Smoothed IV quantile regression and quantile Euler equations

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Outline

1. Consumption Euler equations
2. Smoothed IV quantile regression (SIVQR)
3. Results
4. Conclusion
Standard consumption Euler equation

- Expected utility maximization, \( U(C) = C^{1-\gamma}/(1 - \gamma) \):

\[
0 = \mathbb{E}[\beta(1 + R_{t+1})(C_{t+1}/C_t)^{-\gamma} - 1 | \Omega_t],
\]

\( \Omega_t \): information set at time \( t \)
\( R_{t+1} \): real rate of return of asset
\( C_t \): real consumption at time \( t \)
\( \beta \): discount factor (e.g., \( \beta = 0.99 \))
\( 1/\gamma \): elasticity of intertemporal substitution (EIS)

- Estimation: use variables in \( \Omega_t \) as instruments (inflation, etc.); run GMM, or IV/2SLS after log-linearization
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- Estimation: use variables in $\Omega_t$ as instruments (inflation, etc.); run GMM, or IV/2SLS after log-linearization

- Drawback: no separation of EIS ($1/\gamma$) and risk aversion ($\gamma$)

- Drawback: approximation error from log-linearization
Quantile Euler equation?

- Standard: \( 0 = \mathbb{E}[\beta(1 + R_{t+1})(C_{t+1}/C_t)^{-\gamma} - 1 | \Omega_t] \)
- Replace \( \mathbb{E}[\cdot | \Omega_t] \) with conditional \( \tau \)-quantile \( Q_\tau[\cdot | \Omega_t] \):
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  0 = Q_\tau[\beta(1 + R_{t+1})(C_{t+1}/C_t)^{-\gamma} - 1 | \Omega_t]
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- Advantage: \( 1/\gamma \) is EIS, but both \( \tau \) and \( \gamma \) capture risk attitude
- Advantage: \( \ln(Q_\tau(W)) = Q_\tau(\ln(W)) \), no error
- Advantage: robust to fat tails in consumption
- Application: economically reasonable estimates even when 2SLS unreasonable
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- Application: economically reasonable estimates even when 2SLS unreasonable
- Grounded in decision theory? (next slide)
- Practical to estimate? (SIVQR)
Quantile Euler equation: decision theory

- Quantile utility maximization, static setting: Manski (1988), Chambers (2009), and Rostek (2010) (axiomatization)
- Dynamic setting: de Castro and Galvao (2017) show dynamic consistency and derive Euler equation
Quantile Euler equation: estimation

- Can write as $Q_{\tau}[\epsilon_{t+1} \mid \Omega_t] = 1$, $\epsilon_{t+1} \equiv \beta(1 + R_{t+1})(C_{t+1}/C_t)^{-\gamma}$
- Since $\ln(\cdot)$ is strictly increasing, $Q_{\tau}[\ln(W)] = \ln(Q_{\tau}[W])$
- In contrast, $\mathbb{E}[\ln(W)] \leq \ln(\mathbb{E}[W])$ (Jensen’s); approx error
IV quantile regression (IVQR)

- So we can just run IVQR; but...
IV quantile regression (IVQR)

Chernozhukov and Hong (2003), Figure 1(a)

Criterion for IV-QR
IV quantile regression (IVQR)

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- Chernozhukov and Hansen (2006): iid, linear model, only 1 or 2 endogenous regressors (so can’t have many interactions, polynomial terms, etc.)
- Other methods (also iid): compliers/LQTE (Abadie, Angrist, and Imbens, 2002), MCMC (Chernozhukov and Hong, 2003; Lancaster and Jun, 2010), triangular system (Lee, 2007, and others), very slow MIQP (Chen and Lee, 2017)
IV quantile regression (IVQR)

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- Very difficult to compute IVQR estimator numerically
- Kaplan and Sun (2017), smoothing: iid, linear, but fast and allows many endogenous regressors; also high-order MSE improvement, connection to 2SLS
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de Castro, Galvao, and Kaplan (2017): dependent data, nonlinear model; fast and robust computation, consistency and asymptotic normality

New results underway from Xin Liu: smoothed two-step GMM, computation (code) and asymptotic theory
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Smoothed IVQR (SIVQR): benefits

- Approach: smooth the moment conditions (estimating equations)
- Initially: use just-identified system for numerical robustness; if over-identified, just take linear combination of instruments
Smoothed IVQR (SIVQR): benefits

- **Approach:** smooth the moment conditions (estimating equations)
- **Initially:** use just-identified system for numerical robustness; if over-identified, just take linear combination of instruments
- **Benefit #1:** computation is feasible, fast, scalable (many endogenous regressors), and numerically robust
- **Benefit #2:** often improves MSE (Kaplan and Sun, 2017)
- **Benefit #3:** important first step toward true IV quantile GMM (in progress by Xin Liu)
Smoothed QR (not IV): literature

- Horowitz (1998): smooths criterion fn instead of moments; Studentized bootstrap refinement
- Whang (2006): same moment smoothing used here, but for empirical likelihood QR; also in Otsu (2008)
- Fernandes, Guerre, and Horta (2017): kernel-smoothed QR criterion; FOC same as smoothed moments above
- MaCurdy and Hong (1999): original IVQR smoothing? (unpub’d notes)
What does IVQR estimate?

