It is a consequence of Theorem 1.3.4 that the normed spaces \( \mathcal{M}_p \) are nested, that is, for \( 1 \leq p \leq q \leq 2 \) we have

\[
\mathcal{M}_1 \subseteq \mathcal{M}_p \subseteq \mathcal{M}_q \subseteq \mathcal{M}_2 = L^\infty.
\]

Moreover, if \( m \in \mathcal{M}_p \) and \( 1 \leq p \leq 2 \leq p' \), Theorem 1.3.4 gives

\[
\| T_m \|_{L^2 \to L^2} \leq \| T_m \|_{L^p \to L^p} \| T_m \|_{L^{p'} \to L^{p'}} = \| T_m \|_{L^p \to L^{p'}} \tag{2.5.16}
\]

since \( 1/2 = (1/2)/p + (1/2)/p' \). Theorem 1.3.4 also gives that

\[
\| m \|_{\mathcal{M}_q} \leq \| m \|_{\mathcal{M}_p}
\]

whenever \( 1 \leq q \leq p \leq 2 \). Thus the \( \mathcal{M}_p \)'s form an increasing family of spaces as \( p \) increases from 1 to 2.

**Example 2.5.12.** The function \( m(\xi) = e^{2\pi i \xi \cdot b} \) is an \( L^p \) multiplier for all \( b \in \mathbb{R}^n \), since the corresponding operator \( T_m(f)(x) = f(x+b) \) is bounded on \( L^p(\mathbb{R}^n) \). Clearly \( \| m \|_{\mathcal{M}_p} = 1 \).

**Proposition 2.5.13.** For \( 1 \leq p < \infty \), the normed space \( (\mathcal{M}_p, \| \cdot \|_{\mathcal{M}_p}) \) is a Banach space. Furthermore, \( \mathcal{M}_p \) is closed under pointwise multiplication and is a Banach algebra.

**Proof.** It suffices to consider the case \( 1 \leq p \leq 2 \). It is straightforward that if \( m_1, m_2 \) are in \( \mathcal{M}_p \) and \( b \in \mathbb{C} \) then \( m_1 + m_2 \) and \( bm_1 \) are also in \( \mathcal{M}_p \). Observe that \( m_1 m_2 \) is the multiplier that corresponds to the operator \( T_{m_1} T_{m_2} = T_{m_1 m_2} \) and thus

\[
\| m_1 m_2 \|_{\mathcal{M}_p} = \| T_{m_1} T_{m_2} \|_{L^p \to L^p} \leq \| m_1 \|_{\mathcal{M}_p} \| m_2 \|_{\mathcal{M}_p}.
\]

This proves that \( \mathcal{M}_p \) is an algebra. To show that \( \mathcal{M}_p \) is a complete normed space, consider a Cauchy sequence \( m_j \) in \( \mathcal{M}_p \). It follows from (2.5.16) that \( m_j \) is Cauchy in \( L^\infty \), and hence it converges to some bounded function \( m \) in the \( L^\infty \) norm; moreover all the \( m_j \) are a.e. bounded by some constant \( C \) uniformly in \( j \). We have to show that \( m \in \mathcal{M}_p \). Fix \( f \in \mathcal{M}_p \). We have

\[
T_{m_j}(f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) m_j(\xi) e^{2\pi i \xi \cdot x} d\xi \to \int_{\mathbb{R}^n} \hat{f}(\xi) m(\xi) e^{2\pi i \xi \cdot x} d\xi = T_m(f)(x)
\]

a.e. by the Lebesgue dominated convergence theorem, since \( C|\hat{f}| \) is an integrable upper bound of all integrands on the left in the preceding expression. Since \( \{ m_j \}_j \) is a Cauchy sequence in \( \mathcal{M}_p \), it is bounded in \( \mathcal{M}_p \), and thus \( \sup_j \| m_j \|_{\mathcal{M}_p} < +\infty \). An application of Fatou’s lemma yields that

\[
\int_{\mathbb{R}^n} |T_m(f)|^p \, dx = \int_{\mathbb{R}^n} \liminf_{j \to \infty} |T_{m_j}(f)|^p \, dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} |T_{m_j}(f)|^p \, dx \leq \| m \|_{\mathcal{M}_p} \| f \|_{L^p}^p,
\]