1. Characterization of Deligne-Mumford stacks

Let K be an arbitrary scheme. The purpose of this section is to prove the following theorem characterizing Deligne-Mumford stacks.

Theorem 0.1. *Let F be a stack over K such that*

1) $\Delta : F \to F \times_K F$ is representable, separated, quasi-compact and unramified.

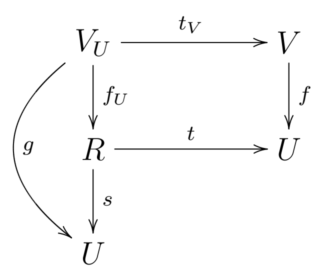
2) There is a scheme U, of finite presentation[1] *over K, and a smooth surjective morphism $U \stackrel{p}{\to} F$*

then F is a Deligne-Mumford stack.

1.1. An étale slice theorem. The key to the proof of Theorem 0.1 is the following "etale slice theorem" for a pair of smooth morphisms.

Notation. We will use the following notation throughout the rest of the section. Let $s, t \colon R \to U$ be a pair of morphisms of K-schemes. Given a morphism $f \colon V \to U$, set $V_U := R \times_{t,U,f} V$ and let $t_V : V_U \to V$ and $f_U \colon V_U \to R$ be the projections. Finally set $g = s \circ f_u$. Then the diagram

(1)



is commutative and the square is cartesian.

Theorem 0.2. *Let R, U be schemes of finite presentation over K. Let $s, t \colon R \to U$ be a pair of smooth morphisms with s also surjective and assume that $R \stackrel{(s,t)}{\to} U \times_K U$ is unramified. Then,with the notation as above, there is a scheme V together with a morphism $f \colon V \to U$ and an open set $B \subset V_U$ such that $t_V(B) = V$ and $g \colon B \to U$ is étale and surjective.*

We begin by stating and proving a local version of Theorem 0.2 for schemes of finite type over $\operatorname{Spec} \mathbb{Z}$.

Lemma 0.3. *Let R and U be schemes of finite type*[2] *over $K = \operatorname{Spec} A$ where A is a finitely generated $\mathbb{Z}$-algebra. Let $s, t \colon R \to U$ be smooth morphisms such that $R \stackrel{(s,t)}{\to} U \times_K U$ is unramified. Then for every closed point $y \in s(U)$ there is a scheme V and a morphism $f \colon V \to U$ such that $g \colon V_U \to U$ is étale at some point $z \in g^{-1}(y)$.*

[1] A morphism of schemes $f \colon X \to Y$ if it is (i) locally of finite presentation, (ii) quasi-compact and (iii) the diagonal $\Delta_f \colon X \to X \times_Y X$ is quasi-compact. Morphisms satisfying (iii) are called *quasi-separated*. Any morphism from a locally Noetherian scheme X is quasi-separated.

[2] Any scheme of finite type over a Noetherian base is automatically of finite presentation.

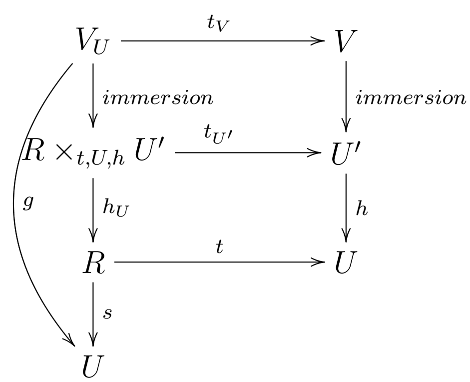
Proof of Lemma 0.3. **Step 1.** Construction of V and z.

Since $R \stackrel{(s,t)}{\to} U \times_K U$ is unramified the map $s^{-1}(y) \stackrel{(s,t)}{\to} y \times_K U$ is unramified. Since U is of finite type over $\operatorname{Spec} \mathbb{Z}$, $k(y)$ is a finite field which must necessarily be a finite, hence separable, extension of the residue field of the image of y in K. Thus the map $y \to K$ is unramified and, by base change so is $y \times_K U \to U$. This means that the the restriction of of t to a morphism $s^{-1}(y) \to U$ is unramified.

Let w be any closed point in $s^{-1}(y)$. By [**EGA4**, Theorem 18.4.7] there is an open neighborhood $W \subset s^{-1}(y)$ containing w such that the unramified map $W \to U$ factors as $W \stackrel{j}{\to} U' \stackrel{h}{\to} U$ with j a closed immersion and h étale.

Since s is a smooth morphism, $s^{-1}(y)$ is a regular scheme, so $\mathcal{O}_{W,w}$ is a regular local ring. Let $t_1, \ldots, t_d$ be a regular system of parameters that generates the maximal ideal of $\mathcal{O}_{W,w}$ and let $g_1, \ldots g_d$ be any sections of $\mathcal{O}_{U'}$ whose images in $\mathcal{O}_{W,w}$ are $t_1, \ldots, t_d$. Let V be the locally closed subscheme of of U' defined by the vanishing of the sections $g_1, \ldots g_d$. and let $f \colon V \to U$ be the restriction of h to V. Since $g_1, \ldots, g_d$ all vanish at w, $j(w) \in V$.

Let $h_U \colon R \times_{t,U,f} U' \to R$ and $t_{U'} \colon R \times_{t,U,f} R \to U'$ be the two projections. Then the diagram



is commutative and the squares are cartesian.

Let $i \colon W \to R$ be the inclusion morphism. Since $W \stackrel{j}{\to} U'$ and $W \stackrel{i}{\to} R$ are immersions such that $h \circ j = t \circ i$, the map $(i,j) \colon W \to R \times_{t,U,h} U'$ is an immersion[3]. Since $j(w) \in V$ then $z = (i,j)(w)$ is a closed point of $V_U = R \times_{t,U,f} V$ such that $g(z) = y$.

