INITIAL $L^2 \times \cdots \times L^2$ BOUNDS FOR MULTILINEAR OPERATORS

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Abstract. The $L^p$ boundedness theory of convolution operators is based on an initial $L^2 \to L^2$ estimate derived from the Fourier transform. The corresponding theory of multilinear operators lacks such a simple initial estimate in view of the unavailability of Plancherel's identity in this setting, and up to now it has not been clear what a natural initial estimate might be. In this work we obtain initial $L^2 \times \cdots \times L^2 \to L^{2/m}$ estimates for three types of important multilinear operators: rough singular integrals, multipliers of Hörmander type, and multipliers whose derivatives satisfy qualitative estimates.

1. INTRODUCTION AND PRELIMINARIES

The systematic study of multilinear operators in harmonic analysis was initiated by Coifman and Meyer in the seventies. Many important multilinear operators can be written as

$$T(f_1, \ldots, f_m)(x) = W \ast (f_1 \otimes \cdots \otimes f_m)(x, \ldots, x), \quad x \in \mathbb{R}^n,$$

where $f_j$ are Schwartz functions on $\mathbb{R}^n$, $W$ is a tempered distribution on $(\mathbb{R}^n)^m$, and $(f_1 \otimes \cdots \otimes f_m)(x_1, \ldots, x_m) = f_1(x_1) \cdots f_m(x_m)$. Alternatively $T(f_1, \ldots, f_m)(x)$ can be expressed as

$$1. \quad \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \hat{f}_1(\xi_1) \cdots \hat{f}_m(\xi_m) \hat{W}(\xi_1, \ldots, \xi_m) e^{2\pi i \langle x, \xi_1 + \cdots + \xi_m \rangle} d\xi_1 \cdots d\xi_n,$$

where $\hat{f}_j(\xi) = \int_{\mathbb{R}^n} f_j(x) e^{-2\pi i \langle x, \xi \rangle} dx$ denotes the Fourier transform of $f_j$ and $\hat{W}$ is the distributional Fourier transform of $W$, which must be an $L^\infty$ function if $T$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for some

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choice of indices that satisfy \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} \). If such an estimate holds for \( T \), then \( \hat{W} \) is called a multilinear Fourier multiplier. The first important result concerning multilinear Fourier multipliers is a non-trivial adaptation of Mihlin’s multiplier condition, obtained by Coifman and Meyer [4]. The proof they gave is based on decomposing the multiplier as a sum of products of linear operators via Littlewood-Paley and Fourier series expansions. This powerful idea has essentially been the only technique available in this area until the appearance of the wave-packet decompositions in the work of Lacey and Thiele [26, 27] on the bilinear Hilbert transform.

If the distribution \( W \) has the form

\[
W = \text{p.v.} \frac{1}{|y_1, \ldots, y_m|^{mn}} \Omega \left( \left( \frac{y_1, \ldots, y_m}{|y_1, \ldots, y_m|} \right) \right)
\]

for some integrable function \( \Omega \) on the sphere \( S^{mn-1} \) with integral zero, then \( T \) is called an \( m \)-linear homogeneous singular. The associated operator is bounded if \( \Omega \) is smooth but it could be unbounded if \( \Omega \) is merely integrable; see [19]. In this paper we focus on the intermediate situation where \( \Omega \) lies in \( L^q \) for some \( q \in (1, \infty] \); these \( \Omega \)'s give rise to rough \( m \)-linear homogeneous singular integrals. The study of bilinear homogeneous singular integrals was initiated by Coifman and Meyer in [5] who addressed the case where \( \Omega \) is a function of bounded variation. The boundedness of \( T \) in the more difficult case when \( \Omega \) is merely in \( L^\infty \) was not solved until four decades later in [16] in terms of wavelet decompositions. Prior to that, the first author and Torres [22] had proved boundedness for \( T \) for any \( m \) when \( \Omega \) satisfies a Lipschitz condition on the sphere. In the case \( m = 1 \) the known results are much better. For instance, Calderón and Zygmund [3] showed that \( T \) is bounded in \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \) if \( \Omega \in L \log L(S^{n-1}) \). This result was improved by Coifman and Weiss [6] under the less restrictive condition that \( \Omega \) belongs to the Hardy space \( H^1(S^{n-1}) \).

One fundamental difference between linear convolution operators and multilinear convolution operators of type (1) is an initial estimate. In the linear case the initial estimate is usually \( L^2 \to L^2 \) and this is obtained by Plancherel’s identity. There is not a straightforward initial estimate for multilinear operators and in most times, it is difficult to find one. Inspired by [16], the first two authors and Slavíková [18] obtained a bilinear substitute of the Plancherel criterion for \( L^2 \times L^2 \to L^1 \) boundedness for multipliers in \( L^q(\mathbb{R}^n) \) \((0 < q < 4)\) with sufficiently many bounded derivatives. This result has also been proved by Kato, Miyachi, and Tomita [24] and has found many applications; see for instance [18, 31].
Overcoming the combinatorial complexity that arises from multilinearity, in this paper we develop a method that yields the crucial estimates for a variety of $m$-linear operators, including rough homogeneous singular integrals and multipliers. Our results contribute to the recent surge of activity in the theory of rough multilinear singular integrals, see for instance [7, 10, 11, 18, 23, 24].

We first present a sharp $L^2 \times \cdots \times L^2 \to L^{2/m}(\mathbb{R}^n)$ boundedness criterion for a multiplier with bounded derivatives up to a certain order. This provides a multilinear extension of the main result in [18].

**Theorem 1.1.** Let $m$ be a positive integer with $m \geq 2$ and $1 < q < \frac{2m}{m-1}$. Set $M_q$ to be a positive integer satisfying
\[ M_q > \frac{m(m-1)n}{2m - (m-1)q}. \]
Suppose that $\sigma \in L^q((\mathbb{R}^n)^m) \cap C^{M_q}((\mathbb{R}^n)^m)$ with
\[ \| \partial^\alpha \sigma \|_{L^\infty((\mathbb{R}^n)^m)} \leq D_0, \quad \text{for } |\alpha| \leq M_q. \]
Then the estimate
\[ \| T\sigma(f_1, \ldots, f_m) \|_{L^{2/m}(\mathbb{R}^n)} \lesssim D_0^{1-\frac{(m-1)q}{2m}} \left( \| \sigma \|_{L^q((\mathbb{R}^n)^m)} \right)^{\frac{(m-1)q}{2m}} \prod_{j=1}^m \| f_j \|_{L^2(\mathbb{R}^n)} \]
is valid for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.

Next we discuss multilinear rough singular integral operators. For a fixed function $\Omega$ on the unit sphere $S^{mn-1}$ and for $\vec{y} := \vec{y}/|\vec{y}| \in S^{mn-1}$ we let
\[ K(\vec{y}) := \frac{\Omega(\vec{y})}{|\vec{y}|^{mn}}. \]
We then define the corresponding multilinear operator
\[ \mathcal{L}_\Omega(f_1, \ldots, f_m)(x) := p.v. \int_{(\mathbb{R}^n)^m} K(\vec{y}) \prod_{j=1}^m f_j(x - y_j) \, d\vec{y}, \quad x \in \mathbb{R}^n \]
for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.

**Theorem 1.2.** Suppose that $\frac{2m}{m+1} < q \leq \infty$ and let $\Omega \in L^q(S^{mn-1})$ satisfying $\int_{S^{mn-1}} \Omega \, d\sigma_{mn-1} = 0$. Then there exists a constant $C = C_{n,m,q} > 0$ such that
\[ \| \mathcal{L}_\Omega(f_1, \ldots, f_m) \|_{L^{2/m}(\mathbb{R}^n)} \leq C \| \Omega \|_{L^q(S^{mn-1})} \prod_{j=1}^m \| f_j \|_{L^2(\mathbb{R}^n)} \]
for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.

The last result of this paper is about the boundedness of multilinear multiplier operators of Hörmander type. The multilinear multiplier operator associated with a bounded function $\sigma$ on $(\mathbb{R}^n)^m$ is defined as in (7):

$$T_\sigma(f_1, \ldots, f_m)(x) := \int_{(\mathbb{R}^n)^m} \sigma(\vec{\xi})(\prod_{j=1}^m \hat{f}_j(\xi_j)) e^{2\pi i (x \cdot \sum_{j=1}^m \xi_j)} d\vec{\xi}$$

for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$. We choose a Schwartz function $\Phi$ on $(\mathbb{R}^n)^m$ having the properties that $\hat{\Phi}$ is positive and supported in the annulus $\{\vec{\xi} \in (\mathbb{R}^n)^m : 1/2 \leq |\vec{\xi}| \leq 2\}$, and $\sum_{\gamma \in \mathbb{Z}} \hat{\Phi}(\vec{\xi}/2^\gamma) = 1$ for $\vec{\xi} \neq \vec{0}$. In the linear case $m = 1$, under the assumption

$$\sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot \hat{\Phi}(1))\|_{L^q(\mathbb{R}^n)} < \infty,$$

the condition

$$s > \max\left(\frac{n}{2} - \frac{n}{p}, \frac{n}{q}\right)$$

implies the boundedness of $T_\sigma$ from $L^p(\mathbb{R}^n)$ to itself. Recently, the bilinear analogue of this result was obtained in the series of papers [17, 20, 30] by Grafakos, He, Honzík, Miyachi, Nguyen, and Tomita; all of these results were inspired by the fundamental work of Tomita [32] in this direction.

**Theorem 1.3.** Let $1 < q < \infty$ and

$$s > \max((m-1)n/2, mn/q).$$

Then there exists an absolute constant $C = C_{n,m,q,s} > 0$ such that

$$\|T_\sigma(f_1, \ldots, f_m)\|_{L^{2/m}(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot \hat{\Phi}(1))\|_{L^q((\mathbb{R}^n)^m)} \prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^n)}$$

for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.

We remark that for $1 < q \leq 2$ this result has been obtained by [32] and [21], so Theorem 1.3 is new only in the case $q > 2$; this corresponds to the classical result of Calderón and Torchinsky [2] in the linear setting. The sharpness of condition (3) was addressed in [17, Theorem 2].

We design two novel ideas to deal with the above results: (I) An innovative decomposition of an $m$-linear multiplier into sums of products so that $l$-linear Plancherel type estimates $(1 \leq l \leq m)$ can be used; see Proposition 2.4. (II) An effective way to split lattice points in $(\mathbb{Z}^n)^m$ into groups of columns for the purposes of obtaining $L^2 \times \cdots \times L^2 \to L^{2/m}$ estimates; for details see Section 3.
It is inevitable to introduce complicated notation in order to comprehensively present the proofs in the general framework of \(m\)-linear operators; for this reason we urge the readers to restrict attention to the case \(m = 3\), which already presents several new ingredients compared with the case \(m = 2\) and contains the main ideas.

**Notation.** \(C\) will denote inessential constants that may vary from occurrence to occurrence. \(A \lesssim B\) means \(A \leq CB\) with \(C\) independent of \(A\) and \(B\), and write \(A \approx B\) if both \(A \lesssim B\) and \(B \lesssim A\) hold. We denote the set of natural numbers by \(\mathbb{N}\) and of integers by \(\mathbb{Z}\); we also set \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\). Throughout this paper, the index \(m \in \mathbb{N}\) will be the degree of multilinearity of operators.

### 2. Plancherel Type Estimates

Let \(\omega\) be a compactly supported smooth function defined on \(\mathbb{R}^n\). For \(\lambda \in \mathbb{N}_0\) let \(\{\omega^k_\lambda\}_{k \in \mathbb{Z}^n}\) be a sequence of compactly supported and smooth functions, defined on \(\mathbb{R}^n\) by the formula \(\omega^k_\lambda(\xi) = 2^{\lambda n/2} \omega(2^\lambda \xi - k)\). These have the following properties:

(i) \(\{\omega^k_\lambda\}_{k \in \mathbb{Z}^n}\) have almost disjoint supports.

(ii) \(\sum_{k \in \mathbb{Z}^n} |\omega^k_\lambda(\xi)| \leq 2^{\lambda n/2}\) for all \(\xi \in \mathbb{R}^n\).

As a consequence of (i) and (ii) we obtain

\[
(\sum_{k \in \mathbb{Z}^n} |\omega^k_\lambda(\xi)|^q)^{1/q} \approx \sum_{k \in \mathbb{Z}^n} |\omega^k_\lambda(\xi)| \leq 2^{\lambda n/2}
\]

for any \(0 < q < \infty\), due to the property of the supports. We define

\[
\omega^\lambda_\vec{k}(\vec{\xi}) := \omega^\lambda_{k_1}(\xi_1) \cdots \omega^\lambda_{k_m}(\xi_m)
\]

where \(\vec{k} := (k_1, \ldots, k_m) \in (\mathbb{Z}^n)^m\) and \(\vec{\xi} = (\xi_1, \ldots, \xi_m) \in (\mathbb{R}^n)^m\). Let \(\mathcal{U}\) be a subset of \((\mathbb{Z}^n)^m\) and \(\{b^\lambda_\vec{k}\}_{\vec{k} \in \mathcal{U}}\) be a sequence of complex numbers. We define

\[
\sigma^\lambda(\vec{\xi}) := \sum_{\vec{k} \in \mathcal{U}} b^\lambda_\vec{k} \omega^\lambda_\vec{k}(\vec{\xi})
\]

and the corresponding \(m\)-linear multiplier operator by

\[
T_{\sigma^\lambda}(f_1, \ldots, f_m)(x) := \int_{(\mathbb{R}^n)^m} \sigma^\lambda(\vec{\xi}) \left( \prod_{j=1}^m f_j(\xi_j) \right) e^{2\pi i \langle x, \sum_{j=1}^m \xi_j \rangle} d\vec{\xi}
\]

for \(x \in \mathbb{R}^n\) and Schwartz functions \(f_1, \ldots, f_m\) on \(\mathbb{R}^n\). This operator coincides with that in (1) when \(\sigma^\lambda = \hat{W}\).
The multiplier $\sigma^\lambda$ defined in (6) appears naturally in the decomposition of many operators and plays a key role in the understanding of their boundedness. Actually, in the bilinear case, one has the estimate ([1, 18])

$$\|T_{\sigma^\lambda} (f, g)\|_{L^1(\mathbb{R}^n)} \leq C \|\{b_k^\lambda\}\|_{\ell^\infty} \|U\|_{L^2(\mathbb{R}^n)}^{1/4} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.$$  

The presence of $\|\{b_k^\lambda\}\|_{\ell^\infty}$ and $|U|^{1/4}$ indicate the contribution of both the height and the support of $\sigma^\lambda$. This phenomenon is dissimilar to the $L^2$ boundedness of linear multipliers, where the support of the multiplier plays no role. Motivated by many applications in which $\sigma^\lambda$ is an important building block, in this work we obtain the $m$-linear version of (8).

