i^{\pm}$ for the Bohl exponents, cf. (4.6). Passing to appropriate coordinates, we will assume that A is in the Jordan normal form. Thus, each $A|_{\text{Im} Q_i}$ is a direct sum of (maybe, several) Jordan blocks $\lambda I + E$ with $|\lambda| = e^{\varkappa_i}$. By Lemma 2.3,

$$(5.4) \quad \|A^j|_{\text{Im} Q_i}\| \leq c|j|^m e^{j\varkappa_i}, \quad i = 1, \dots, d_0, \quad j \in \mathbb{Z}.$$

Equations (4.10) - (4.11) now become:

$$(5.5) \quad \mathbf{Y}^{(i)} - (A^j Q_i)_{j>N} = T_i \mathbf{Y}^{(i)}, \quad i = 1, \dots, n,$$

$$(5.6) \quad \mathbf{Y}^{(i)} - (A^j Q_i)_{j \leq -N} = T_i \mathbf{Y}^{(i)}, \quad i = n+1, \dots, d_0,$$

where

$$(T_i \mathbf{u})_j = - \sum_{k=j}^{\infty} A^{j-k-1} (Q_i + \dots + Q_{d_0}) B_k C_k u_k \\ + \sum_{k=N}^{j-1} A^{j-k-1} (I - (Q_i + \dots + Q_{d_0})) B_k C_k u_k, \quad \mathbf{u} = (u_j)_{j>N}, i = 1, \dots, n, \\ (T_i \mathbf{u})_j = \sum_{k=-\infty}^{j-1} A^{j-k-1} (Q_1 + \dots + Q_i) B_k C_k u_k \\ (5.7) \quad - \sum_{k=j}^{-N} A^{j-k-1} (I - (Q_1 + \dots + Q_i)) B_k C_k u_k, \quad \mathbf{u} = (u_j)_{j \leq -N}, i = n+1, \dots, d_0.$$

LEMMA 5.1. Assume (5.1) - (5.2). Then for a sufficiently large N there exist solutions $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j>N}$, $i = 1, \dots, n$, resp. $\mathbf{Y}^{(i)} = (Y_j^{(i)})_{j \leq -N}$, $i = n+1, \dots, d_0$, of (5.5), resp. (5.6) such that $Y_j^{(i)} = Y_j^{(i)} Q_i$ and the following assertions hold:

$$(5.8) \quad (a) \quad \sup_{j>N} |j|^{-m} e^{-j\kappa_i} \|Y_j^{(i)}\| < \infty, \quad i = 1, \dots, n, \\ \sup_{j \leq -N} |j|^{-m} e^{-j\kappa_i} \|Y_j^{(i)}\| < \infty, \quad i = n+1, \dots, d_0,$$

$$(b) \quad e^{-j\kappa_i} \|Y_j^{(i)} - A^j Q_i\| \rightarrow 0$$

as $j \rightarrow \infty$ for $i = 1, \dots, n$ and as $j \rightarrow -\infty$ for $i = n+1, \dots, d_0$.

Proof. We prove the lemma for $i = n+1, \dots, d_0$; the proof for $i = 1, \dots, n$ is similar. Using (5.3) - (5.4), we infer:

$$(5.9) \quad \|A^{j-k-1} (Q_1 + \dots + Q_i)\| \leq c |j-k|^m e^{(j-k)\kappa_i}, \quad 0 \geq j \geq k,$$

$$(5.10) \quad \|A^{j-k-1} (I - (Q_1 + \dots + Q_i))\| \leq c |j-k|^m e^{(j-k)\kappa_{i+1}} \\ \leq c' |j-k|^m e^{(j-k)\kappa_i}, \quad 0 \geq k \geq j.$$