- Chernozhukov and Hansen (2005): identification of (conditional) \( \tau \)-quantile treatment effects, or “structural quantile effects”
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- Simplistic example: linear structural random coefficient model, $Y = \mathbf{X}'\beta(U)$, assume $\mathbf{X}'\beta(U)$ monotonic in unobserved $U \sim \text{Unif}(0,1)$

- If $Y$ is wage, $U$ is “ability”: $\mathbf{X}'\beta(0.5)$ traces out potential wage outcomes (given different $\mathbf{X}$) for individual with median ability ($P(U \leq 0.5) = 0.5$)
What does IVQR estimate?

- Chernozhukov and Hansen (2005): identification of (conditional) \( \tau \)-quantile treatment effects, or “structural quantile effects”

- Simplistic example: linear structural random coefficient model, \( Y = X' \beta(U) \), assume \( X' \beta(U) \) monotonic in unobserved \( U \sim \text{Unif}(0, 1) \)

- If \( Y \) is wage, \( U \) is “ability”: \( X' \beta(0.5) \) traces out potential wage outcomes (given different \( X \)) for individual with median ability \( \left( P(U \leq 0.5) = 0.5 \right) \)

- If instrument vector \( Z \perp \perp U \), then \( P(Y \leq X' \beta(\tau) \mid Z) = P(U \leq \tau \mid Z) = P(U \leq \tau) = \tau \): a conditional quantile restriction on the observables \( Y, X, \) and \( Z \), and the parameter vector \( \beta(\tau) \)
Alternative quantile models with endogeneity

- Triangular system: compared in Chernozhukov, Hansen, and Wüthrich (2017, §2.5); like Chesher (2003), Lee (2007), et al.
- LQTE (like LATE): compared in Melly and Wüthrich (2017, §§5–6); like Abadie et al. (2002), Frölich and Melly (2013), et al.
IVQR moment conditions

\[ \tau = P(Y \leq h(X, \tau) \mid Z) \]

(linear IV)

(linear IVQR)
IVQR moment conditions

\( \tau = P(Y \leq h(X, \tau) \mid Z), \quad h(X, \tau) = X'\beta_{0\tau}, \quad P(\cdot) = \mathbb{E}(1\{\cdot\}) \rightleftharpoons 0 = \mathbb{E}[Z(1\{Y - X'\beta_{0\tau} \leq 0\} - \tau)] \)
IVQR moment conditions

\[ Y = h(X) + U, \quad \mathbb{E}(U \mid Z) = 0, \quad h(X) = X'\beta_0 \implies 0 = \mathbb{E}[Z(Y - X'\beta_0)] \] (linear IV)

\[ \tau = P(Y \leq h(X, \tau) \mid Z), \quad h(X, \tau) = X'\beta_{0\tau}, \quad P(\cdot) = \mathbb{E}(1\{\cdot\}) \implies 0 = \mathbb{E}[Z(1\{Y - X'\beta_{0\tau} \leq 0\} - \tau)] \] (linear IVQR)
IVQR moment conditions

\[ Y = h(X) + U, \quad \mathbb{E}(U \mid Z) = 0, \quad h(X) = X'\beta_0 \implies 0 = \mathbb{E}[Z(Y - X'\beta_0)], \quad \beta_0 = \left[ \mathbb{E}(ZX') \right]^{-1} \mathbb{E}(ZY) \quad (\text{linear IV}) \]

\[ \tau = \mathbb{P}(Y \leq h(X, \tau) \mid Z), \quad h(X, \tau) = X'\beta_{0\tau}, \quad \mathbb{P}(\cdot) = \mathbb{E}(1\{\cdot\}) \implies 0 = \mathbb{E}[Z(1\{Y - X'\beta_{0\tau} \leq 0\} - \tau)] \quad (\text{linear IVQR}) \]
Just run GMM?