Proposition 2.1. Let $N$ be a positive integer and $\mathcal{U}$ be a subset of $(\mathbb{Z}^n)^m$ with $|\mathcal{U}| \leq N$. For $\lambda \geq 0$, let $\{b_k^\lambda\}_{k \in (\mathbb{Z}^n)^m}$ be a sequence of complex numbers satisfying $\|\{b_k^\lambda\}\|_{\ell^\infty} \leq A_\lambda$. Let $\sigma^\lambda$ be defined as in (6). Then there exists a constant $C = C_{n, m} > 0$ such that

$$\|T_{\sigma^\lambda} (f_1, \ldots, f_m)\|_{L^{2/m}(\mathbb{R}^n)} \leq CA_\lambda N^{m-1} 2^{\lambda m} \prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^n)}$$

for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.

When $m = 1$, Proposition 2.1 follows from Plancherel’s identity and yields a bound on the $L^2$ norm of the corresponding linear operator $T_{\sigma}$ that depends only on the height of the multiplier $\sigma$. For $m = 2$, it coincides with (8). Below we focus on the consequences of Proposition 2.1 while its proof is postponed until the next section.

Remark 1. After completing this paper, we were informed that Kato, Miyachi, and Tomita [25] recently obtained a result that implies Proposition 2.1. Their proof is independent of ours and builds on their previous work in [24].

The restriction $|\mathcal{U}| \leq N$ in Proposition 2.1 can be interpreted in terms of the compact support condition of $\sigma^\lambda$. Indeed, the support of $\sigma^\lambda$ has measure bounded by a constant times $N$. As we have seen in the proof of Proposition 2.1, the $L^2 \times \cdots \times L^2 \to L^{2/m}$ boundedness of $m$-linear multiplier operator $T_{\sigma}$, $m \geq 2$, may be affected by the support of $\sigma$ while the $L^2$ boundedness depends only on $\|\sigma\|_{L^\infty}$ in the linear setting.

On the other hand, the following “support-independent” result could be obtained from Proposition 2.1 under an extra $\ell^q$ condition which is satisfied in many applications.

Proposition 2.2. Let $m \in \mathbb{N}$ and $0 < q < \frac{2m}{m-1}$. Fixing $\lambda \in \mathbb{N}_0$, let $\{\omega_k^\lambda\}_{k \in (\mathbb{Z}^n)^m}$ be wavelets of level $\lambda$. Suppose $\{b_k^\lambda\}_{k \in (\mathbb{Z}^n)^m}$ is a sequence of complex numbers satisfying $\|\{b_k^\lambda\}_{k \in (\mathbb{Z}^n)^m}\|_{\ell^\infty} \leq A_\lambda$ and $\|\{b_k^\lambda\}_{k \in (\mathbb{Z}^n)^m}\|_{\ell^q} \leq B_{\lambda, q'}$. Then
the $m$-linear multiplier $\sigma^\lambda$, defined in (6) with $\mathcal{U} = (\mathbb{Z}^n)^m$, satisfies
\[
\| T_{\sigma^\lambda} (f_1, \ldots, f_m) \|_{L^2/m} \lesssim A_\lambda^{1 - \frac{(m-1)q}{2m}} B_{\lambda, q}^{\frac{(m-1)q}{2m}} 2^{\lambda mn/2} \prod_{j=1}^m \| f_j \|_{L^2}
\]
for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.

Proof. When $m = 1$, it is clear from Plancherel’s identity. Therefore we assume $m \geq 2$.

For $r \in \mathbb{N}$ let
\[
\mathcal{U}_r^\lambda := \{ \vec{k} \in (\mathbb{Z}^n)^m : A2^{-r} < |b^\lambda_k| \leq A2^{-r+1} \}.
\]
As $\| \{ b^\lambda_k \}_{k \in (\mathbb{Z}^n)^m} \|_{\ell^n} \leq A_\lambda m^{(\mathbb{Z}^n)^m}$ can be written as the disjoint union of $\mathcal{U}_r^\lambda$, $r \in \mathbb{N}$, and thus we may decompose $\sigma^\lambda$ as
\[
\sigma^\lambda = \sum_{r \in \mathbb{N}} \sigma_r^\lambda
\]
where $\sigma_r^\lambda := \sum_{k \in \mathcal{U}_r^\lambda} b^\lambda_k \omega^\lambda_k$. Observe that
\[
2^{-r} A_\lambda |\mathcal{U}_r^\lambda|^{1/q} \leq \left( \sum_{k \in \mathcal{U}_r^\lambda} |b_k^\lambda|^q \right)^{1/q} \leq B_{\lambda, q},
\]
which implies
\[
|\mathcal{U}_r^\lambda| \leq \left( \frac{B_{\lambda, q}}{2^{-r} A_\lambda} \right)^q.
\]
Applying Proposition 2.1 and (10) to each $\sigma_r^\lambda$, we obtain
\[
\| T_{\sigma_r^\lambda} (f_1, \ldots, f_m) \|_{L^2/m} \lesssim |\mathcal{U}_r^\lambda|^{(m-1)/2m} 2^{\lambda mn/2} (A_\lambda 2^{-r}) \prod_{j=1}^m \| f_j \|_{L^2}
\]
\[
\leq (A_\lambda 2^{-r})^{1 - \frac{(m-1)q}{2m}} B_{\lambda, q}^{\frac{(m-1)q}{2m}} 2^{\lambda mn/2} \prod_{j=1}^m \| f_j \|_{L^2}.
\]
Taking $\ell^2/m$-norm over $r \in \mathbb{N}$, we have
\[
\| T_{\sigma^\lambda} (f_1, \ldots, f_m) \|_{L^2/m} \lesssim \left( \sum_{r \in \mathbb{N}} \| T_{\sigma_r^\lambda} (f_1, \ldots, f_m) \|_{L^2/m} \right)^{m/2}
\]
\[
\lesssim A_\lambda^{1 - \frac{(m-1)q}{2m}} B_{\lambda, q}^{\frac{(m-1)q}{2m}} 2^{\lambda mn/2} \prod_{j=1}^m \| f_j \|_{L^2} \cdot \| f_m \|_{L^2}
\]
since $1 - \frac{(m-1)q}{2m} > 0$ and $2/m \leq 1$. \hfill \square
For the case $m \geq 2$ and $q \geq \frac{2m}{m-1}$, we have the following substitute under an extra condition that all $\vec{k}$ belong to a ball of radius $C2^{\lambda}$, centered at the origin, which means that $\sigma^\lambda$ is contained in a ball of radius comparable to 1.

**Proposition 2.3.** Let $m$ be a positive integer with $m \geq 2$ and $\frac{2m}{m-1} \leq q < \infty$. For each $\lambda \in \mathbb{N}_0$ let $\{\omega_k^\lambda\}$ be wavelets of level $\lambda$. Let $U^\lambda := \{\vec{k} \in (\mathbb{Z}^n)^m : |\vec{k}| \leq C2^{\lambda}\}$ for some $C > 0$. Suppose $\{b_k^\lambda\}_{k \in (\mathbb{Z}^n)^m}$ is a sequence of complex numbers with $\|\|\{b_k^\lambda\}_{k \in (\mathbb{Z}^n)^m}\|_{\ell^q} \leq B_{\lambda,q}$. Then the m-linear multiplier $\sigma^\lambda$, defined in (6) with $\mathcal{U} = U^\lambda$, satisfies

$$\left\| T_{\sigma^\lambda} (f_1, \ldots, f_m) \right\|_{L^{2/m}} \lesssim B_{\lambda,q} D_{\lambda,q,m} \left( \prod_{j=1}^m \left\| f_j \right\|_{L^2} \right)$$

for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$, where

$$D_{\lambda,q,m} := \begin{cases} \lambda^{m/2} 2^{\lambda mn/2}, & q = \frac{2m}{m-1} \\ 2^{\lambda n \left( \frac{2m-1}{2} - \frac{m}{q} \right)}, & q > \frac{2m}{m-1} \end{cases}$$

**Proof.** Pick $r_{\max} \in \mathbb{N}$ satisfying $\frac{\lambda mn}{q} \leq r_{\max} < \frac{\lambda mn}{q} + 1$. Define

$$U^\lambda_{\max} := \{\vec{k} \in U^\lambda : |b_k^\lambda| \leq 2^{-r_{\max}+1} B_{\lambda,q}\}$$

and for $1 \leq r < r_{\max}$

$$U^\lambda_r := \{\vec{k} \in U^\lambda : 2^{-r} B_{\lambda,q} < |b_k^\lambda| \leq 2^{-r+1} B_{\lambda,q}\}.$$

Since $|b_k^\lambda| \leq B_{\lambda,q}$ for all $\vec{k} \in (\mathbb{Z}^n)^m$, we can write

$$\sigma^\lambda = \sum_{r=1}^{r_{\max}} \sigma^\lambda_r$$

where $\sigma^\lambda_r := \sum_{\vec{k} \in U^\lambda_r} b_{\vec{k}}^\lambda \omega_{\vec{k}}^\lambda$. Using the same argument in (9), we see that

$$|U^\lambda_r| \leq 2^{|q|}, \quad 1 \leq r < r_{\max}$$

and

$$|U^\lambda_{\max}| \leq |U^\lambda| \lesssim 2^{|\lambda mn|} \leq 2^{r_{\max}|q|}.$$

Applying Proposition 2.1, (11), and (12) to each $\sigma^\lambda_r$, we obtain

$$\left\| T_{\sigma^\lambda} (f_1, \ldots, f_m) \right\|_{L^{2/m}} \lesssim |U^\lambda_r|^{(m-1)/2m} 2|\lambda mn/2| \left\| \{b_k^\lambda\}_{k \in U^\lambda_r} \right\|_{\ell^q} \prod_{j=1}^m \left\| f_j \right\|_{L^2} \lesssim 2^{rq(m-1)/2m} 2^{|\lambda mn/2|} B_{\lambda,q} \prod_{j=1}^m \left\| f_j \right\|_{L^2}$$
We note that
\[ \ell \] which completes the proof.

Taking the \( \ell^{2/m} \) norm over \( 1 \leq r \leq r_{\max} \), we have
\[
\| T_{\sigma} \ell (f_1, \ldots, f_m) \|_{L^{2/m}} \leq \left( \sum_{r=1}^{r_{\max}} \| T_{\sigma} \ell (f_1, \ldots, f_m) \|_{L^{2/m}}^{2/r} \right)^{m/2} \\
\lesssim B_{\lambda, q} 2^{\lambda mn/2} \left( \sum_{r=1}^{r_{\max}} 2^{r(\frac{q(m-1)}{2m} - 1)} \right)^{m/2} \prod_{j=1}^{m} \| f_j \|_{L^2}.
\]

We note that
\[
\left( \sum_{r=1}^{r_{\max}} 2^{r(\frac{q(m-1)}{2m} - 1)} \right)^{m/2} \approx \begin{cases} 2^{m/2}, & q = \frac{2m}{m-1} \\ 2^{m/2} \left( \frac{q}{2m} \right), & q > \frac{2m}{m-1} \end{cases}
\]
which completes the proof.

In the study of bilinear rough singular integrals and bilinear Hörmander multipliers, an argument splitting the problem to diagonal and off-diagonal cases is utilized. The off-diagonal case uses Plancherel’s theorem and a pointwise control. In the diagonal case, we employ the bilinear result of Plancherel type, which is actually the driving force of this work. We now present a multilinear version generalizing and combining these two parts, which shows that all \( l \)-linear Plancherel type result, \( 1 \leq l \leq m \), is necessary in the study of many \( m \)-linear multipliers.

For \( \mu \in \mathbb{N}_0 \) let \( \mathcal{V}_\mu \) be a subset of \( \{ \vec{k} \in (\mathbb{Z}^n)^m : 2^{\mu-c_0} \leq \| \vec{k} \| \leq 2^{\mu+c_0} \} \) for some \( c_0 \geq 1 \). Let \( M \) be a positive constant and for each \( 1 \leq l \leq m \) let
\[
\mathcal{V}_l^\mu := \{ \vec{k} \in \mathcal{V}_\mu : |k_1|, \ldots, |k_l| \geq M > |k_{l+1}|, \ldots, |k_m| \}.
\]
We also define \( L_k^\lambda f := (\omega_k^\lambda \hat{f})^\vee \) and \( L_k^{\lambda, \gamma, \mu} f := (\omega_k^\lambda \cdot |2^\gamma \hat{f})^\vee \) for \( k \in \mathbb{Z}^n \).