Introduce the Banach space $\ell_{\infty}^{\infty} = \{\mathbf{u} = (u_j)_{j \leq -N} : \|\mathbf{u}\|_{\ell_{\infty}^{\infty}} := \sup_{j \leq -N} |j|^{-m} e^{-j\kappa_i} \|u_j\| < \infty\}$. Denote $q_N = \sum_{k=-\infty}^{-N} |k|^{2m} \|B_k C_k\|$ and remark that $q_N \rightarrow 0$ as $N \rightarrow \infty$ by (5.2). Using (5.9) - (5.10) and the fact that $\sup_{k \geq 0} |k|^m e^{-k(\kappa_{i+1} - \kappa_i)} < \infty$, we have for $j \leq -N$ and the operator $T = T_i(N)$ defined in (5.7):

$$(5.11) \quad |j|^{-m} e^{-j\kappa_i} \|(T \mathbf{u})_j\| \leq c \|\mathbf{u}\|_{\ell_{\infty}^{\infty}} \left(\sum_{k=-\infty}^{j-1} |j^{-1} (j-k) k|^m \|B_k C_k\| \right. \\ \left. + \sum_{k=j}^{-N} (|j-k|^m e^{(j-k)(\kappa_{i+1} - \kappa_i)}) |j^{-1} k|^m \|B_k C_k\| \right) \leq c' q_N \|\mathbf{u}\|_{\ell_{\infty}^{\infty}}.$$

Note that $|k| \geq |j| \geq N \geq 1$ in the first sum and $|j| \geq |k| \geq N \geq 1$ in the second sum. Thus, $\|T\mathbf{u}\|_{\ell^\infty} \leq cq_N\|\mathbf{u}\|_{\ell^\infty}$, and $T = T_i(N)$ is a contraction on ℓ^∞ for a sufficiently large N . Since $(A^j Q_i)_{j \leq -N} \in \ell^\infty$ by (5.4), assertion (5.8) follows. To show (b) in the lemma, we follow the proof of the celebrated Levinson Theorem in [E, p.14]. Indeed, for $j < -M < -N$ and $\mathbf{u} = (Y_j^{(i)})_{j \leq -N} \in \ell^\infty$, similarly to (5.11), we infer:

$$\begin{aligned}
& e^{-j\kappa_i} \|(T\mathbf{u})_j\| \leq \|e^{-j\kappa_i} (T_i(M)\mathbf{u})_j\| \\
& + \sum_{k=-M+1}^{-N} \|A^{j-k-1} (I - (Q_1 + \cdots + Q_i)) B_k C_k u_k\| \\
& \leq c\|\mathbf{u}\|_{\ell^\infty} \left(\sum_{k=-\infty}^{j-1} |(j-k)k|^m \|B_k C_k\| + \sum_{k=j}^{-M} |j-k|^m e^{(j-k)(\kappa_{i+1}-\kappa_i)} |k|^m \|B_k C_k\| \right) \\
& + \sum_{k=-M+1}^{-N} (|j-k|^m e^{\frac{1}{2}(j-k)(\kappa_{i+1}-\kappa_i)}) e^{\frac{1}{2}(j-k)(\kappa_{i+1}-\kappa_i)} |k|^m \|B_k C_k\| \\
& \leq c'\|\mathbf{u}\|_{\ell^\infty} (q_M + c(M, N) e^{\frac{1}{2}j(\kappa_{i+1}-\kappa_i)}).
\end{aligned}$$

Choosing first a sufficiently large M , and then letting $j \rightarrow -\infty$, we have (b) in the lemma. ■

As soon as the existence of the solutions of (5.5) - (5.6) is established, we can use formulas (4.12) to obtain the matrix solutions $\mathbf{Y}^+ = \mathbf{Y}^{(1)} + \cdots + \mathbf{Y}^{(n)}$ on \mathbb{Z}_+ and $\mathbf{Y}^- = \mathbf{Y}^{(n+1)} + \cdots + \mathbf{Y}^{(d_0)}$ on \mathbb{Z}_- of the perturbed equation $x_{j+1} = (A + B_j C_j)x_j$, and then define the Evans determinant, \mathcal{E} , as indicated in (4.18). Note that the equality $\mathcal{E} = \mathcal{D}$ for the ‘‘standard’’ Evans determinant \mathcal{D} also holds as in Proposition 4.5 since \mathbf{Y}^\pm enjoy the properties listed in Lemma 5.1. The proof of Theorem 4.6 remains unchanged with a natural replacement of the exponential term in (4.24) by k^{2m} . Thus, we have the following result.

