\[ s_m + t_m = \sum_{n=0}^{\infty} a_n = s \implies t_m = s - s_m. \]

Let \( m \to \infty \) and use \( s_m - s \to 0 \) to obtain that \( t_m \to 0 \), as desired.

**Definition 1.5.19.** A complex series \( \sum_{n=0}^{\infty} a_n \) is said to be **absolutely convergent** if the series \( \sum_{n=0}^{\infty} |a_n| \) is convergent.

A well-known consequence of the completeness property of real numbers is that every bounded monotonic sequence (increasing or decreasing) converges. Since the partial sums of a series with nonnegative terms are increasing, we conclude that if these partial sums are bounded, then the series is convergent. Thus, if for a complex series we have \( \sum_{n=1}^{N} |a_n| \leq M \) for all \( N \), then the series \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.

Recall that, for series with real terms, absolute convergence implies convergence. The same is true for complex series.

**Theorem 1.5.20.** Absolutely convergent series are convergent, i.e., for \( a_n \in \mathbb{C} \)

\[ \sum_{n=0}^{\infty} |a_n| < \infty \implies \sum_{n=0}^{\infty} a_n \text{ converges.} \]

**Proof.** Let \( s_n = a_0 + a_1 + \cdots + a_n \) and \( v_n = |a_0| + |a_1| + \cdots + |a_n| \). By Theorem 1.5.11, it is enough to show that the sequence of partial sums \( \{s_n\}_{n=0}^{\infty} \) is Cauchy.

For \( n > m \geq 0 \), using the triangle inequality, we have

\[ |s_n - s_m| = \left| \sum_{j=m+1}^{n} a_j \right| \leq \sum_{j=m+1}^{n} |a_j| = v_n - v_m. \]

Since \( \sum_{n=0}^{\infty} |a_n| \) converges, the sequence \( \{v_n\}_{n=0}^{\infty} \) converges and hence it is Cauchy. Thus, given \( \epsilon > 0 \) we can find \( N \) so that, \( v_n - v_m < \epsilon \) for \( n > m \geq N \), implying that \( |s_n - s_m| < \epsilon \) for \( n > m \geq N \). Hence \( \{s_n\}_{n=0}^{\infty} \) is a Cauchy sequence.

For a complex series \( \sum_{n=0}^{\infty} a_n \), consider the series \( \sum_{n=0}^{\infty} |a_n| \) whose terms are real and nonnegative. If we can establish the convergence of the series \( \sum_{n=0}^{\infty} |a_n| \) using one of the tests of convergence for series with nonnegative terms, then using Theorem 1.5.20, we can infer that the series \( \sum_{n=0}^{\infty} a_n \) is convergent. Thus, all known tests of convergence for series with nonnegative terms can be used to test the (absolute) convergence of complex series. We list a few such convergence theorems.

**Theorem 1.5.21.** Suppose that \( a_n \) are complex numbers, \( b_n \) are real numbers, \( |a_n| \leq b_n \) for all \( n \geq n_0 \), and \( \sum_{n=0}^{\infty} b_n \) is convergent. Then \( \sum_{n=0}^{\infty} a_n \) is absolutely convergent.

**Proof.** By the comparison test for real series, we have that \( \sum_{n=0}^{\infty} |a_n| \) is convergent. By Theorem 1.5.20, it follows that \( \sum_{n=0}^{\infty} a_n \) is convergent.