**Proposition 2.4.** Let \( m \) be a positive integer with \( m \geq 2 \) and \( 0 < q < \frac{2m}{m-1} \). For each \( \lambda \in \mathbb{N}_0 \), let \( \{ \omega_k^\lambda \} k \) be wavelets of level \( \lambda \). Suppose that \( \{ b_k^{\lambda, \gamma, \mu} \}_{\gamma, \mu \in \mathbb{Z}, k \in (\mathbb{Z}^n)^m} \) is a sequence of complex numbers satisfying
\[
\sup_{\gamma, \mu \in \mathbb{Z}} \| \{ b_k^{\lambda, \gamma, \mu} \}_{k \in (\mathbb{Z}^n)^m} \|_{\ell^\omega} \leq A_{\lambda, \mu}
\]
and
\[
\sup_{\gamma, \mu \in \mathbb{Z}} \| \{ b_k^{\lambda, \gamma, \mu} \}_{k \in (\mathbb{Z}^n)^m} \|_{\ell^q} \leq B_{\lambda, \mu, q}.
\]
Then the following statements hold:
(1) For $1 \leq r \leq 2$ there exists a constant $C > 0$, independent of $\lambda, \mu$, such that

$$\left\| \left( \sum_{\gamma \in \mathbb{Z}} \sum_{\bar{k} \in \gamma_{\bar{
u}}^{\lambda+\mu}} b_{\bar{k}}^\lambda \gamma \mu L_{k_1}^\lambda \gamma f_1 \prod_{j=2}^{m} L_{k_j}^\lambda \gamma f_j \right)^{1/r} \right\|_{L^{2/m}} \leq CA_{\lambda, \mu} 2^{\lambda mn/2} \left( \sum_{\gamma \in \mathbb{Z}} \left\| f_1^\lambda \gamma \mu \right\|_{L^2} \right)^{1/r} \prod_{i=2}^{m} \left\| f_i \right\|_{L^2}$$

for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$.

(2) For $2 \leq l \leq m$ there exists a constant $C > 0$, independent of $\lambda, \mu$, such that

$$\left\| \sum_{\gamma \in \mathbb{Z}} \sum_{\bar{k} \in \gamma_{\bar{
u}}^{\lambda+\mu}} b_{\bar{k}}^\lambda \gamma \mu \left( \prod_{j=1}^{l} L_{k_j}^\lambda \gamma f_j \right) \prod_{j=l+1}^{m} L_{k_j}^\lambda \gamma f_j \right\|_{L^{2/m}} \leq CA_{\lambda, \mu} \frac{1}{2^{l(l-1)/2}} B_{\lambda, \mu, q} 2^{\lambda mn/2} \left[ \prod_{j=1}^{l} \left( \sum_{\gamma \in \mathbb{Z}} \left\| f_j^\lambda \gamma \mu \right\|_{L^2}^{2} \right)^{1/2} \right] \prod_{j=l+1}^{m} \left\| f_j \right\|_{L^2}$$

for Schwartz functions $f_1, \ldots, f_m$ on $\mathbb{R}^n$, where $\prod_{m+1}^{m}$ is understood as empty.

The proof of Proposition 2.4 is given in the next section following that of Proposition 2.1.

3. PROOFS OF PROPOSITION 2.1 AND PROPOSITION 2.4

When $m = 1$, Proposition 2.1 follows immediately from Plancherel’s identity. Thus, we will be concerned only with the case $m \geq 2$. For the bilinear case $m = 2$, a concept called column is used; see, for instance, [1, 18]. A column $Col_k^l$ with $k \in \mathbb{Z}$ related to a subset $U \subset \mathbb{Z}^2$ is defined as $U \cap \{\{k\} \times \mathbb{Z}\}$. We generalize this concept to higher dimensions in least two ways, expressed in terms of the dimension and the codimension of the set. A column related to $U \subset \mathbb{Z}^3$ could be $U \cap \{(k,l) \times \mathbb{Z}\}$ of dimension 1, or $U \cap \{\{k\} \times \mathbb{Z}^2\}$ of dimension 2. It turns out that both notions are needed to handle the case $m = 3$.

We introduce several notions and study their combinatorial properties. For a fixed $\bar{k} \in (\mathbb{Z}^n)^m$, $1 \leq l \leq m$, and $1 \leq j_1 \leq \cdots \leq j_l \leq m$ let

$$\bar{k}^{j_1, \ldots, j_l} := (k_{j_1}, \ldots, k_{j_l})$$

denote the vector in $(\mathbb{Z}^n)^l$ consisting of $j_1, \ldots, j_l$ components of $\bar{k}$ and $\bar{k}^{+j_1, j_2, \ldots, j_l}$ stand for the vector in $(\mathbb{Z}^n)^{m-l}$, consisting of $\bar{k}$ excepting $j_1, \ldots,
For a fixed \( j \) \( k \) components (e.g. \( k^{1,\ldots,j} = \bar{k}^{j+1,\ldots,m} = (k_{j+1}, \ldots, k_m) \in (\mathbb{Z}^n)^{m-j} \)). For any sets \( U \) in \((\mathbb{Z}^n)^m\), \( 1 \leq j \leq m \), and \( 1 \leq j_1 \leq \cdots \leq j_l \leq m \) let
\[
\mathcal{P}_j U := \{ k_j \in \mathbb{Z}^n : \bar{k} \in U \text{ for some } \bar{k}^{j} \in (\mathbb{Z}^n)^{m-1} \}
\]
\[
\mathcal{P}_{j_1, \ldots, j_l} U := \{ \bar{k}^{j_1, \ldots, j_l} \in (\mathbb{Z}^n)^{m-1} : \bar{k} \in U \text{ for some } k_{j_1}, \ldots, k_{j_l} \in \mathbb{Z}^n \}
\]
be the projections of \( U \) onto the \( j \)-coordinate and \( \bar{k}^{j_1, \ldots, j_l} \)-plane, respectively. For a fixed \( \bar{k}^{j_1, \ldots, j_l} \in \mathcal{P}_{j_1, \ldots, j_l} U \), we define
\[
\text{Col}^{U}_{k^{j_1, \ldots, j_l}} := \{ \bar{k}^{j_1, \ldots, j_l} \in (\mathbb{Z}^n)^l : \bar{k} = (k_1, \ldots, k_m) \in U \}.
\]
Then we observe that
\[
\sum_{k \in U} \cdots = \sum_{\bar{k}^{j_1, \ldots, j_l} \in \mathcal{P}_{j_1, \ldots, j_l} U} \left( \sum_{k^{j_1, \ldots, j_l} \in \text{Col}^{U}_{k^{j_1, \ldots, j_l}}} \cdots \right).
\]
Furthermore, for each \( \bar{k}^{j_1, \ldots, j_l} \) we have
\[
\text{Col}^{U}_{k^{j_1, \ldots, j_l}} = \bigcup_{k_{j_l} \in \mathcal{P}_l \text{Col}^{U}_{k^{j_1, \ldots, j_l-1}}} \text{Col}^{U}_{k^{j_1, \ldots, j_l}} \times \{ k_{j_l} \}
\]
and this allows us to write
\[
\sum_{k \in U} \cdots = \sum_{\bar{k}^{j_1, \ldots, j_l} \in \mathcal{P}_{j_1, \ldots, j_l} U} \left( \sum_{k_{j_l} \in \mathcal{P}_l \text{Col}^{U}_{k^{j_1, \ldots, j_l-1}}} \left( \sum_{k_{j_l} \in \text{Col}^{U}_{k^{j_1, \ldots, j_l}}} \cdots \right) \right).
\]
To make it easier to understand, let us think about the case \( n = 1 \) and \( m = 3 \). \( \mathcal{P}_1 U \) is the projection of \( U \) to the first coordinate. \( \mathcal{P}_{1,1} U \) is the projection of \( U \) to the \((k_2, k_3)\)-plane. \( \text{Col}^{U}_{k^{1}} \) is an 1-column in \( \mathbb{Z}^3 \cap \mathcal{P}_1 U \) with \((k_2, k_3) = \bar{k}^{1} \) fixed. When \( j_1 = 1 \), identity (13) says that
\[
\sum_{k \in U} \cdots = \sum_{(k_2, k_3) \in \mathcal{P}_{1,1} U} \left( \sum_{k_1 \in \text{Col}^{U}_{k^{1,2,3}}} \cdots \right).
\]
When \((j_1, \ldots, j_l) = (1, 2)\), identity (14) says that
\[
\sum_{k \in U} \cdots = \sum_{k_3 \in \mathcal{P}_3 U} \left( \sum_{k_2 \in \mathcal{P}_2 \text{Col}^{U}_{k^{1,2,3}}} \left( \sum_{k_1 \in \text{Col}^{U}_{k^{1,2,3}}} \cdots \right) \right).
\]

The proof of Proposition 2.1 is based on the decompositions in (13) and (14) and on the following lemma.
Lemma 3.1. Let $m \geq 2$ and $\mathcal{U}$ be a subset of $(\mathbb{Z}^n)^m$. Let $\lambda \in \mathbb{N}_0$ and $\{\omega_k^{\lambda}\}_{k \in (\mathbb{Z}^n)^m}$ be wavelets whose level is $\lambda$. Let $\sigma^{\lambda} = \sum_{k \in \mathcal{U}} b_k^{\lambda} \omega_k^{\lambda}$, where $\{b_k^{\lambda}\}_{k \in \mathcal{U}}$ is a sequence of complex numbers satisfying $\|\{b_k^{\lambda}\}_{k \in (\mathbb{Z}^n)^m}\|_{\ell^{\infty}} \leq A_{\lambda}$. Then there exists a constant $C > 0$ such that

$$\left\| T_{\sigma^{\lambda}}(f_1, \ldots, f_m) \right\|_{L^{2/m}} \leq CA_{\lambda}2^{\lambda(m-1)/2} \left( \prod_{i \neq j, 1 \leq i \leq m} \|f_i\|_{L^2} \right) \times \left( \int_{\mathbb{R}^n} |\hat{f}_j(\xi)|^2 \sum_{k \in \mathcal{U}} |\omega_k^{\lambda}(\xi)|^2 d\xi \right)^{1/2}$$

for each $1 \leq j \leq m$.

Proof. Without loss of generality, we may assume $j = 1$. In view of (13), $\sigma^{\lambda}$ can be written as

$$\sigma^{\lambda}(\xi) = \sum_{\vec{k} \in \mathcal{P}_+ \mathcal{U}} \omega_{k_2}^{\lambda}(\xi_2) \ldots \omega_{k_m}^{\lambda}(\xi_m) \sum_{k_1 \in \text{Col}_{k+1}^L} h_{k_1} b_{k_1}^{\lambda} \omega_{k_1}^{\lambda}(\xi_1),$$

and this yields that

$$T_{\sigma^{\lambda}}(f_1, \ldots, f_m)(x) = \sum_{\vec{k} \in \mathcal{P}_+ \mathcal{U}} \left( \prod_{i=2}^m |L_{k_i}^{\lambda} f_i(x)| \right) \sum_{k_1 \in \text{Col}_{k+1}^L} b_{k_1}^{\lambda} |L_{k_1}^{\lambda} f_1(x)|,$$

where $L_{k}^{\lambda} f := (\omega_k^{\lambda} \hat{f})^\vee$ for $k \in \mathbb{Z}^n$. Using the Cauchy-Schwarz inequality and Hölder’s inequality, we obtain that

$$\left\| T_{\sigma^{\lambda}}(f_1, \ldots, f_m) \right\|_{L^{2/m}} \leq \left\| \left( \sum_{\vec{k} \in (\mathbb{Z}^n)^{m-1}} \left( \prod_{i=2}^m |L_{k_i}^{\lambda} f_i| \right)^2 \right)^{1/2} \right\|_{L^{2/(m-1)}} \times \left\| \left( \sum_{\vec{k} \in (\mathbb{Z}^n)^{m-1}} \sum_{k_1 \in \text{Col}_{k+1}^L} b_{k_1}^{\lambda} |L_{k_1}^{\lambda} f_1|^2 \right)^{1/2} \right\|_{L^2}$$

$$=: I \times II.$$

As a direct consequence of Plancherel’s identity and (4), we have

$$\left\| \left\{ L_{k}^{\lambda} f \right\}_{k \in \mathbb{Z}^n} \right\|_{L^2(\mathbb{C})} \leq 2^{\lambda n/2} \|f\|_{L^2}$$

and thus,

$$I = \prod_{i=2}^m \left\| \sum_{k_i \in \mathbb{Z}^n} |L_{k_i}^{\lambda} f_i|^2 \right\|_{L^{2/(m-1)}}^{1/2} \leq \prod_{i=2}^m \left\| \left\{ L_{k_i}^{\lambda} f_i \right\}_{k_i \in \mathbb{Z}^n} \right\|_{L^2(\mathbb{C})}^{1/2} \approx 2^{\lambda (m-1)n/2} \prod_{i=2}^m \|f_i\|_{L^2},$$

$$\leq 2^{\lambda (m-1)n/2} m \prod_{i=2}^m \|f_i\|_{L^2},$$
where the first inequality is obtained by Hölder’s inequality. Moreover, it follows from Plancherel’s identity and the disjoint compact support property of \( \{ \omega_{k_1}^\lambda \}_{k_1 \in \mathbb{Z}^n} \) that

\[
II \lesssim \left( \sum_{k_1 \in P_1 \cup \mathcal{U}} \left\| \hat{f}_1 \sum_{k_1 \in \text{Col}_{k_1}^{\mathcal{U}}} b_k^\lambda \omega_{k_1}^\lambda \right\|_{L^2}^2 \right)^{1/2}
\]

\[
\approx \left( \int_{\mathbb{R}^n} |\hat{f}_1(\xi)|^2 \sum_{k_1 \in \mathcal{U}} \sum_{k_1 \in \text{Col}_{k_1}^{\mathcal{U}}} |b_k^\lambda|^2 |\omega_{k_1}^\lambda(\xi)|^2 \, d\xi \right)^{1/2}
\]

and this is controlled by a constant multiple of

\[
A_\lambda \left( \int_{\mathbb{R}^n} |\hat{f}_1(\xi)|^2 \sum_{k_1 \in \mathcal{U}} |\omega_{k_1}^\lambda(\xi)|^2 \, d\xi \right)^{1/2}
\]

where (13) is applied. This completes the proof. \( \square \)