PROPOSITION 5.2. *Formula (4.22) holds provided $A = A_j$, $j \in \mathbb{Z}$, assumption (3.4) is satisfied, and assumptions (3.1) - (3.2) and (4.8) - (4.9) in Theorem 4.6 are replaced, respectively, by (5.1) and (5.2).*

Next, as an illustration, we will consider a particularly important class of second order difference equations, the discrete Schrödinger equation, and show how to specialize our results for the corresponding (2×2) first order system. Given a real-valued potential $\mathbf{v} = (v_j)_{j \in \mathbb{Z}} \in \ell^1(\mathbb{Z}; \mathbb{R})$, consider on $\ell^2(\mathbb{Z}; \mathbb{C})$ a bounded self-adjoint operator, L , defined by $(L\mathbf{y})_j = y_{j+1} + y_{j-1} + v_j y_j$, $\mathbf{y} \in \ell^2(\mathbb{Z}; \mathbb{C})$. If $(S\mathbf{y})_j = y_{j-1}$ denotes the shift operator, and $z \in \mathbb{R}$, then the following identity, cf. [LS, Exmp. 1.6], holds on $(\ell^2(\mathbb{Z}; \mathbb{C}))^2 = \ell^2(\mathbb{Z}; \mathbb{C}^2)$ for the operator $L = \mathcal{S} + \mathcal{S}^{-1} + D_{\mathbf{v}}$, cf. (2.6):

$$(5.12) \quad \begin{bmatrix} 0 & I \\ I & \mathcal{S} \end{bmatrix} \begin{bmatrix} L - zI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & \mathcal{S} \\ I & -\mathcal{S} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & -I \\ I & D_{\mathbf{v}} - zI \end{bmatrix} \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{bmatrix}.$$

Writing $v_j = e^{i \arg v_j} |v_j|$, and introducing (2×2) matrices

$$(5.13) \quad A = \begin{bmatrix} 0 & 1 \\ -1 & z \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -e^{i \arg v_j} |v_j|^{\frac{1}{2}} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & |v_j|^{\frac{1}{2}} \end{bmatrix},$$

and $A_j^\times = A + B_j C_j$, we observe from (5.12) that the operator $L - zI$ has a bounded inverse on $\ell^2(\mathbb{Z}; \mathbb{C})$ if and only if the operator $I - D_{\mathbf{A}^\times} \mathcal{S}$, $\mathbf{A}^\times = (A_j^\times)_{j \in \mathbb{Z}}$, has a bounded inverse on $\ell^2(\mathbb{Z}; \mathbb{C}^2)$ since the (2×2) operator matrices containing \mathcal{S} in the left-hand side of (5.12) are invertible operators on $\ell^2(\mathbb{Z}; \mathbb{C}^2)$, and the right-hand side of (5.12) is $I - D_{\mathbf{A}^\times} \mathcal{S}$. Recalling the operators $G_{\mathbf{A}}$ and $G_{\mathbf{A}^\times}$, see the Introduction, corresponding to the difference equations $x_{j+1} = Ax_j$ and $x_{j+1} = A_j^\times x_j$, $x_j \in \mathbb{C}^2$, $j \in \mathbb{Z}$, by the formulas $(G_{\mathbf{A}} \mathbf{x})_j = x_{j+1} - Ax_j$ and $(G_{\mathbf{A}^\times} \mathbf{x})_j = x_{j+1} - A_j^\times x_j$, observe that $(I - D_{\mathbf{A}^\times} \mathcal{S}) \mathcal{S}^{-1} = G_{\mathbf{A}^\times}$. Note that $\det A_j^\times = 1$ and thus assumption (3.4) holds. The eigenvalues of $A = A(z)$ will be denoted by $\lambda = \lambda(z)$ and λ^{-1} ; these are the roots of the equation $\lambda^2 - z\lambda + 1 = 0$. If $|z| \leq 2$ then $|\lambda| = 1$; if $|z| > 2$ we choose λ so that $|\lambda| < 1$ and denote $\Lambda = \lambda^{-1} - \lambda$. Thus, if $|z| > 2$ then $A = A(z)$ satisfies assumptions (5.1). Then $G_{\mathbf{A}}$ is invertible on $\ell(\mathbb{Z}; \mathbb{C}^2)$, and the kernel of the operator $\mathcal{K} = G_{\mathbf{A}}^{-1}$ is given by (1.2) where $U_j = A^j$ and $P = P(z)$ is the spectral projection for A so that $\sigma(A|_{\text{Im } P}) = \{\lambda^{-1}\}$, explicitly, $P = \frac{1}{\Lambda} \begin{bmatrix} -\lambda & 1 \\ -1 & \lambda^{-1} \end{bmatrix}$. Using the identity $G_{\mathbf{A}^\times} = G_{\mathbf{A}}(I - G_{\mathbf{A}}^{-1} D_{\mathbf{B}} D_{\mathbf{C}})$ with $\mathbf{B} = (B_j)_{j \in \mathbb{Z}} \in \ell^2$ and $\mathbf{C} = (C_j)_{j \in \mathbb{Z}} \in \ell^2$, recalling that $\mathcal{T} = \mathcal{T}(z)$ from (1.1) satisfies $\mathcal{T} = D_{\mathbf{C}} G_{\mathbf{A}}^{-1} D_{\mathbf{B}}$, and noting that the operators $I - G_{\mathbf{A}}^{-1} D_{\mathbf{B}} D_{\mathbf{C}}$ and $I - \mathcal{T}$ are invertible at the same time, we have the following corollary of Theorem 4.6.

