6.1.2 Vector-Valued Analogues

We now obtain a vector-valued extension of Theorem 6.1.2. We have the following.

Proposition 6.1.4. Let \( \Psi \) be an integrable \( C^1 \) function on \( \mathbb{R}^n \) with mean value zero that satisfies (6.1.3) and let \( \Delta_j \) be the Littlewood–Paley operator associated with \( \Psi \). Then there exists a constant \( C_n < \infty \) such that for all \( 1 < p, r < \infty \) and all sequences of \( L^p \) functions \( f_j \) we have

\[
\left\| \left( \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} |\Delta_k(f_j)|^2 \right)^{\frac{r}{2}} \right)^{\frac{1}{r}} \right\|_{L^p(\mathbb{R}^n)} \leq C_n \tilde{C}_{p,r} \left\| \left( \sum_{j \in \mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(\mathbb{R}^n)},
\]

where \( \tilde{C}_{p,r} = \max(p, (p-1)^{-1}) \max(r, (r-1)^{-1}) \). Moreover, for some \( C'_n > 0 \) and all sequences of \( L^1 \) functions \( f_j \) we have

\[
\left\| \left( \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} |\Delta_k(f_j)|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{r}} \right\|_{L^1(\mathbb{R}^n)} \leq C'_n \max(r, (r-1)^{-1}) \left\| \left( \sum_{j \in \mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^1(\mathbb{R}^n)}.
\]

In particular,

\[
\left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_j(f_j)|^r \right)^{\frac{1}{r}} \right\|_{L^p(\mathbb{R}^n)} \leq C_n \tilde{C}_{p,r} \left\| \left( \sum_{j \in \mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(\mathbb{R}^n)}.
\]

Proof. We introduce Banach spaces \( \mathcal{B}_1 = C \) and \( \mathcal{B}_2 = \ell^2 \) and for \( f \in L^p(\mathbb{R}^n) \) define an operator

\[
\tilde{T}(f) = \{\Delta_k(f)\}_{k \in \mathbb{Z}}.
\]

In the proof of Theorem 6.1.2 we showed that \( \tilde{T} \) has a kernel \( \tilde{K} \) that satisfies condition (6.1.16). Furthermore, \( \tilde{T} \) obviously maps \( L'(\mathbb{R}^n, \mathcal{B}_1) \) to \( L'(\mathbb{R}^n, \mathcal{B}_2) \). Applying Proposition 5.6.4, we obtain the first two statements of the proposition. Restricting to \( k = j \) yields (6.1.22).

6.1.3 \( L^p \) Estimates for Square Functions Associated with Dyadic Sums

Let us pick a Schwartz function \( \Psi \) whose Fourier transform is compactly supported in the annulus \( 2^{-1} \leq |\xi| \leq 2 \) such that (6.1.6) is satisfied. (Clearly (6.1.6) has no chance of being satisfied if \( \hat{\Psi} \) is supported only in the annulus \( 1 \leq |\xi| \leq 2 \).) The Littlewood–Paley operation \( f \mapsto \Delta_j(f) \) represents the smoothly truncated frequency localization of a function \( f \) near the dyadic annulus \( |\xi| \approx 2^j \). Theorem 6.1.2 says that the square function formed by these localizations has \( L^p \) norm comparable to that of the original function. In other words, this square function characterizes the \( L^p \) norm of a function. This is the main feature of Littlewood–Paley theory.