it suffices to obtain $L^2 \to L^{2,\infty}$ bounds for the one-sided maximal operators

$$C_1(f)(x) = \sup_{N>0} \left| \int_{-\infty}^{N} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \right|,$$

$$C_2(f)(x) = \sup_{N>0} \left| \int_{-\infty}^{-N} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \right|,$$

acting on a Schwartz function $f$ (with bounds independent of $f$). Note that

$$C_2(f)(x) \leq |f(x)| + C_1(\tilde{f})(-x),$$

where $\tilde{f}(x) = f(-x)$ is the usual reflection operator. Therefore, it suffices to obtain bounds only for $C_1$.

For $a > 0$ and $y \in \mathbb{R}$ we define the translation operator $\tau^y$, the modulation operator $M^a$, and the dilation operator $D^a$ as follows:

$$\tau^y(f)(x) = f(x-y),$$

$$D^a(f)(x) = a^{-\frac{1}{2}} f(a^{-1}x),$$

$$M^y(f)(x) = f(x)e^{2\pi iyx}.$$  

These operators are isometries on $L^2(\mathbb{R})$.

We break down the proof of Theorem 6.1.1 into several steps.

### 6.1.1 Preliminaries

We denote rectangles of area 1 in the $(x, \xi)$ plane by $s, t, u$, etc. All rectangles considered in the sequel have sides parallel to the axes. We think of $x$ as the time coordinate and of $\xi$ as the frequency coordinate. For this reason we refer to the $(x, \xi)$ coordinate plane as the time–frequency plane. The projection of a rectangle $s$ on the time axis is denoted by $I_s$, while its projection on the frequency axis is denoted by $\omega_s$. Thus a rectangle $s$ is just $s = I_s \times \omega_s$. Rectangles with sides parallel to the axes and area equal to one are called tiles.

The center of an interval $I$ is denoted by $c(I)$. Also for $a > 0$, $aI$ denotes an interval with the same center as $I$ whose length is $a|I|$. Given a tile $s$, we denote by $s(1)$ its bottom half and by $s(2)$ its upper half defined by

$$s(1) = I_s \times (\omega_s \cap (-\infty, c(\omega_s))), \quad s(2) = I_s \times (\omega_s \cap [c(\omega_s), +\infty)).$$

These sets are called semi-tiles. The projections of these sets on the frequency axes are denoted by $\omega_s(1)$ and $\omega_s(2)$, respectively. See Figure 6.1.

A dyadic interval is an interval of the form $[m2^k, (m+1)2^k)$, where $k$ and $m$ are integers. We denote by $D$ the set of all rectangles $I \times \omega$ with $I, \omega$ dyadic intervals and $|I||\omega| = 1$. Such rectangles are called dyadic tiles.
all dyadic tiles. For every integer \( m \), we denote by \( D_m \) the set of all tiles \( s \in D \) such that \( |I_s| = 2^m \). We call these dyadic tiles of scale \( m \).

**Fig. 6.1** The lower and the upper parts of a tile \( s \).

We fix a Schwartz function \( \phi \) such that \( \hat{\phi} \) takes values in \([0, 1]\) and supported in the interval \([-1/10, 1/10]\), and equal to 1 on the interval \([-9/100, 9/100]\). For each tile \( s \), we introduce a function \( \phi_s \) as follows:

\[
\varphi_s(x) = |I_s|^{-\frac{1}{2}} \varphi \left( \frac{x - c(I_s)}{|I_s|} \right) e^{2\pi i c(\omega_s(1)) x}. \tag{6.1.4}
\]

This function is localized in frequency near \( c(\omega_s(1)) \). Using the previous notation, we have

\[
\varphi_s = M_{c(\omega_s(1))} \tau_{c(I_s)} D_{|I_s|} (\varphi).
\]

Observe that

\[
\hat{\varphi}_s(\xi) = |\omega_s|^{-\frac{1}{2}} \hat{\varphi} \left( \frac{\xi - c(\omega_s(1))}{|\omega_s|} \right) e^{2\pi i c(\omega_s(1)) - \xi c(I_s)}, \tag{6.1.5}
\]

from which it follows that \( \hat{\varphi}_s \) is supported in \( \frac{2}{5}|\omega_s(1)| \). Also observe that the functions \( \varphi_s \) have the same \( L^2(\mathbb{R}) \) norm.

Recall the complex inner product notation for \( f, g \in L^2(\mathbb{R}) \):

\[
\langle f \mid g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} \, dx. \tag{6.1.6}
\]

Given a real number \( \xi \) and \( m \in \mathbb{Z} \), we introduce an operator

\[
A_{\xi}^m(f) = \sum_{s \in D_m} \chi_{\omega_s(2)}(\xi) \langle f \mid \varphi_s \rangle \varphi_s, \tag{6.1.7}
\]

for functions \( f \in \mathcal{S}(\mathbb{R}) \). The series in (6.1.7) converges absolutely and in \( L^2 \) for \( f \) in the Schwartz class (see Exercise 6.1.9) and thus \( A_{\xi}^m \) is well defined on \( \mathcal{S}(\mathbb{R}) \). Note that for a fixed \( m \), the sum in (6.1.7) is taken over the row of dyadic rectangles of size \( 2^m \times 2^{-m} \) whose tops contain the horizontal line at height \( \xi \). The Fourier transforms of the operators \( A_{\xi}^m \) are supported in a horizontal strip contained in \(( -\infty, \xi ] \) of width \( \frac{2}{5}2^{-m} \). Notice that if the characteristic function were missing in (6.1.7), then for a suitable function \( \varphi \), the sum would be equal to a multiple of \( f(x) \); cf. Exercise 6.1.9. Thus for each \( m \in \mathbb{Z} \) the operator \( A_{\xi}^m(f) \) may be viewed as a “piece” of the multiplier operator \( f \mapsto \left( f \chi_{(-\infty, \xi]} \right)^\vee \). Summing over \( m \) yields a better approximation to this half-line multiplier operator.