PROPOSITION 5.3. *Assume $\mathbf{v} = (v_j)_{j \in \mathbb{Z}} \in \ell^1(\mathbb{Z}; \mathbb{R})$. If $|z| > 2$ then $z \in \sigma(L)$ on $\ell^2(\mathbb{Z}; \mathbb{C})$ if and only if $\mathcal{E}(z) = 0$ for the Evans function $\mathcal{E}(z) = \det(I - \mathcal{T}(z))$.*

We remark that the Evans determinant $\mathcal{E}(z)$ here is defined, cf. (4.18), as $\mathcal{E}(z) = \det[Y_0^+ + Y_0^-]$, where $\mathbf{Y}^\pm = \mathbf{Y}^\pm(z)$ are the (2×2) matrix solutions of the following system:

$$(5.14) \quad \begin{aligned} Y_j^+ - A^j(z)(I - P(z)) &= - \sum_{k=j}^{\infty} A^{j-k-1}(z) B_k C_k Y_k^+, \quad j \in \mathbb{Z}, \\ Y_j^- - A^j(z)P(z) &= \sum_{k=-\infty}^{j-1} A^{j-k-1}(z) B_k C_k Y_k^-, \quad j \in \mathbb{Z}. \end{aligned}$$

Recall that the solutions satisfy $Y_j^+ = Y_j^+(I - P(z))$, $Y_j^- = Y_j^- P(z)$. Also, because (5.14) are Volterra-type equations, passing to $\lambda^{\mp j} Y_j^\pm$ and using (2.12), one can easily see that $\mathbf{Y}^\pm = (Y_j^\pm)$ are indeed defined for all $j \in \mathbb{Z}$.

Next, we will use the special structure of A , B_j , and C_j to simplify system (5.14). Introduce matrices

$$W = \begin{bmatrix} 1 & 1 \\ \lambda & \lambda^{-1} \end{bmatrix}, \quad W^{-1} = \Lambda^{-1} \begin{bmatrix} \lambda^{-1} & -1 \\ -\lambda & 1 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix},$$

and remark that $\tilde{A} = W^{-1} A W$ and $\Lambda^{-1} v_j V \tilde{A} = W^{-1} B_j C_j W$ for the matrices A , B_j , and C_j introduced in (5.13). The change of variables $\tilde{x}_j = W^{-1} x_j$ transforms the equation $x_{j+1} = A_j^\times x_j$ to the equation $\tilde{x}_{j+1} = (\tilde{A} + \Lambda^{-1} v_j V \tilde{A}) \tilde{x}_j$ with the diagonal unperturbed coefficient \tilde{A} whose dichotomy projection $\tilde{P} = W^{-1} P W$ is given by $\tilde{P} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Similarly,