3.1. Proof of Proposition 2.1. Let \( N_1, \ldots, N_{m-1} \) be positive numbers less than \( N \), which will be chosen later. We separate \( \mathcal{U} \) into \( m \) disjoint subsets

\[
\mathcal{U}^1 := \{ \vec{k} \in \mathcal{U} : |\text{Col}_{k_1}^{\mathcal{U}}| > N_1 \}
\]

\[
\mathcal{U}^2 := \{ \vec{k} \in \mathcal{U} \setminus \mathcal{U}^1 : |\text{Col}_{k_1}^{\mathcal{U}}| > N_2 \}
\]

\[
: \quad \mathcal{U}^{m-1} := \{ \vec{k} \in \mathcal{U} \setminus (\mathcal{U}^1 \cup \cdots \cup \mathcal{U}^{m-2}) : |\text{Col}_{k_1}^{\mathcal{U}}| > N_{m-1} \}
\]

\[
\mathcal{U}^m := \mathcal{U} \setminus (\mathcal{U}^1 \cup \cdots \cup \mathcal{U}^{m-1})
\]

and write

\[
\sigma^\lambda = \sum_{j=1}^{m} \sum_{\vec{k} \in \mathcal{U}^j} b_k^\lambda \omega_{k_1}^\lambda =: \sum_{j=1}^{m} \sigma_{(j)}^\lambda.
\]

Observe that for \( 1 \leq j \leq m-1 \), due to (13),

\[
N \geq |\mathcal{U}^j| > N_j |P_{j+1} \cup \cdots \cup \mathcal{U}^j|,
\]

which implies

\[
|P_{j+1} \cup \cdots \cup \mathcal{U}^j| < NN_j^{-1}.
\]

Moreover, for \( 2 \leq j \leq m \) and \( \vec{k} \in P_{j+1} \cup \cdots \cup \mathcal{U}^j \),

\[
|\text{Col}_{k_1}^{\mathcal{U}}| \leq N_{j-1},
\]

which follows from the fact \( \mathcal{U}^j \subset \mathcal{U} \setminus \mathcal{U}^{j-1} \).
We now apply Lemma 3.1 to each $\sigma_{(j)}$, $1 \leq j \leq m$, to obtain

$$
(17) \quad \|T_{\sigma_{(j)}}(f_1, \ldots, f_m)\|_{L^{2/m}} \leq CA_{\lambda}2^{\lambda(m-1)n/2} \prod_{i \neq j, 1 \leq i \leq m} \|f_i\|_{L^2} \times \left( \int_{\mathbb{R}^n} |\tilde{f}_j(\xi)|^2 \sum_{k \in U^j} |\omega^\lambda_{kj}(\xi)|^2 d\xi \right)^{1/2}.
$$

Note that

$$
\sum_{k \in U^j} |\omega^\lambda_{k1}(\xi)|^2 = \sum_{k^1 \in P_{*1}U^1} \left( \sum_{k \in \text{Col}_{k_1}U^j} |\omega^\lambda_{kj}(\xi)|^2 \right) \leq 2^{\lambda n}|P_{*1}U^1| < 2^{\lambda n}N_1^{-1}
$$

where (13), (4), and (15) are applied. Similarly, when $2 \leq j \leq m - 1$, we have

$$
\sum_{k \in U^j} |\omega^\lambda_{kj}(\xi)|^2 = \sum_{k^1 \in P_{*1}U^1} \left( \sum_{k \in \text{Col}_{k_1}U^j} |\omega^\lambda_{kj}(\xi)|^2 |\text{Col}_{k^1_{*1}, \ldots, j-1}U^j| \right) \leq 2^{\lambda n}N_{j-1}|P_{*1, \ldots, j}U^j| \leq 2^{\lambda n}NN_{j-1}N_j^{-1},
$$

using (14), (4), (16), and (15). For the last case $j = m$, it follows from (13), (4), and (16) that

$$
\sum_{k \in U^m} |\omega^\lambda_{km}(\xi)|^2 = \sum_{k \in \text{Col}_{k_1}U^m} |\omega^\lambda_{km}(\xi)|^2 |\text{Col}_{k^1_{m}}U^m| \leq 2^{\lambda n}N_{m-1}.
$$

Now we choose $N_1, \ldots, N_{m-1}$ satisfying the identity

$$
(18) \quad NN_1^{-1} = NN_1N_2^{-1} = NN_2N_3^{-1} = \cdots = NN_{m-2}N_{m-1}^{-1} = N_{m-1}.
$$

Solving (18), we have

$$
N_j = N_{j/m}, \quad 1 \leq j \leq m - 1
$$

and this establishes

$$
\sum_{k \in U^j} |\omega^\lambda_{kj}(\xi)|^2 \leq 2^{\lambda n}N^{(m-1)/m}, \quad 1 \leq j \leq m,
$$

which further implies

$$
\left( \int_{\mathbb{R}^n} |\tilde{f}_j(\xi)|^2 \sum_{k \in U^j} |\omega^\lambda_{kj}(\xi)|^2 d\xi \right)^{1/2} \leq 2^{\lambda n/2}N^{(m-1)/2m} \|f_j\|_{L^2}.
$$

Then this, together with (17), proves

$$
\|T_{\sigma_{(j)}}(f_1, \ldots, f_m)\|_{L^{2/m}} \lesssim A_{\lambda}N^{(m-1)/2m} \sum_{i=1}^{m} \|f_i\|_{L^2}
$$

as desired.
3.2. Proof of Proposition 2.4. We first observe that

\begin{equation}
\left| \mathcal{P}_{\lambda_1, \ldots, \lambda_l, \mu} \right| \leq M^{n(m-l)} \quad \text{for } \mu \geq 0,
\end{equation}

and

\begin{equation}
L_k^{\lambda, \gamma} f(x) = L_k^\lambda \left( f(\cdot/2^\gamma) \right) (2^\gamma x),
\end{equation}

and

\begin{equation}
|L_k^{\lambda, \gamma} f(x)| \lesssim 2^{\lambda n/2} M f(x) \quad \text{for } k \in \mathbb{Z}^n.
\end{equation}

Here $M$ is the Hardy-Littlewood maximal operator, defined by $M f(x) := \sup_{Q \ni x} \left| Q \right|^{-1} \int_Q |f(y)| dy$, where the supremum is taken over all cubes containing $x$. Then in view of (13) we can write

\begin{equation}
\sum_{k \in \mathcal{P}_{\lambda_1, \ldots, \lambda_l}} b_k^{\lambda, \gamma, \mu} \left( \prod_{j=1}^l L_{k_j}^{\lambda_j, \gamma_j, \mu_j} (x) \right) \left( \prod_{j=l+1}^m L_{k_j}^{\lambda_j, \gamma_j} (x) \right)
\end{equation}

When $l = 1$, using (20), Hölder’s inequality, (Minkowski inequality for $r < 2$), the $L^2$ boundedness of $M$, and (19), we obtain

\begin{align}
\left\| \sum_{\gamma \in \mathbb{Z}} \max_{k \in \mathcal{P}_{1, \ldots, 1}} b_k^{\lambda, \gamma, \mu} L_k^{\lambda, \gamma} f_1 \left( \prod_{j=2}^m L_{k_j}^{\lambda_j, \gamma_j} f_j \right) \right\|_{L^r}^{1/r} & \leq 2^{2^{(m-1)n/2}} \left( \prod_{j=2}^m \| f_j \|_{L^2} \right) \\
& \times \sum_{k \in \mathcal{P}_{1, \ldots, 1}} b_k^{\lambda, \gamma, \mu} L_k^{\lambda} f_1 \left( \prod_{j=2}^m L_{k_j}^{\lambda_j, \gamma_j} f_j \right) \left( \prod_{j=1}^m L_{k_j}^{\lambda_j, \gamma_j} (2^\gamma x) \right) \left( \prod_{j=1}^m L_{k_j}^{\lambda_j, \gamma_j} (2^\gamma x) \right) \left( \prod_{j=1}^m L_{k_j}^{\lambda_j, \gamma_j} (2^\gamma x) \right)
\end{align}

A change of variables and Plancherel’s identity yield that

\begin{equation}
\left\| \sum_{k \in \mathcal{P}_{1, \ldots, 1}} b_k^{\lambda, \gamma, \mu} L_k^{\lambda} f_1 \left( \prod_{j=2}^m L_{k_j}^{\lambda_j, \gamma_j} (2^\gamma x) \right) \left( \prod_{j=1}^m L_{k_j}^{\lambda_j, \gamma_j} (2^\gamma x) \right) \left( \prod_{j=1}^m L_{k_j}^{\lambda_j, \gamma_j} (2^\gamma x) \right) \|_{L^2} \leq A_{\lambda, \mu} 2^{2^{\lambda n/2}} \| f_1^{\lambda, \gamma, \mu} \|_{L^2},
\end{equation}

which proves the first estimate.
Similarly, for \(2 \leq l \leq m\), we can see
\[
\left\| \sum_{\gamma \in \mathbb{Z}} \left( \sum_{k \in \mathcal{V}_l^+} b^\lambda, \gamma, \mu \left( \prod_{j=1}^l L^\lambda, \gamma, f^\lambda j \right) \prod_{j=l+1}^m L^\lambda, \gamma, f_j \right) \right\|_{L^{2/m}} \leq 2^{(m-l)n/2} \left( \prod_{j=l+1}^m \|f_j\|_{L^2} \right) \times \\
\sum_{\kappa^1, \ldots, \kappa^l \in \mathcal{P} \subset \mathbb{Z}} \left( \sum_{\gamma \in \mathbb{Z}} \left( \sum_{\kappa^1, \ldots, \kappa^l \in \mathcal{V}_l^+} b^\lambda, \gamma, \mu \left[ \prod_{j=1}^l L^\lambda, \gamma, f^\lambda j \right] \left( \prod_{j=l+1}^m \|f^\lambda j \|_{L^2} \right) \right) \right)_{L^{2/l}}.
\]

The \(L^{2/l}\) norm is clearly dominated by
\[
\left( \sum_{\gamma \in \mathbb{Z}} \left( \sum_{k \in \mathcal{V}^+} b^\lambda, \gamma, \mu \left( \prod_{j=1}^l L^\lambda, \gamma, f^\lambda j \right) \right) \right)^{2/l}_{L^{2/l}} \leq 2^{(m-l)n/2} \left( \prod_{j=l+1}^m \|f_j\|_{L^2} \right) \times \\
\sum_{\kappa^1, \ldots, \kappa^l \in \mathcal{P} \subset \mathbb{Z}} \left( \sum_{\gamma \in \mathbb{Z}} \left( \sum_{\kappa^1, \ldots, \kappa^l \in \mathcal{V}_l^+} b^\lambda, \gamma, \mu \left[ \prod_{j=1}^l L^\lambda, \gamma, f^\lambda j \right] \left( \prod_{j=l+1}^m \|f^\lambda j \|_{L^2} \right) \right) \right)_{L^{2/l}}.
\]

since \(2/l \leq 1\), and now we apply a change of variables, Proposition 2.2, and Hölder’s inequality to obtain that the above expression is less than
\[
A_1 \gamma \mu \frac{(l-1)q}{2r} B_1 \frac{(l-1)q}{2r} 2^{l \ln 2} \left( \sum_{\gamma \in \mathbb{Z}} \left( \prod_{j=1}^l \|f^\lambda j \|_{L^2} \right) \right)^{1/2} \leq A_1 \gamma \mu \frac{(l-1)q}{2r} B_1 \frac{(l-1)q}{2r} 2^{l \ln 2} \left( \sum_{\gamma \in \mathbb{Z}} \left( \prod_{j=1}^l \|f^\lambda j \|_{L^2} \right) \right)^{1/2}.
\]

This completes the proof.

