5.6 Vector-Valued Singular Integrals

In view of Corollary 5.6.5, it suffices to check that for such a \( \Phi \), (5.6.18) holds. First observe that in view of the decreasing character of \( \Phi \), we have

\[
\sup_{j} |f| * \Phi_{2^j} \leq M_{\Phi}(|f|) \leq 2^n \sup_{j} |f| * \Phi_{2^j},
\]

and for this reason we choose to work with the easier dyadic maximal operator

\[
M^d_{\Phi}(f) = \sup_{j} |f| * \Phi_{2^j}.
\]

We observe the validity of the simple inequalities

\[
2^{-n} M(f) \leq M(f) \leq \frac{1}{V_n} M_{\Phi}(|f|) \leq \frac{2^n}{V_n} M^d_{\Phi}(|f|). \tag{5.6.26}
\]

If we can show that

\[
\sup_{x \in \mathbb{R}^n \setminus \{0\}} \int_{|x| > 2|y|} |\Phi_{2^j}(x) - \Phi_{2^j}(x)| \, dx = C_n < \infty, \tag{5.6.27}
\]

then (5.6.22) and (5.6.23) are satisfied with \( M^d_{\Phi} \) replacing \( M_{\Phi} \). We therefore turn our attention to (5.6.27). We have

\[
\int_{|x| > 2|y|} \sup_{j \in \mathbb{Z}} |\Phi_{2^j}(x) - \Phi_{2^j}(x)| \, dx \leq \sum_{j \in \mathbb{Z}} \int_{|x| > 2|y|} |\Phi_{2^j}(x) - \Phi_{2^j}(x)| \, dx
\]

\[
\leq \sum_{2^j > |y|} \int_{|x| > 2|y|} |y| \left| \nabla \Phi \left( \frac{x - \theta y}{2^{(n+1)j}} \right) \right| \, dx + \sum_{2^j \leq |y|} \int_{|x| > 2|y|} \left( |\Phi_{2^j}(x) - \Phi_{2^j}(x)| + |\Phi_{2^j}(x)| \right) \, dx
\]

\[
\leq \sum_{2^j > |y|} \int_{|x| > 2|y|} \frac{|y| C_N}{2^{(n+1)j} (1 + |2^{-j} x - \theta y|)^N} \, dx + 2 \sum_{2^j \leq |y|} \int_{|x| > 2|y|} |\Phi_{2^j}(x)| \, dx
\]

\[
\leq \sum_{2^j > |y|} \int_{|x| > 2^{-j}|y|} \frac{|y| C_N}{2^{j} (1 + |x|)^N} \, dx + 2 \sum_{2^j \leq |y|} C_N (2^{-j}|y|)^{-N}
\]

\[
\leq CN \sum_{2^j > |y|} \frac{|y|}{2^j} + C_N
\]

\[
\leq 3C_N,
\]

where \( C_N > 0 \) depends on \( N > n, \theta \in [0,1], \) and \( |x - \theta y| \geq |x||2| \) when \( |x| \geq 2|y| \).

Now apply (5.6.22) and (5.6.23) to \( M^d_{\Phi} \) and use (5.6.26) to obtain the desired vector-valued inequalities.