passing in equations (5.14) to the new unknowns $\tilde{\mathbf{Y}}^\pm = W^{-1} \mathbf{Y}^\pm W$, we may use the fact that $\mathbf{Y}^+ = \mathbf{Y}^+(I - P)$ and $\mathbf{Y}^- = \mathbf{Y}^- P$ imply $\tilde{\mathbf{Y}}^+ = \tilde{\mathbf{Y}}^+(I - \tilde{P})$ and $\tilde{\mathbf{Y}}^- = \tilde{\mathbf{Y}}^- \tilde{P}$ to

conclude that the second column in $\tilde{\mathbf{Y}}_j^+$ and the first column in $\tilde{\mathbf{Y}}_j^-$ are equal to zero. Let $u_j^+ \in \mathbb{C}^2$, resp. $u_j^- \in \mathbb{C}^2$ denote the first column of $\tilde{\mathbf{Y}}_j^+$, resp. the second column of $\tilde{\mathbf{Y}}_j^-$. Then equations (5.14) are equivalent to the following (2×1) vector equations:

$$(5.15) \quad \begin{aligned} u_j^+ &= [\lambda^j \quad 0]^\top - \sum_{k=j}^{\infty} \Lambda^{-1} v_k \tilde{A}^{j-k-1} V \tilde{A} u_k^+, \quad j \in \mathbb{Z}, \\ u_j^- &= [0 \quad \lambda^{-j}]^\top - \sum_{k=-\infty}^{j-1} \Lambda^{-1} v_k \tilde{A}^{j-k-1} V \tilde{A} u_k^-, \quad j \in \mathbb{Z}. \end{aligned}$$

Introducing (2×1) vectors $x_j^\pm = W u_j^\pm$ given, in components, by $x_j^\pm = [x_j^\pm(1) \quad x_j^\pm(2)]^\top$ and passing in (5.15) to the new unknowns x_j^\pm , we have the following system of equations equivalent to (5.14):

$$(5.16) \quad \begin{aligned} x_j^+ &= [\lambda^j \quad \lambda^{j+1}]^\top - \sum_{k=j}^{\infty} \Lambda^{-1} v_k [\lambda^{-j+k+1} - \lambda^{j-k-1} \quad \lambda^{-j+k} - \lambda^{j-k}]^\top x_k^+(2), \\ x_j^- &= [\lambda^{-j} \quad \lambda^{-j-1}]^\top + \sum_{k=-\infty}^{j-1} \Lambda^{-1} v_k [\lambda^{-j+k+1} - \lambda^{j-k-1} \quad \lambda^{-j+k} - \lambda^{j-k}]^\top x_k^-(2). \end{aligned}$$

Recall that $\mathbf{y} = (y_j)_{j \in \mathbb{Z}}$, $y_j \in \mathbb{C}$, is a solution of the difference equation $L\mathbf{y} = z\mathbf{y}$ if and only if $\mathbf{x} = (x_j)_{j \in \mathbb{Z}}$ with $x_j = [y_{j-1} \quad y_j]^\top \in \mathbb{C}^2$ is a solution of the equation $x_{j+1} = A_j^\times x_j$. Moreover, since $\det A_j^\times = 1$, if \mathbf{y}^+ and \mathbf{y}^- are two solutions of the equation $L\mathbf{y} = z\mathbf{y}$ with the corresponding \mathbf{x}^+ and \mathbf{x}^- , then the Wronskian

$$(5.17) \quad \mathcal{W}(\mathbf{y}^+, \mathbf{y}^-) := y_{j-1}^+ y_j^- - y_j^+ y_{j-1}^- = \det[x_j^+ \quad x_j^-]$$

does not depend on $j \in \mathbb{Z}$.