4. COMPACTLY SUPPORTED WAVELETS

Typical functions possessing properties (i) and (ii) in Section 2 are the compactly supported wavelets constructed by Daubechies [8]; their construction is contained in the books of Meyer [28] and Daubechies [9]. Wavelets have been used to study singular integrals in different settings; see for instance [29], [14], [12], and [16]. For the purposes of this paper, we need smooth wavelets with compact supports but also of product type, like (5). The construction of such orthonormal bases is carefully presented in Triebel [34], but for the reader’s sake we provide an outline. For any fixed \(M \in \mathbb{N}\) there exist real compactly supported functions \(\psi_F, \psi_M \in C^0(\mathbb{R})\) satisfying the following properties:

(a) \(\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1\)
(b) \(\int_{\mathbb{R}} x^\alpha \psi_M(x) dx = 0\) for all \(0 \leq \alpha \leq M\)
(c) If $\Psi_{\vec{G}}$ is a function on $\mathbb{R}^{mn}$, defined by $$\Psi_{\vec{G}}(\vec{x}) := \Psi_{g_1}(x_1) \cdots \Psi_{g_{mn}}(x_{mn})$$ for $\vec{x} := (x_1, \ldots, x_{mn}) \in \mathbb{R}^{mn}$ and $\vec{G} := (g_1, \ldots, g_{mn})$ in the set $$\mathcal{I} := \left\{ \vec{G} := (g_1, \ldots, g_{mn}) : g_i \in \{ F, M \} \right\},$$ then the family of functions $$\bigcup_{\lambda \in \mathbb{N}_0} \bigcup_{\vec{k} \in \mathbb{Z}^{mn}} \{ 2^{\lambda mn/2} \Psi_{\vec{G}}(2^\lambda \vec{x} - \vec{k}) : \vec{G} \in \mathcal{I}^\lambda \}$$ forms an orthonormal basis of $L^2(\mathbb{R}^{mn})$, where $\mathcal{I}^0 := \mathcal{I}$ and $\mathcal{I}^\lambda := \mathcal{I} \setminus \{(F, \ldots, F)\}$ for $\lambda \geq 1$.

Fix $1 < q < \infty$ and $s \geq 0$. Let $\| F \|_{L^q(\mathbb{R}^{mn})}$ denote the Sobolev space norm defined as the $L^q((\mathbb{R}^n)^m)$ norm of $(\vec{I} - \vec{\Delta})^{s/2} F$, where $\vec{\Delta}$ is the Laplacian of a function $F$ on $(\mathbb{R}^n)^m$. It is also shown in [33] that if $M$ is sufficiently large and $F$ is a tempered distribution on $\mathbb{R}^{mn}$ lying in $L^q_s(\mathbb{R}^{mn})$, then $F$ can be represented as

$$F(\vec{x}) = \sum_{\lambda \in \mathbb{N}_0} \sum_{\vec{G} \in \mathcal{I}^\lambda} \sum_{\vec{k} \in \mathbb{Z}^{mn}} b^\lambda_{\vec{G}, \vec{k}} 2^{\lambda mn/2} \Psi_{\vec{G}}(2^\lambda \vec{x} - \vec{k})$$

and

$$\left\| \left( \sum_{\vec{G}, \vec{k}} |b^\lambda_{\vec{G}, \vec{k}} \Psi^\lambda_{\vec{G}, \vec{k}}(\vec{x})|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^{mn})} \leq C 2^{-s\lambda} \| F \|_{L_s^q(\mathbb{R}^{mn})},$$

where $\Psi^\lambda_{\vec{G}, \vec{k}}(\vec{x}) = 2^{\lambda mn/2} \Psi_{\vec{G}}(2^\lambda \vec{x} - \vec{k})$, and

$$b^\lambda_{\vec{G}, \vec{k}} := \int_{\mathbb{R}^{mn}} F(\vec{x}) \Psi^\lambda_{\vec{G}, \vec{k}}(\vec{x}) d\vec{x}.$$

Moreover, it follows from the last estimate and disjoint supports of $\Psi^\lambda_{\vec{G}, \vec{k}}$ that

$$\left\| \left\{ b^\lambda_{\vec{G}, \vec{k}} \right\}_{\vec{k} \in \mathbb{Z}^{mn}} \right\|_{L^q(\mathbb{R}^{mn})} \leq \left( \frac{2^{\lambda mn(1-q/2)}}{\int_{\mathbb{R}^{mn}} \left( \sum_{\vec{k}} |b^\lambda_{\vec{G}, \vec{k}} \Psi^\lambda_{\vec{G}, \vec{k}}(\vec{x})|^2 \right)^{q/2} d\vec{x} \right)^{1/q} \approx 2^{\lambda(mn-q/2)} \| F \|_{L_s^q(\mathbb{R}^{mn})},$$

(22)

We will write $\vec{G} := (G_1, \ldots, G_m) \in (\{ F, M \}^n)^m$, and $$\Psi_{\vec{G}}(\vec{\xi}) = \Psi_{G_1}(\xi_1) \cdots \Psi_{G_m}(\xi_m).$$

For each $\vec{k} := (k_1, \ldots, k_m) \in (\mathbb{Z}^n)^m$ and $\lambda \in \mathbb{N}_0$, let $$\Psi^\lambda_{G_i, k_i}(\xi_i) := 2^{\lambda n/2} \Psi_{G_i}(2^\lambda \xi_i - k_i), \quad 1 \leq i \leq m.$$
We first observe that (22) yields that
\[ \Psi_{G,k}^\lambda(\vec{\xi}) := \Psi_{G_1,k_1}^\lambda(\xi_1) \cdots \Psi_{G_m,k_m}^\lambda(\xi_m). \]
We also assume that the support of \( \psi \) is contained in \( \{ \xi \in \mathbb{R} : |\xi| \leq C_0 \} \) for some \( C_0 > 1 \), which implies that
\[ \text{Supp}(\Psi_{G_i,k_i}^\lambda) \subset \{ \xi_i \in \mathbb{R}^n : |2^\lambda \xi_i - k_i| \leq C_0 \sqrt{n} \}. \]
In other words, the support of \( \Psi_{G_i,k_i}^\lambda \) is contained in the ball centered at \( 2^{-\lambda}k_i \) and radius \( C_0\sqrt{n}2^{-\lambda} \).

5. Proof of Theorem 1.1

Using (21) with \( s = 0 \), we decompose \( \sigma \) as
\[ \sigma(\vec{\xi}) = \sum_{\lambda \in \mathbb{N}_0} \sum_{G \in \mathcal{I}^\lambda} \sum_{k \in (2^\lambda \mathbb{Z})^m} b_{G,k}^\lambda \Psi_{G_1,k_1}^\lambda(\xi_1) \cdots \Psi_{G_m,k_m}^\lambda(\xi_m) =: \sum_{\lambda \in \mathbb{N}_0} \sum_{G \in \mathcal{I}^\lambda} \sigma_{G,k}^\lambda(\vec{\xi}), \]
where \( b_{G,k}^\lambda := \int_{(\mathbb{R}^n)^m} \sigma(\vec{\xi}) \Psi_{G,k}^\lambda(\vec{\xi}) d\vec{\xi} \). As an immediate consequence of Proposition 2.2, we have
\[ \| T_{\sigma_G^\lambda}(f_1, \ldots, f_m) \|_{L^{2/m}} \lesssim \| \{ b_{G,k}^\lambda \} \|_{L^2} \| \sigma \|_{L^q((\mathbb{R}^n)^m)}. \]

We first observe that (22) yields that
\[ \| \{ b_{G,k}^\lambda \} \|_{L^q} \lesssim 2^{\lambda mn(1/q - 1/2)} \| \sigma \|_{L^q((\mathbb{R}^n)^m)}. \]

In addition, as \( \sigma \in C^{M_q}((\mathbb{R}^n)^m) \), using this property, the \( M_q \) vanishing moment condition of \( \Psi_{G,k}^\lambda \) in conjunction with Taylor’s formula, an argument similar to [18, Lemma 2.1] and [16, Lemma 7] yields
\[ \| \{ b_{G,k}^\lambda \} \|_{L^\infty} \lesssim 2^{-\lambda(M_q + mn/2)} D_0. \]

Therefore, we finally arrive at the estimate
\[ \| T_{\sigma_G^\lambda}(f_1, \ldots, f_m) \|_{L^{2/m}} \lesssim 2^{-\lambda \left( M_q \left( 1 - \frac{(m-1)q}{2m} \right) \right)} D_0 \frac{(m-1)q}{2m} \| \sigma \|_{L^q((\mathbb{R}^n)^m)} \prod_{j=1}^m \| f_j \|_{L^2}, \]
which in turn implies that
\[ \| T_{\sigma}(f_1, \ldots, f_m) \|_{L^{2/m}} \leq \left( \sum_{\lambda \in \mathbb{N}_0} \sum_{G \in \mathcal{I}^\lambda} \| T_{\sigma_G^\lambda}(f_1, \ldots, f_m) \|_{L^{2/m}}^{2/m} \right)^{m/2}. \]
\[
\lesssim D_0^{1 - \frac{(m-1)q}{2mn}} \|\sigma\|_{L^{\frac{m}{q}}((\mathbb{R}^n)^m)} \left( \sum_{\lambda \in \mathbb{N}_0} 2^{-\lambda(M_q(1 - \frac{(m-1)q}{2mn}) - \frac{n(m-1)}{2m})^\frac{1}{2}} \right)^{m/2} \prod_{j=1}^m \|f_j\|_{L^2}.
\]

Since \(M_q > \frac{m(m-1)n}{2m(m-1)q}\), the sum over \(\lambda\) converges and completes the proof.

6. PROOF OF THEOREM 1.2

Without loss of generality, we may assume \(\frac{2m}{m+1} < q < 2\) as \(L^r(\mathbb{S}^{mn-1}) \subset L^q(\mathbb{S}^{mn-1})\) for \(r \geq q\). We first utilize a dyadic decomposition introduced by Duoandikoetxea and Rubio de Francia [13]. Recall that \(\Phi^{(m)}\) is a Schwartz function such that \(\Phi^{(m)}\) is supported in the annulus \(\{\xi \in (\mathbb{R}^n)^m : 1/2 \leq |\xi| \leq 2\}\) and \(\sum_{j \in \mathbb{Z}} \Phi^{(m)}(2^j \xi) = 1\) for \(\xi \neq 0\) where \(\Phi^{(m)}(\xi) := \Phi^{(m)}(\xi/2^j)\).

For \(\gamma \in \mathbb{Z}\) let

\[
K^{\gamma}(\bar{\gamma}) := \Phi^{(m)}(2^\gamma \bar{\gamma}) K(\bar{\gamma}), \quad \bar{\gamma} \in (\mathbb{R}^n)^m
\]

and then we observe that \(K^{\gamma}(\bar{\gamma}) = 2^{\gamma mn} K^0(2^\gamma \bar{\gamma})\). For \(\mu \in \mathbb{Z}\) we define

\[
K^{\gamma}_\mu(\gamma) := \Phi^{(m)}_{\mu+\gamma} K^\gamma(y) = 2^{\gamma mn} [\Phi^{(m)}_{\mu} + K^0](2^\gamma y).
\]

It follows from this definition that

\[
\hat{K}_\mu^{\gamma}(\bar{\xi}) = \Phi^{(m)}(2^{-(\mu+\gamma)} \bar{\xi}) \hat{K}^0(2^{-\gamma} \bar{\xi}) = \hat{K}_\mu^0(2^{-\gamma} \bar{\xi}),
\]

which implies that \(\hat{K}_\mu^{\gamma}\) is bounded uniformly in \(\gamma\) while they have almost disjoint supports, so it is natural to add them together as follows,

\[
K^{\gamma}_\mu(\bar{\gamma}) := \sum_{\gamma \in \mathbb{Z}} K^{\gamma}_\mu(\bar{\gamma}).
\]

We define

\[
\mathcal{L}_\mu(f_1, \ldots, f_m)(x) := \int_{(\mathbb{R}^n)^m} \mathcal{K}_\mu(\bar{\gamma}) \prod_{j=1}^m f_j(x - y_j) \, d\bar{\gamma}, \quad x \in \mathbb{R}^n
\]

and write

\[
\|\mathcal{L}_\Omega(f_1, \ldots, f_m)\|_{L^{2/m}} \lesssim \left\| \sum_{\mu \in \mathbb{Z}, 2^{-10} \leq C_0 \sqrt{mn}} \mathcal{L}_\mu(f_1, \ldots, f_m) \right\|_{L^{2/m}}
\]

(24)

\[
+ \left\| \sum_{\mu \in \mathbb{Z}, 2^{-10} > C_0 \sqrt{mn}} \mathcal{L}_\mu(f_1, \ldots, f_m) \right\|_{L^{2/m}}
\]

where \(C_0\) is the constant that appeared in (23).