Chernozhukov and Hong (2003), Figure 1(a)

Criterion for IV-QR
Smoothing the indicator function
Smoothing the indicator function
Connections: IV, Winsorized mean

\[ 0 = n^{-1} \sum_{i=1}^{n} Z_i \left[ \tilde{I} \left( \frac{X_i' \hat{\beta}_\tau - Y_i}{h_n} \right) - \tau \right] \]

- As \( h_n \to \infty \), \( \tilde{I}(\cdot/h_n) \) approx linear:
Connections: IV, Winsorized mean

\[ 0 = n^{-1} \sum_{i=1}^{n} Z_i \left[ \tilde{I} \left( \frac{X_i \hat{\beta}_\tau - Y_i}{h_n} \right) - \tau \right] \]

- As \( h_n \to \infty \), \( \tilde{I}(\cdot/h_n) \) approx linear:
  - IVQR
  - Smoothed IVQR
  - IV (mean)

\( h_n = 0 \) \quad \text{or} \quad h_n > 0 \quad \text{or} \quad h_n \to \infty \)
Connections: IV, Winsorized mean

\[ 0 = n^{-1} \sum_{i=1}^{n} Z_i \left[ \tilde{I} \left( \frac{X_i \hat{\beta} - Y_i}{h_n} \right) - \tau \right] \]

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\( h_n = 0 \quad h_n > 0 \quad h_n \to \infty \)

- Can try to pick \( h_n \) to improve efficiency (like median vs. mean); maybe even better to explicitly average IVQR and IV (and QR) like Hansen (2017)?
Connections: IV, Winsorized mean

\[ 0 = n^{-1} \sum_{i=1}^{n} Z_i \left[ \tilde{I} \left( \frac{X_i \hat{\beta}_\tau - Y_i}{h_n} \right) - \tau \right] \]

- As \( h_n \to \infty \), \( \tilde{I}(\cdot/h_n) \) approx linear:

  - IVQR
  - smoothed IVQR
  - IV (mean)

- Can try to pick \( h_n \) to improve efficiency (like median vs. mean); maybe even better to explicitly average IVQR and IV (and QR) like Hansen (2017)?

- Special case:
  \( X_i = Z_i = 1 \),
  \[ \tilde{I}'(u) = \mathbb{1}\{-1 \leq u \leq 1\}/2, \]
  \( \tau = 0.5 \) \( \Rightarrow \) “Winsorized” mean (Huber, 1964, Ex. iii, p. 79)
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JTPA: context

- Abadie, Angrist, and Imbens (2002), 5102 adult men
- Randomized offer of services to individuals ($Z_i$), 62% uptake: $P(D_i = 1 \mid Z_i = 1) = 0.62$.
- Other regressors: age, race, etc.
- Endogeneity from self-selection into treatment; OLS estimate twice as big as IV est
- $Y_i$: 30-month earnings (US dollars) in “after” period
## JTPA: results

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<th>Regressor</th>
<th>Method</th>
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<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
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<td>381</td>
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<td>tiny (h)</td>
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<td>Training</td>
<td>huge (h)</td>
<td>1579</td>
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<tr>
<td>Training</td>
<td>2SLS</td>
<td></td>
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<tr>
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<td>AAI</td>
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<td>7683</td>
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<tr>
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<td>10,174</td>
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<td></td>
<td></td>
<td></td>
<td>6647</td>
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</tr>
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</table>
Quantile Euler equation estimates

- Same data, model, instruments as Yogo (2004) Table 2 (but with quantiles); “weak instruments are not a problem” (Yogo, 2004, p. 805)
- Very little smoothing (for estimation)
- $\beta$: discount factor (1 = no discount)
- $1/\gamma$: EIS, elasticity of intertemporal substitution
Quantile Euler equation estimates: UK

![Quantile Euler equation estimates graph](chart.png)
Quantile Euler equation estimates: USA

\[ \hat{\beta}(\tau) \quad \Delta \quad 1/\hat{\gamma}(\tau) \]
Quantile Euler equation estimates: Netherlands

![Chart showing Quantile Euler equation estimates for Netherlands.](chart)

- **Estimate**: \( \hat{\beta}(\tau) \) and \( 1/\hat{\gamma}(\tau) \)

Kaplan and: Sun, de Castro, Galvao, Liu
Quantile Euler equation estimates: Sweden

\[ \widehat{\beta}(\tau) \quad \Delta \quad 1/\widehat{\gamma}(\tau) \]
Quantile Euler equation estimates

<table>
<thead>
<tr>
<th>τ</th>
<th>US</th>
<th></th>
<th>UK</th>
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<th>NTH</th>
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<th>SWE</th>
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<tr>
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<td>̂γ_τ</td>
<td>̂β_τ</td>
<td>̂γ_τ</td>
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<td>̂γ_τ</td>
<td>̂β_τ</td>
<td>̂γ_τ</td>
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<tr>
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<td>0.80</td>
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<td>5.2</td>
<td>0.98</td>
<td>3.9</td>
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<td>6.0</td>
<td>0.96</td>
<td>−6.8</td>
<td>0.27</td>
<td>−544.4</td>
</tr>
</tbody>
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- First theoretical results for feasible IVQR with dependent data (and nonlinear model)
- Quantile Euler equations: decouple EIS and risk attitude, robust to fat tails, no error in log-linearization, more reasonable estimates than 2SLS (in our example)
  determination of \( \tau \)?
Conclusion

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- Thank you!
- (And further questions or comments are welcome)


References VI