A direct calculation shows that solutions $\mathbf{x}^\pm = (x_j^\pm)$ of (5.16) have the property $x_{j+1}^\pm(1) = x_j^\pm(2)$. Also, \mathbf{x}^\pm satisfy the equation $x_{j+1}^\pm = A_j^\times x_j^\pm$ because \mathbf{Y}^\pm in (5.14) satisfy this equation. Thus, letting $y_j^\pm = \lambda^{\mp 1} x_j^\pm(2)$ and keeping only the second component of the vectors in (5.16), we have obtained the solutions $\mathbf{y}^\pm = (y_j^\pm)$ of the second order difference equation $L\mathbf{y} = z\mathbf{y}$ that satisfy the following equations:

$$(5.18) \quad \begin{aligned} y_j^+ &= \lambda^j - \sum_{k=j}^{\infty} \Lambda^{-1} v_k (\lambda^{-j+k} - \lambda^{j-k}) y_k^+, \quad j \in \mathbb{Z}, \\ y_j^- &= \lambda^{-j} + \sum_{k=-\infty}^{j-1} \Lambda^{-1} v_k (\lambda^{-j+k} - \lambda^{j-k}) y_k^-, \quad j \in \mathbb{Z}. \end{aligned}$$

Introducing $\varkappa < 0$ so that $\lambda = e^\varkappa$, (5.18) could be rewritten as follows:

$$\begin{aligned} y_j^+ &= e^{j\varkappa} - \sum_{k=j}^{\infty} v_k \frac{\sinh(j-k)\varkappa}{\sinh \varkappa} y_k^+, \quad j \in \mathbb{Z}, \\ y_j^- &= e^{-j\varkappa} + \sum_{k=-\infty}^{j-1} v_k \frac{\sinh(j-k)\varkappa}{\sinh \varkappa} y_k^-, \quad j \in \mathbb{Z}. \end{aligned}$$

The solutions $\mathbf{y}^\pm = (y_j^\pm)_{j \in \mathbb{Z}_\pm}$ are asymptotic to $\lambda^{\pm j} = e^{\pm j\kappa}$ as $j \rightarrow \pm\infty$; these are the *Jost solutions* of $L\mathbf{y} = z\mathbf{y}$, cf. [CS, XVII.1.9-10], [GH, Sec. 2], [GM, (4.52)] for the continuous and [C, C1], [FT, Sec. III.2], [GH, (6.6)], [HKS], [T, (10.3)], and [To, (3.3.1)] for the discrete models. Recalling that $\mathbf{y}^\pm = \mathbf{y}^\pm(z)$ and definition (5.17) of the Wronskian of the solutions, we define the *Jost function*, $\mathcal{J} = \mathcal{J}(z)$, by $\mathcal{J}(z) := \Lambda^{-1}\mathcal{W}(\mathbf{y}^+(z), \mathbf{y}^-(z))$, cf. [T, Sec. 10.2] and [To, (3.3.19)]. Our last claim is that, in fact, the Evans function for the Schrödinger equation is the same as the Jost function.

PROPOSITION 5.4. *If $\mathbf{v} = (v_j)_{j \in \mathbb{Z}} \in \ell^1(\mathbb{Z}; \mathbb{R})$ and $|z| > 2$ then $\mathcal{E}(z) = \mathcal{J}(z)$.*

Proof. Using the choice of transformations converting (5.14) to (5.18), we infer:

$$\begin{aligned} \mathcal{E}(z) &= \det[Y_0^+ + Y_0^-] = \det[W\tilde{Y}_0^+W^{-1} + W\tilde{Y}_0^-W^{-1}] = \det W \det[\tilde{Y}_0^+ + \tilde{Y}_0^-] \det W^{-1} \\ &= \det[u_0^+ \quad u_0^-] = \det[W^{-1}x_0^+ \quad W^{-1}x_0^-] = \Lambda^{-1} \det[x_0^+ \quad x_0^-] \\ &= \Lambda^{-1}\mathcal{W}(\mathbf{y}^+, \mathbf{y}^-) = \mathcal{J}(z), \end{aligned}$$