First of all, using the argument in [16, Proposition 3], we can prove that

\[
\|\mathcal{L}_\mu(f_1, \ldots, f_m)\|_{L^p} \lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} \left( \prod_{j=1}^m \|f_j\|_{L^{p_j}} \right) \left\{ \begin{array}{ll}
2^{(mn-\delta)\mu}, & \mu \geq 0 \\
2^{(1-\delta)\mu}, & \mu < 0
\end{array} \right.
\]

(25)
for $0 < \delta < 1/q'$, and this implies that

$$
\left\| \sum_{\mu \in \mathbb{Z}^m, 2\mu^{-10} \leq C_0 \sqrt{mn}} \mathcal{L}_\mu (f_1, \ldots, f_m) \right\|_{L^2/m} \lesssim \| \Omega \|_{L^q(\mathbb{Z}^{mn-1})} \prod_{j=1}^m \| f_j \|_{L^2}.
$$

It remains to bound the term (24), but this can be reduced to proving that for $2\mu^{-10} > C_0 \sqrt{mn}$, there exists $\varepsilon_0 > 0$ such that

$$
(26) \quad \left\| \mathcal{L}_\mu (f_1, \ldots, f_m) \right\|_{L^2/m} \lesssim 2^{-\varepsilon_0 \mu} \| \Omega \|_{L^q(\mathbb{Z}^{mn-1})} \prod_{j=1}^m \| f_j \|_{L^2},
$$

which compensates the estimate (25) for $\mu \geq 0$. Recall that

$$
\hat{K}_\mu (\xi) = \sum_{\gamma \in \mathbb{Z}} \hat{K}_\mu^0 (\xi / 2^\gamma)
$$

and

$$
(27) \quad \text{Supp} \hat{K}_\mu^0 \subset \{ \xi \in (\mathbb{R}^n)^m : 2\mu^{-1} \leq |\xi| \leq 2\mu + 1 \}.
$$

Using (21), $\hat{K}_\mu^0$ can be written as

$$
(28) \quad \hat{K}_\mu^0 (\xi) = \sum_{\lambda \in \mathbb{N}_0} \sum_{\tilde{G} \in \mathbb{Z}^l} \sum_{\tilde{k} \in (\mathbb{Z}^n)^m} b^\lambda, \mu_{\tilde{G}, \tilde{k}} \Psi^\lambda \tilde{G}_1, \tilde{k}_1 (\xi_1) \cdots \Psi^\lambda \tilde{G}_m, \tilde{k}_m (\xi_m)
$$

where

$$
b^\lambda, \mu_{\tilde{G}, \tilde{k}} := \int_{(\mathbb{R}^n)^m} \hat{K}_\mu^0 (\xi) \Psi^\lambda \tilde{G}_\tilde{k} (\xi) d\xi.
$$

It is already known in [16, Lemma 7] that

$$
(29) \quad \left\| \{ b^\lambda, \mu_{\tilde{G}, \tilde{k}} \}_{\tilde{k} \in (\mathbb{Z}^n)^m} \right\|_{L^q} \lesssim 2^{-\delta \mu} 2^{\lambda (M + 1 + mn)} \| \Omega \|_{L^q(\mathbb{Z}^{mn-1})}
$$

where $M$ is the number of vanishing moments of $\Psi_{\tilde{G}}$ and $0 < \delta < 1/q'$. In addition, (22), the Hausdorff-Young inequality, and Young’s inequality prove that

$$
(30) \quad \left\| \{ b^\lambda, \mu_{\tilde{G}, \tilde{k}} \}_{\tilde{k} \in (\mathbb{Z}^n)^m} \right\|_{L'^q} \lesssim 2^{-\lambda mn (1/2 - 1/q')} \| \hat{K}_\mu^0 \|_{L^q}
$$

Furthermore, if $2\mu^{-10} > C_0 \sqrt{mn}$, then we may replace $\tilde{k} \in (\mathbb{Z}^n)^m$ in (28) by $2^{\lambda + \mu - 2} \leq |\tilde{k}| \leq 2^{\lambda + \mu + 2}$ due to (27) and the compact support condition of $\Psi^\lambda_{\tilde{G}, \tilde{k}}$. Therefore the proof of (26) can be reduced to the inequality

$$
\left\| \sum_{\lambda \in \mathbb{N}_0} \sum_{\tilde{G} \in \mathbb{Z}^l} \sum_{\gamma \in \mathbb{Z}} \sum_{\tilde{k} \in (\mathbb{Z}^n)^m} b^\lambda, \mu_{\tilde{G}, \tilde{k}} \prod_{j=1}^m \hat{K}^\lambda, \gamma_{\tilde{G}_j, \tilde{k}_j} f_j \right\|_{L^2/m}.
$$
\begin{align}
(31) \quad & \lesssim 2^{-\varepsilon_0 \mu} \|\Omega\|_{L^q(\mathbb{Z}^{m-1})} \prod_{j=1}^{m} \|f_j\|_{L^2} \\
\end{align}

where

\[ \mathcal{U}^{\lambda, \mu} := \{ \vec{k} \in (\mathbb{Z}^n)^m : 2^{\lambda + \mu - 2} \leq |\vec{k}| \leq 2^{\lambda + \mu + 2}, |k_1| \geq \cdots \geq |k_m| \} \]

and

\begin{align}
(32) \quad & L_{G, k}^{\lambda, \mu} := (\Psi_{G, k}(\cdot / 2^\lambda) \hat{f})^\vee.
\end{align}

Here, it is additionally assumed that $|k_1| \geq \cdots \geq |k_m|$ in $\mathcal{U}^{\lambda, \mu}$ as the remaining cases follow by symmetry. Then we note that $\mathcal{U}^{\lambda, \mu}$ can be expressed as the union of $m$ disjoint subsets

\begin{align*}
\mathcal{U}_1^{\lambda, \mu} := & \{ \vec{k} \in \mathcal{U}^{\lambda, \mu} : |k_1| \geq 2C_0 \sqrt{n} > |k_2| \geq \cdots \geq |k_m| \} \\
\mathcal{U}_2^{\lambda, \mu} := & \{ \vec{k} \in \mathcal{U}^{\lambda, \mu} : |k_1| \geq |k_2| \geq 2C_0 \sqrt{n} > |k_3| \geq \cdots \geq |k_m| \} \\
& \vdots \\
\mathcal{U}_m^{\lambda, \mu} := & \{ \vec{k} \in \mathcal{U}^{\lambda, \mu} : |k_1| \geq \cdots \geq |k_m| \geq 2C_0 \sqrt{n} \}.
\end{align*}

The function in the left-hand side of (31) could be written as

\[
\sum_{l=1}^{m} \sum_{\lambda \in \mathbb{N}_0} \sum_{1 \leq \gamma, \delta \leq 2} \sum_{\gamma \in \mathbb{Z}} T_{G, l}^{\lambda, \gamma, \mu} (f_1, \ldots, f_m)
\]

where

\begin{align}
(33) \quad & T_{G, l}^{\lambda, \gamma, \mu} (f_1, \ldots, f_m) := \sum_{\vec{k} \in \mathcal{U}_l^{\lambda, \mu}} b_{G, \vec{k}}^{\lambda, \gamma, \mu} \left( \prod_{j=1}^{m} L_{G, k_j}^{\lambda, \gamma, \mu} f_j \right).
\end{align}

Observe that when $\vec{k} \in \mathcal{U}_l^{\lambda, \mu}$,

\begin{align}
(34) \quad & L_{G, k_j}^{\lambda, \gamma, \mu} f_j = L_{G, k_j}^{\lambda, \gamma} f_j^{\lambda, \gamma, \mu} \quad \text{for} \quad 1 \leq j \leq l
\end{align}

due to the support of $\Psi_{G, k_j}^{\lambda, \gamma}$, where $f_j^{\lambda, \gamma, \mu}(\xi_j):= \hat{f}_j(\xi_j) \chi_{C_0 \sqrt{2} \gamma - \lambda \leq |\xi_j| \leq 2^{\gamma+\mu+2}}$. Moreover, it is easy to show that for $\mu \geq 10$ and $\lambda \in \mathbb{N}_0$,

\begin{align}
(35) \quad & \left( \sum_{\gamma \in \mathbb{Z}} \|f_j^{\lambda, \gamma, \mu}\|_{L^2}^2 \right)^{1/2} \lesssim (\mu + \lambda)^{1/2} \|f_j\|_{L^2} \lesssim \mu^{1/2} (\lambda + 1)^{1/2} \|f_j\|_{L^2}
\end{align}

where Plancherel’s identity is applied in the first inequality.

Now we claim that for each $1 \leq l \leq m$ there exists $\varepsilon_0, M_0 > 0$ such that

\begin{align}
(36) \quad & \left\| \sum_{\gamma \in \mathbb{Z}} T_{G, l}^{\lambda, \gamma, \mu} (f_1, \ldots, f_m) \right\|_{L^{2/m}} \lesssim \|\Omega\|_{L^q(\mathbb{Z}^{m-1})} 2^{-\varepsilon_0 \mu} 2^{-\lambda M_0} \prod_{j=1}^{m} \|f_j\|_{L^2}.
\end{align}
Then the left-hand side of (31) is controlled by a constant times
\[
\left( \sum_{l=1}^{m} \sum_{\lambda \in \mathbb{N}_0} \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\tilde{g},l}^{\lambda,\gamma,\mu} (f_1, \ldots, f_m) \right\|_{L^{2/m}}^{2/m} \right)^{m/2} \lesssim 2^{-\ell_0\mu} \left\| \Omega \right\|_{L^q(\mathbb{R}^{m-1})} \prod_{j=1}^{m} \left\| f_j \right\|_{L^2},
\]
which completes the proof of (31). Therefore, it remains to prove (36).

6.1. The case \( l = 1 \). The proof relies on the fact that if \( \hat{g}_{\gamma} \) is supported on \( \{ \xi \in \mathbb{R}^n : C^{-1} 2^{\gamma+\mu} \leq |\xi| \leq C 2^{\gamma+\mu} \} \) for some \( C > 1 \) and \( \mu \in \mathbb{Z} \), then
\[
(37) \quad \left\| \left\{ \Phi_j^{(1)} \ast \left( \sum_{\gamma \in \mathbb{Z}} g_{\gamma} \right) \right\}_{j \in \mathbb{Z}} \right\|_{L^p(\mathbb{R}^n)} \lesssim C \left\| \{ g_j \}_{j \in \mathbb{Z}} \right\|_{L^p(\ell^2)} \quad \text{uniformly in } \mu
\]
for \( 0 < p < \infty \). The proof of (37) is elementary and standard, so it is omitted here. See [16, (13)] and [35, Theorem 3.6] for a related argument. Note that
\[
2^{\lambda+\mu-3} \leq 2^{\lambda+\mu-2} - 2C_0 \sqrt{mn} \leq |k| - (|k_2| + \cdots + |k_m|) \leq 2^{\lambda+\mu+2}
\]
and this implies that
\[
\text{Supp}(\Psi_{\lambda,\mu}^{G_1,k_1}(\cdot/2^\gamma)) \subset \{ \xi \in \mathbb{R}^n : 2^{\gamma+\mu-4} \leq |\xi| \leq 2^{\gamma+\mu+3} \}.
\]
Moreover, since \( |k_j| \leq 2C_0 \sqrt{n} \) for \( 2 \leq j \leq m \) and \( 2^{\mu-10} > C_0 \sqrt{mn} \),
\[
\text{Supp}(\Psi_{\lambda,\mu}^{G_1,k_j}(\cdot/2^\gamma)) \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq m^{-1/2} 2^{\gamma+\mu-8} \}.
\]
Therefore, the Fourier transform of \( \mathcal{T}_{\tilde{g},1}^{\lambda,\gamma,\mu} (f_1, \ldots, f_m) \) is supported in the set \( \{ \xi \in \mathbb{R}^n : 2^{\gamma+\mu-5} \leq |\xi| \leq 2^{\gamma+\mu+4} \} \). Using the Littlewood-Paley theory for Hardy spaces [15, Theorem 2.2.9], there exists a unique polynomial \( Q_{\lambda,\mu}^{G}(x) \) such that
\[
\left\| \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\tilde{g},1}^{\lambda,\gamma,\mu} (f_1, \ldots, f_m) - Q_{\lambda,\mu}^{G} \right\|_{L^{2/m}} \lesssim \left\| \left\{ \Phi_j^{(1)} \ast \left( \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\tilde{g},1}^{\lambda,\gamma,\mu} (f_1, \ldots, f_m) \right) \right\}_{j \in \mathbb{Z}} \right\|_{L^{2/m}(\ell^2)}
\]
and then (37) and (34) yield that the above \( L^{2/m}(\ell^2) \)-norm is dominated by a constant multiple of
\[
\left\| \left( \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\tilde{g},1}^{\lambda,\gamma,\mu} (f_1^{\lambda,\gamma,\mu}, f_2, \ldots, f_m)^2 \right)^{1/2} \right\|_{L^{2/m}}.
\]
We now apply Proposition 2.4, (29), and (35) to bound the $L^{2/m}$-norm by
\[
\left\| \{ b_{\mathcal{G},k}^{\lambda,\mu} \} \right\|_{\ell^\infty} 2^{\lambda mn/2} \left( \sum_{\gamma \in \mathbb{Z}} \left\| f_1^{\lambda,\gamma,\mu} \right\|_{L^2} \right)^{1/2} \prod_{j=2}^m \left\| f_j \right\|_{L^2} 
\]
\[
\lesssim \left\| \Omega \right\|_{L^q(S^m)} 2^{-\delta \mu} 2^{-\lambda(M+1+mn/2)} (\lambda + 1)^{1/2} \prod_{j=1}^m \left\| f_j \right\|_{L^2}
\]
This implies that the left-hand side of (38) is bounded by
\[
\left\| \Omega \right\|_{L^q(S^m)} 2^{-\delta \mu} 2^{-\lambda M_0} \prod_{j=1}^m \left\| f_j \right\|_{L^2}
\]
for some $0 < \delta_0 < \delta$ and $0 < M_0 < M + 1 + mn/2$, and thus
\[
(39) \quad \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\mathcal{G},1}^{\lambda,\gamma,\mu} (f_1, \ldots, f_m) - Q^{\lambda,\mu,\hat{G}} \in L^{2/m}.
\]
Furthermore, it follows from Proposition 2.4 that
\[
\left\| \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\mathcal{G},1}^{\lambda,\gamma,\mu} (f_1, \ldots, f_m) \right\|_{L^{2/m}} \leq \left\| \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\mathcal{G},1}^{\lambda,\gamma,\mu} (f_1, \ldots, f_m) \right\|_{L^{2/m}}
\]
\[
\lesssim \left\{ b_{\mathcal{G},k}^{\lambda,\mu} \right\} \left\| \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\mathcal{G},1}^{\lambda,\gamma,\mu} (f_1, \ldots, f_m) \right\|_{L^{2/m}} \left( \sum_{\gamma \in \mathbb{Z}} \left\| f_1^{\lambda,\gamma,\mu} \right\|_{L^2} \right)^{1/2} \prod_{j=1}^m \left\| f_j \right\|_{L^2} 
\]
\[
\lesssim 2^{-\delta \mu} 2^{-\lambda(M+1+mn/2)} \left\| \Omega \right\|_{L^q(S^m)} \left( \sum_{\gamma \in \mathbb{Z}} \left\| f_1^{\lambda,\gamma,\mu} \right\|_{L^2} \right)^{1/2} \prod_{j=1}^m \left\| f_j \right\|_{L^2}.
\]
Since $f_1$ is a Schwartz function, we have
\[
\left\| f_1^{\lambda,\gamma,\mu} \right\|_{L^2} = \left\| f_1^{\lambda,\gamma,\mu} \right\|_{L^2} = \left( \int_{\mathbb{R}^n} \left| \mathcal{F}_{f_1}^{\gamma,\lambda,\mu} (\xi) \right|^2 d\xi \right)^{1/2} 
\]
\[
\lesssim N \begin{cases} 2^{(\gamma+\mu)n/2}, & \gamma < 0 \\ 2^{-(\gamma-\lambda)(N-n/2)}, & \gamma \geq 0 \end{cases}
\]
for sufficiently large $N > n/2$, which yields that
\[
\sum_{\gamma \in \mathbb{Z}} \left\| f_1^{\lambda,\gamma,\mu} \right\|_{L^2}
\]
is finite (of course, this depends on $\gamma$, $\mu$, and $f_1$). Therefore, we also have
\[
(41) \quad \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\mathcal{G},1}^{\lambda,\gamma,\mu} (f_1, \ldots, f_m) \in L^{2/m}
\]
and thus the polynomial $Q^{\lambda,\mu,\hat{G}}$ in (39) should be zero. In conclusion,
\[
\left\| \sum_{\gamma \in \mathbb{Z}} \mathcal{T}_{\mathcal{G},1}^{\lambda,\gamma,\mu} (f_1, \ldots, f_m) \right\|_{L^{2/m}}
\[ \| \Omega \|_{L^q(S^{m-1})} 2^{-\epsilon_0} 2^{-\lambda M_0} \prod_{j=1}^m \| f_j \|_{L^2}. \]