which concludes the proof. ■

References

- [AGJ] J. Alexander, R. Gardner and C. Jones, *A topological invariant arising in the stability analysis of travelling waves*, J. Reine Angew. Math. 410 (1990), 167–212.
- [BCK] N. Balmforth, R. Craster and P. Kevrekidis, *Being stable and discrete*, Phys. D 135 (2000), 212–232.
- [C] K. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Math. Phys. 15 (1974), 2166–2174.
- [C1] K. Case, *On discrete inverse scattering problems. II*, J. Math. Phys. 14 (1973), 916–920.
- [CS] K. Chadan and P. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed., Springer-Verlag, New York, 1989.
- [CL] C. Chicone and Y. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, Math. Surv. Monogr. 70, Amer. Math. Soc., Providence, 1999.
- [Co] W. A. Coppel, *Dichotomies in Stability Theory*, Lect. Notes in Math. 629, Springer-Verlag, New York, 1978.
- [DK] J. Daleckij and M. Krein, *Stability of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, RI, 1974.
- [E] M. Eastham, *The Asymptotic Solution of Linear Differential Systems. Applications of the Levinson Theorem*, Oxford Univ. Press, New York, 1989.
- [FT] L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag, Berlin, 1987.
- [GH] F. Gesztesy and H. Holden, *Trace formulas and conservation laws for nonlinear evolution equations*, Rev. Math. Phys. 6 (1994) 51–95.
- [GML] F. Gesztesy, Y. Latushkin and K. A. Makarov, *Fredholm determinants and the Evans function*, in preparation.
- [GML1] F. Gesztesy, Y. Latushkin and K. A. Makarov, *Evans functions and modified Fredholm determinants*, Oberwolfach Reports, 2005 (to appear).

- [GM] F. Gesztesy and K. A. Makarov, *(Modified) Fredholm determinants for operators with matrix-valued semi-separable integral kernels revisited*, Integral Equations Operator Theory 47 (2003), 457–497; erratum: 48 (2004) 425–426; corrected (electronic only) version: 48 (2004) 561–602, DOI 10.1007/s00020-003-1279-z.
- [GGK] I. Gohberg, S. Goldberg and N. Krupnik, *Traces and Determinants of Linear Operators*, Operator Theory Adv. Appl. 116, Birkhäuser, Basel, 2000.
- [GKvS] I. Gohberg, M. Kaashoek and F. van Schagen, *Noncompact integral operators with semiseparable kernels and their discrete analogues: inversion and Fredholm properties*, Integral Equations Operator Theory 7 (1984), 642–703.
- [HKS] D. Hinton, M. Klaus and J. Shaw, *Half-bound states and Levinson’s theorem for discrete systems*, SIAM J. Math. Anal. 22 (1991), 754–768.
- [JK] C. Jones and T. Kapitula, *Stability of fronts, pulses, and wave-trains*, preprint, 2003.
- [KK] T. Kapitula and P. Kevrekidis, *Stability of waves in discrete systems*, Nonlinearity 14 (2001), 533–566.
- [KS] T. Kapitula and B. Sandstede, *Eigenvalues and resonances using the Evans function*, Discrete Contin. Dyn. Syst. 10 (2004), 857–869.
- [KS1] T. Kapitula and B. Sandstede, *Edge bifurcations for near integrable systems via Evans function techniques*, SIAM J. Math. Anal. 33 (2002) 1117–1143.
- [LS] Y. Latushkin and A. M. Stepin, *Weighted shift operators, spectral theory of linear extensions and the multiplicative ergodic theorem*, Matem. Sbornik 70 (1991), 143–163.
- [SS] R. Sacker and G. Sell, *A spectral theory for linear differential systems*, J. Differential Equations 27 (1978), 320–358.
- [S] B. Sandstede, *Stability of travelling waves*, in: Handbook of Dynamical Systems, Vol. 2, North-Holland, Amsterdam, 2002, 983–1055.
- [Si] B. Simon, *Trace Ideals and their Applications*, Cambridge Univ. Press, Cambridge, 1979.
- [T] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. Monogr. 72, Amer. Math. Soc., Providence, RI, 2000.
- [To] M. Toda, *Theory of Nonlinear Lattices*, 2nd ed., Springer-Verlag, Berlin, 1989.