This proves (36) for \( l = 1 \).

6.2. The case \( 2 \leq l \leq m \). We apply (34), Proposition 2.4 with \( 2 < q' < \frac{2m}{m-1} \), (35), (29), and (30) to obtain that

\[
\begin{align*}
\bigg\| \sum_{\gamma \in \mathbb{Z}} T_{G,l}^{\lambda, \gamma, \mu} (f_1, \ldots, f_m) \bigg\|_{L^2/m} & \leq \bigg\| \sum_{\gamma \in \mathbb{Z}} T_{G,l}^{\lambda, \gamma, \mu} (f_1, \ldots, f_m) \bigg\|_{L^2/m} \\
& \lesssim \bigg\| \{ b_{G,k}^{\lambda, \gamma, \mu} \}_{\gamma \in \mathbb{Z}} \bigg\|_{L^2} \bigg\| b_{G,k}^{\lambda, \mu} \bigg\|_{L^2} \prod_{j=1}^m \| f_j \|_{L^2} \\
& \lesssim \| \Omega \|_{L^q(S^{m-1})} 2^{-\delta \mu (1 - \frac{(m-1)q' - 2m}{2m})} \prod_{j=1}^m \| f_j \|_{L^2} \\
& \lesssim \bigg\| \Omega \bigg\|_{L^q(S^{m-1})} 2^{-\delta \mu (1 - \frac{(m-1)q'}{2m})} \prod_{j=1}^m \| f_j \|_{L^2}.
\end{align*}
\]

(42)

where

\[
C_{M,m,n,q} := (M + 1 + mn)(1 - \frac{(m-1)q'}{2m}) + mn(1/q - 1/2) \left( \frac{m-1}{2m} \right) - \frac{mn}{2}.
\]

Here we used the embedding \( L^q \hookrightarrow L^\infty \) and the fact that \( \frac{m-1}{2m} \leq \frac{m-1}{2m}. \) Then (36) follows from choosing \( M \) sufficiently large so that \( C_{M,m,n,q} > 0 \) since \( 1 - \frac{(m-1)q'}{2m} > 0. \)

7. Proof of Theorem 1.3

The strategy in this section is similar to that used in handling multilinear rough singular integrals in Section 6, but the decomposition is more delicate. We describe the decomposition first. Write

\[
\sigma(\xi) = \sum_{\gamma \in \mathbb{Z}} \sigma_\gamma(\xi/2^\gamma)
\]

where \( \sigma_\gamma(\xi) := \sigma(2^\gamma \xi(\xi) \Phi(\xi) \big( \xi \big)). \) Clearly,

\[
\text{Supp}(\sigma_\gamma) \subset \{ \xi \in (\mathbb{Z}^n)^m : 1/2 \leq |\xi| \leq 2 \}
\]

and according to (21),

\[
\sigma_\gamma(\xi) = \sum_{\lambda \in \mathbb{N}_0} \sum_{G \in \mathcal{T}^\lambda} \sum_{k \in (\mathbb{Z}^n)^m} b_{G,k}^{\lambda, \gamma, \mu} \Psi_{G_1, k_1}(\xi_1) \cdots \Psi_{G_m, k_m}(\xi_m)
\]

where \( b_{G,k}^{\lambda, \gamma, \mu} := \int_{(\mathbb{R}^n)^m} \sigma_\gamma(\xi) \Psi_{G,k}(\xi) \Phi(\xi) \big( \xi \big) d\xi. \) Moreover, it follows from (22) that for \( 1 < q < \infty \) and \( s \geq 0 \)

\[
\bigg\| \{ b_{G,k}^{\lambda, \gamma, \mu} \}_{k \in (\mathbb{Z}^n)^m} \bigg\|_{L^q} \lesssim 2^{-\lambda(s-mn/q+mn/2)} \bigg\| \sigma(2^\gamma \xi) \Phi(\xi) \big( \xi \big) \bigg\|_{L^q((\mathbb{R}^n)^m)}.
\]

(45)
As we did in the proof of Theorem 1.2, it is enough to consider only the case $|k_1| \geq \cdots \geq |k_m|$. Therefore, we replace $\bar{k} \in (\mathbb{Z}^n)^m$ in (44) by $\bar{k} \in \mathcal{U} := \{\bar{k} \in (\mathbb{Z}^n)^m : |k_1| \geq \cdots \geq |k_m|\}$ and write

$$
\sigma_{\gamma}^{(1)}(\vec{\xi}) = \sum_{\lambda \in \mathbb{N}_0 : 2^\lambda \geq 2^6C_0\sqrt{n}} \sum_{G \in \Lambda^\lambda} \sum_{\bar{k} \in \mathcal{U}} b_{G,\bar{k}}^{\lambda,\gamma} \psi_{G_{1,k_1}}^\lambda(\xi_1) \cdots \psi_{G_{m,k_m}}^\lambda(\xi_m)
$$

$$+ \sum_{\lambda \in \mathbb{N}_0 : 2^\lambda < 2^6C_0\sqrt{n}} \sum_{G \in \Lambda^\lambda} \sum_{\bar{k} \in \mathcal{U}} b_{G,\bar{k}}^{\lambda,\gamma} \psi_{G_{1,k_1}}^\lambda(\xi_1) \cdots \psi_{G_{m,k_m}}^\lambda(\xi_m)
$$

$$=: \sigma_{\gamma}^{(1)}(\vec{\xi}) + \sigma_{\gamma}^{(2)}(\vec{\xi}).$$

We are only concerned with $\sigma_{\gamma}^{(1)}$ as a similar and simpler argument is applicable to the other one since the sum over $\lambda$ in $\sigma_{\gamma}^{(2)}$ is finite sum.

If $2^6C_0\sqrt{n} \leq 2^\lambda$, then $b_{G,\bar{k}}^{\lambda,\gamma}$ vanishes unless $2^{\lambda-2} \leq |\bar{k}| \leq 2^{\lambda+2}$ due to (43) and the compact support of $\psi_{G_{\bar{k}}}$. Thus, letting

$$\mathcal{U}^\lambda := \{\bar{k} \in \mathcal{U} : 2^{\lambda-2} \leq |\bar{k}| \leq 2^{\lambda+2}\},$$

we write

$$\sigma_{\gamma}^{(1)}(\vec{\xi}) = \sum_{\lambda : 2^\lambda \geq 2^6C_0\sqrt{n}} \sum_{G \in \Lambda^\lambda} \sum_{\bar{k} \in \mathcal{U}^\lambda} b_{G,\bar{k}}^{\lambda,\gamma} \psi_{G_{1,k_1}}^\lambda(\xi_1) \cdots \psi_{G_{m,k_m}}^\lambda(\xi_m).$$

Now we split $\mathcal{U}^\lambda$ into $m$ disjoint subsets

$$\mathcal{U}_1^\lambda := \{\bar{k} \in \mathcal{U}^\lambda : |k_1| \geq 2C_0\sqrt{n} > |k_2| \geq \cdots \geq |k_m|\},$$

$$\mathcal{U}_2^\lambda := \{\bar{k} \in \mathcal{U}^\lambda : |k_1| \geq |k_2| \geq 2C_0\sqrt{n} > |k_3| \geq \cdots \geq |k_m|\},$$

$$\vdots$$

$$\mathcal{U}_m^\lambda := \{\bar{k} \in \mathcal{U}^\lambda : |k_1| \geq \cdots \geq |k_m| \geq 2C_0\sqrt{n}\}$$

and accordingly,

$$\sigma_{\gamma}^{(1)}(\vec{\xi}) = \sum_{l=1}^m \sigma_{\gamma,l}^{(1)}(\vec{\xi})$$

where

$$\sigma_{\gamma,l}^{(1)}(\vec{\xi}) := \sum_{\lambda : 2^\lambda \geq 2^6C_0\sqrt{n}} \sum_{G \in \Lambda^\lambda} \sum_{\bar{k} \in \mathcal{U}_l^\lambda} b_{G,\bar{k}}^{\lambda,\gamma} \psi_{G_{1,k_1}}^\lambda(\xi_1) \cdots \psi_{G_{m,k_m}}^\lambda(\xi_m).$$

Then it is enough to show that for each $1 \leq l \leq m$

$$\left\| \sum_{\lambda : 2^\lambda \geq 2^6C_0\sqrt{n}} \sum_{G \in \Lambda^\lambda} \sum_{\bar{k} \in \mathcal{U}_l^\lambda} b_{G,\bar{k}}^{\lambda,\gamma} \left( \prod_{j=1}^m L_{G_{j,k_j}}^{\lambda,\gamma} f_j \right) \right\|_{L^{2/m}} \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^l \cdot) \Phi^{(m)}(\vec{f}) \right\|_{L^q_\chi((\mathbb{R}^n)^m)} \prod_{j=1}^m \| f_j \|_{L^2}$$

(46)
Then, due to (47), the left-hand side of (46) is less than
\[
C_0 \sqrt{n} \leq |k| - C_0 \sqrt{n} \leq 2^{\lambda - \gamma} |\xi| \leq |k| + C_0 \sqrt{n} \leq 2^{\lambda + 2} + C_0 \sqrt{n} \leq 2^{\lambda + 3},
\]
which implies
\[
(47) \quad L_{G,k}^{\lambda,\gamma} f(x) = L_{G,k}^{\lambda,\gamma} f^{\lambda,\gamma}(x)
\]
where \( f^{\lambda,\gamma} := (\mathcal{F} \chi_{C_0 \sqrt{n} 2^{\gamma-\lambda} \leq |\cdot| \leq 2^{\gamma+1}})^\vee \). Furthermore, a direct computation with Plancherel’s identity proves
\[
(48) \quad \left( \sum_{y \in \mathbb{Z}} \| f^{\lambda,\gamma} \|_{L^2}^2 \right)^{1/2} \lesssim C_0 (\lambda + 3)^{1/2} \| f \|_{L^2}.
\]
Let
\[
\Sigma_{1,\hat{G}}^{\lambda,\gamma}(f_1, \ldots, f_m)(x) := \sum_{k \in \mathcal{U}_k^l} b_{G,k}^{\lambda,\gamma} \left( \prod_{j=1}^l L_{G,k}^{\lambda,\gamma} f_{j}(x) \right) \left( \prod_{j=l+1}^m L_{G,k}^{\lambda,\gamma} f_{j}(x) \right).
\]
Then, due to (47), the left-hand side of (46) is less than
\[
(49) \quad \left( \sum_{\lambda : 2^\lambda \geq 2^8 C_0 m \sqrt{n}} \sum_{G \in \mathcal{I}^\lambda} \left\| \sum_{y \in \mathbb{Z}} \Sigma_{1,\hat{G}}^{\lambda,\gamma}(f_1, \ldots, f_m) \right\|_{L^2/m}^{2/m} \right)^{m/2}.
\]
We claim that for \( 1 \leq l \leq m \) there exists a constant \( C > 0 \) such that
\[
(50) \quad \left\| \sum_{y \in \mathbb{Z}} \Sigma_{1,\hat{G}}^{\lambda,\gamma}(f_1, \ldots, f_m) \right\|_{L^2/m} \leq C 2^{-\lambda (s - \max \left( \frac{(m-1)n}{2}, \frac{mn}{q} \right))} (\lambda + 3)^m
\]
\[
\times \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j, x) \hat{\Phi}(m) \right\|_{L^q((\mathbb{R}^n)^m)} \left( \prod_{j=1}^m \| f_j \|_{L^2} \right),
\]
which clearly implies that (49) is majorized by the right-hand side of (46) as
\[
\left( \sum_{\lambda : 2^\lambda \geq 2^8 C_0 m \sqrt{n}} 2^{-\lambda (s - \max \left( \frac{(m-1)n}{2}, \frac{mn}{q} \right))} (\lambda + 3)^2 \right)^{m/2} < \infty,
\]
which is due to the assumption \( s > \max \left( \frac{(m-1)n}{2}, \frac{mn}{q} \right) \).

Therefore, let us prove (50).

7.1. The case \( l = 1 \). We utilize the Littlewood-Paley theory for Hardy spaces as in Section 6. There exists a unique polynomial \( Q^{\lambda,\hat{G}}(x) \) such that
\[
\left\| \sum_{y \in \mathbb{Z}} \Sigma_{1,\hat{G}}^{\lambda,\gamma}(f_1, \ldots, f_m) - Q^{\lambda,\hat{G}} \right\|_{L^2/m}
\]
Moreover, since
and this proves that
Furthermore, Proposition 2.4, together with (52), yields that
\( \text{Supp}(\Psi_{G_j,k_j}(\cdot/2^2)) \subset \{ \xi \in \mathbb{R}^n : 2^j - 2 \leq |\xi| \leq 2^{j+3} \}. \)
Moreover, since \(|k_j| \leq 2C_0 \sqrt{n} \) for \( 2 \leq j \leq m, \)
\( \text{Supp}(\Psi_{G_j,k_j}(\cdot/2^2)) \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{-6} m^{-1} 2^j \}. \)
Therefore, the Fourier transform of \( \mathcal{X}_{1,G}^\lambda \gamma(f_1, \ldots, f_m) \) is supported in the set
\( \{ \xi \in \mathbb{R}^n : 2^j - 5 \leq |\xi| \leq 2^{j+4} \} \) and the technique of (37) yields that the right-hand side of (51) is dominated by a constant times
\[
\left\| \left( \sum_{\gamma \in \mathbb{Z}} |\mathcal{X}_{1,G}^\lambda \gamma(f_1, \ldots, f_m)|^2 \right)^{1/2} \right\|_{L^{2/m}}.
\]
The \( L^{2/m} \)-norm is bounded by
\[
\sup_{\gamma \in \mathbb{Z}} \left\| \left( b_{G,k}^\lambda \gamma(f_1, \ldots, f_m) \right) \right\|_{L^{2\lambda mn/2}} \leq 2^\lambda \| \mathcal{F} \|_{L^2} \prod_{j=2}^m \| f_j \|_{L^2}
\]
thanks to Proposition 2.4. The embedding \( \ell^q \hookrightarrow \ell^\infty \) and (45) imply
\[
\sup_{\gamma \in \mathbb{Z}} \left\| \left( b_{G,k}^\lambda \gamma(f_1, \ldots, f_m) \right) \right\|_{L^{2\lambda mn/2}} \leq 2^{-\lambda(s - mn/q + mn/2)} \sup_{\gamma \in \mathbb{Z}} \| \sigma(2^j \gamma) \hat{\Phi}(m) \|_{L^q((\mathbb{R}^n)^q)}.
\]
This, together with (48), finally proves that the left-hand side of (51) is dominated by a constant multiple of
\[
2^{-\lambda(s - mn/q) + 3/2} \sup_{\gamma \in \mathbb{Z}} \| \sigma(2^j \gamma) \hat{\Phi}(m) \|_{L^q((\mathbb{R}^n)^q)} \prod_{j=1}^m \| f_j \|_{L^2}
\]
and accordingly,
\[
\sum_{\gamma \in \mathbb{Z}} \mathcal{X}_{1,G}^\lambda \gamma(f_1, \ldots, f_m) - \mathcal{Q}^\lambda \mathcal{G} \in L^{2/m}.
\]
Moreover, Proposition 2.4, together with (52), yields that
\[
\left\| \left( \sum_{\gamma \in \mathbb{Z}} \mathcal{X}_{1,G}^\lambda \gamma(f_1, \ldots, f_m) \right) \right\|_{L^{2/m}} \leq \left\| \left( \sum_{\gamma \in \mathbb{Z}} |\mathcal{X}_{1,G}^\lambda \gamma(f_1, \ldots, f_m)| \right) \right\|_{L^{2/m}} \leq 2^{-\lambda(s - mn/q)} \left( \sum_{\gamma \in \mathbb{Z}} \| f_1^\lambda \gamma \|_{L^2} \right) \prod_{j=2}^m \| f_j \|_{L^2}.
and, similarly to (40), we have

$$
\|f_1^{\lambda, \gamma}\|_{L^2} = \left[ \int_{|x| \leq 2^{\gamma+3}} |\hat{f}_1(\xi)|^2 \, d\xi \right]^{1/2} \lesssim N \begin{cases} 
2^{(\gamma+3)n/2}, & \gamma < 0 \\
2^{-(\gamma-\lambda)(N-n/2)}, & \gamma \geq 0 
\end{cases}
$$

for sufficiently large $N > n/2$. Using the argument that led to (41), we see that

$$
\sum_{\gamma \in \mathbb{Z}} \sigma_{\mathcal{L}, \gamma}^\lambda(f_1, \ldots, f_m) \in L^2/m
$$

and thus $Q^{\lambda, \tilde{G}} = 0$. Then the inequality (50) for $l = 1$ follows.

7.2. The case $2 \leq l \leq m$. If $0 < q < \frac{2l}{l+1}$, we simply apply Proposition 2.4 to have

$$
\left\| \sum_{\gamma \in \mathbb{Z}} \sigma_{\mathcal{L}, \gamma}^\lambda(f_1, \ldots, f_m) \right\|_{L^2/m}
\lesssim \sup_{\gamma \in \mathbb{Z}} \| \{ b^\lambda, \gamma \}_{\mathcal{L}, k}^{(x)} \|_{L^2} \left[ \prod_{j=1}^l \left( \sum_{\gamma \in \mathbb{Z}} \| f_j^{\lambda, \gamma} \|_{L^2}^2 \right)^{1/2} \right] \left[ \prod_{j=l+1}^m \| f_j \|_{L^2} \right]
$$

where the embedding $\ell^q \hookrightarrow \ell^\infty$ is applied. Then the last expression is no more than a constant multiple of

$$
2^{-\lambda(s-mn/q)}(\lambda + 3)^{l/2} \sup_{j \in \mathbb{Z}} \| \sigma(2^{j-\gamma}) \Phi(\gamma) \|_{L^q((\mathbb{R}^n)^m)} \prod_{j=1}^m \| f_j \|_{L^2}
$$

by using (45) and (48). Then (50) follows.

If $\frac{2l}{l+1} \leq q < \infty$, using the argument in proving (20), we obtain

$$
\left| \sum_{\gamma \in \mathbb{Z}} \sigma_{\mathcal{L}, \gamma}^\lambda(f_1, \ldots, f_m)(x) \right|
\lesssim 2^{\lambda(m-l)n/2} \left( \prod_{j=l+1}^m \mathcal{M} f_j(x) \right)
$$

$$
\times \sum_{k_1, \ldots, k_l \in \mathcal{P}_{1, \ldots, l}} \sum_{k_{l+1, \ldots, m} \in \mathcal{U}^l_{k_{l+1, \ldots, m}}} \left| \sum_{\gamma \in \mathbb{Z}} b^\lambda, \gamma \left( \prod_{j=1}^l \mathcal{L}^{\lambda, \gamma}_{\mathcal{L}, k_j} \left( f_j^{\lambda, \gamma} (\cdot / 2^\gamma) \right) (2^\gamma x) \right) \right|
$$

and then Hölder’s inequality and the $L^2$ boundedness of $\mathcal{M}$ yield that

$$
\left\| \sum_{\gamma \in \mathbb{Z}} \sigma_{\mathcal{L}, \gamma}^\lambda(f_1, \ldots, f_m) \right\|_{L^2/m}
\lesssim 2^{\lambda(m-l)n/2} \left( \prod_{j=l+1}^m \| f_j \|_{L^2} \right)
$$

$$
\times \sum_{k_1, \ldots, k_l \in \mathcal{P}_{1, \ldots, l}} \sum_{k_{l+1, \ldots, m} \in \mathcal{U}^l_{k_{l+1, \ldots, m}}} \left\| \sum_{\gamma \in \mathbb{Z}} b^\lambda, \gamma \left( \prod_{j=1}^l \mathcal{L}^{\lambda, \gamma}_{\mathcal{L}, k_j} \left( f_j^{\lambda, \gamma} (\cdot / 2^\gamma) \right) (2^\gamma x) \right) \right\|_{L^2/m}
$$
where we used the fact \(|P_{*1,\ldots,j}U^\lambda| \lesssim 1\). Due to Proposition 2.3, a change of variables, and (45), the \(L^{2/l}\)-norm is less than

\[
\left( \sum_{\gamma \in \mathbb{Z}} \left\| \frac{b_{G,k}(\prod_{j=1}^l L_{G_j,k_j}(f_j^\lambda \cdot \gamma \cdot (\cdot / 2^\gamma)))} {L_{2/l}} \right\|^{2/l} \right)^{1/2} = \left( \sum_{\gamma \in \mathbb{Z}} 2^{-\gamma} \left\| \frac{b_{G,k}(\prod_{j=1}^l L_{G_j,k_j}(f_j^\lambda \cdot \gamma \cdot (\cdot / 2^\gamma)))} {L_{2/l}} \right\|^{2/l} \right)^{1/2} \lesssim E_{q,l,\lambda} 2^{-\lambda(s-mn/q+mn/2)} 2^{\lambda ln/2} \sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot \gamma) \Phi^j(m) \|_{L^q_2((\mathbb{R}^n)^m)} \times \left( \sum_{\gamma \in \mathbb{Z}} 2^{-\gamma} \prod_{j=1}^l \| f_j^\lambda \cdot \gamma \cdot (\cdot / 2^\gamma) \|_{L^2_2} \right)^{1/2} \]

where

\[
E_{q,l,\lambda} := \begin{cases} 
\frac{\lambda^{1/2}}{2}, & q = \frac{2l}{l-1} \\
2^{\lambda(n(2-l)/q-1/2)}, & q > \frac{2l}{l-1}.
\end{cases}
\]

Since

\[
\left( \sum_{\gamma \in \mathbb{Z}} 2^{-\gamma} \prod_{j=1}^l \| f_j^\lambda \cdot \gamma \cdot (\cdot / 2^\gamma) \|_{L^2_2} \right)^{1/2} \leq \prod_{j=1}^l \left( \sum_{\gamma \in \mathbb{Z}} 2^{-\gamma} \| f_j^\lambda \cdot \gamma \cdot (\cdot / 2^\gamma) \|_{L^2_2} \right)^{1/2} \leq \prod_{j=1}^l \left( \sum_{\gamma \in \mathbb{Z}} \| f_j^\lambda \cdot \gamma \|_{L^2_2} \right)^{1/2} \lesssim (\lambda + 3)^{1/2} \prod_{j=1}^l \| f_j \|_{L^2_2},
\]

we finally obtain that

\[
\left\| \sum_{\gamma \in \mathbb{Z}} \sigma_{l,G}^\lambda(f_1, \ldots, f_m) \right\|_{L^2/m} \lesssim F_{q,l,\lambda}^{(s,m,n)} \sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot \gamma) \Phi^j(m) \|_{L^q_2((\mathbb{R}^n)^m)} \prod_{j=1}^m \| f_j \|_{L^2}\]

where

\[
F_{q,l,\lambda}^{(s,m,n)} := E_{q,l,\lambda} 2^{-\lambda(s-mn/q)} (\lambda + 3)^{1/2}.
\]

It is easy to check that for \(2 \leq l \leq m\) and \(\frac{2l}{l-1} \leq q\)

\[
F_{q,l,\lambda}^{(s,m,n)} \lesssim 2^{-\lambda(s-\max\left(\frac{(m-1)n}{2}, \frac{mn}{q}\right))} (\lambda + 3)^m
\]

and the proof of (50) is complete.
8. CONCLUDING REMARKS

In this article we focused on the $L^2 \times \cdots \times L^2 \to L^{2/m}$ boundedness for several fundamental $m$-linear operators. In future work we plan to obtain similar initial estimates for maximal singular integrals and maximal multipliers.

The $L^2 \times \cdots \times L^2$ estimates obtained in this paper provide crucial initial bounds that provide the cornerstone needed to launch a complete boundedness study on general products of Lebesgue spaces. Certainly our initial estimates can be extended to include points obtained by duality and interpolation; these are called local $L^2$ points. For the remaining points there are techniques available, for instance, interpolation between dyadic pieces of an operator between good local $L^2$ points and bad points near the boundary of the region $1 < p_1, \ldots, p_m < \infty$, $1/m < p < \infty$; this technique was developed in [16] in the bilinear case. We chose not to pursue this line of investigation here in order to direct our focus on the idea of wavelet expansions and shorten the exposition. We plan to pursue general $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ boundedness for many multilinear operators in subsequent work. It should be mentioned that in a recent manuscript of Heo, Lee, Hong, Yang, Lee, and Park [23] the extension to the full range of indices was obtained for Theorem 1.3, when $q = 2$, although the case of general $q$ remains unresolved.

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