[3]This follows because $(i,j) \colon W \to R \times_{t,U,h} U'$ is the composition of the product $i \times j \colon W \times_U W \to R \times_{t,U,h} U'$ with the diagonal $W \to W \times_U W$.

Step 2. Proof that $g\colon V_U \to U$ is étale z.
Consider the diagram of morphisms and fibers

$$\begin{array}{ccccccc} & & & \xrightarrow{(i,j)} & & & \\ & & (s\circ h_U)^{-1}(y) & \subset & R\times_{t,U,h} U' & & \\ & & \downarrow & & \downarrow h_U & & \\ W & \overset{open}{\subset} & s^{-1}(y) & \subset & R & & \\ & & \downarrow & & \downarrow s & & \\ & & y & \in & U & & \end{array}$$

Since $(s\circ h_U)((i,j)(W)) = s(W) = y$, the immersion $(i,j)\colon W \to R\times_{t,U,h} U'$ factors through the inclusion $(s\circ h_U)^{-1}(y) \subset R$. Thus (i,j) defines a section of the étale morphism $h_U^{-1}(W) \to W$. (Recall that h_U is étale since it is obtained by base change from the the étale morphism $h\colon U' \to U$.) Since a section of an étale morphism is an open immersion [**EGA4**, Cor 17.9.3] $(i,j)(W)$ is open in $(s\circ h_U)^{-1}(y)$.

By construction, V_U is the subscheme of $R\times_{t,U,h} U'$ cut out by the local equations $t^*_{U'}g_1, \dots t^*_{U'}g_d$. Under the identification $\mathcal{O}_{(s\circ h_U)^{-1}(y),z} = \mathcal{O}_{W,w}$ these equations restrict to the sequence $(i,j)^*t^*_{U'}g_1, \dots (i,j)^*t^*_{U'}g_d$. Since $t_{U'}\circ(i,j) = j$ and $j^*g_i = t_i$, the local equations for V_U restrict to the regular seqeunce $t_1,\dots,t_d$ in the local ring $\mathcal{O}_{W,w}$. Therefore, by [**EGA4**, Theorem 11.3.8] applied to the flat morphism $s\circ h_U$, the quotient $\mathcal{O}_{V_U} = \mathcal{O}_{U'}/(t^*_{U'}g_1,\dots,t^*_{U'}g_d)$ is $(s\circ h_U)$-flat at z; i.e. $g\colon V_U \to U$ is flat at z. Since $t_1,\dots,t_d$ generate the maximal ideal of $(s\circ h_U)^{-1}(y)$ at z we see that $\mathcal{O}_{g^{-1}(y),z} = k(z)$. Since $k(y)$ and $k(z)$ are finite fields, $\operatorname{Spec} k(z) \to \operatorname{Spec} k(y)$ is unramified. Therefore, g is flat and unramified, and thus étale, at z □

Proof of Theorem 0.2. **Step 1.** Reduction to schemes which are of finite type over $\operatorname{Spec}\mathbb{Z}$.

Since all morphisms in (1) are K-morphisms, we may construct V and B by working locally in the ground scheme K. Thus we may assume that $K = \operatorname{Spec} A$.

Let $(A_\lambda)_{\lambda\in L}$ be the inductive system of subrings of A which are $\mathbb{Z}$-algebras of finite type (ordered by inclusion). Then $A = \lim_{\to} A_\lambda$. For each $\lambda \in L$ set $K_\lambda = \operatorname{Spec} A_\lambda$. Then $K = \operatorname{Spec} A$ is the limit of the projective system $(K_\lambda)_{\lambda\in L}$ of schemes of finite type over $\operatorname{Spec}\mathbb{Z}$. By [**EGA4**, Theorem 8.8.2(ii)] there is a scheme U_λ (resp. $R_{\lambda'}$) finitely presented over K_λ (resp. $K_{\lambda'}$) such that $U \simeq U_\lambda \times_{K_\lambda} K$ (resp. $R \simeq R_{\lambda'} \times_{K_{\lambda'}} K$) as K-schemes. Let $\alpha \in L$ be any index such that $\alpha \geq \lambda, \lambda'$ and set $R_\alpha = R_{\lambda'} \times_{K_{\lambda'}} K_\alpha$ and $U_\alpha = U_\lambda \times_{K_\lambda} K_\alpha$. Then R_α and U_α are of finite presentation over K_α and $U \simeq U_\alpha \times_{K_\alpha} K$ and $R \simeq R_\alpha \times_{K_\alpha} K$.

If we replace the original inductive system $(A_\lambda)_{\lambda\in L}$ by the system of subrings of A (of finite type over $\mathbb{Z}$) which contain A_α then by [**EGA4**, Proposition 8.2.5] the isomorphisms $U \simeq U_\alpha \times_{K_\alpha} K$ and $R \simeq R_\alpha \times_{K_\alpha} K$ induce isomorphisms $U \simeq \lim_{\to} U_\lambda$ and $R \simeq \lim_{\to} U_\lambda$ where U_λ is defined to be $U_\alpha \times_{K_\alpha} K_\lambda$ and R_λ is defined to be $R_\alpha \times_{K_\alpha} K_\lambda$. Since the K_α-schemes U_α and R_α are finitely presented over the affine scheme K_α they are also quasi-compact and quasi-separated. By theorem [**EGA4**, Theorem 8.8.2(i)] the natural map $\lim_{\to} \operatorname{Hom}_{K_\lambda}(R_\lambda, U_\lambda) \to \operatorname{Hom}_K(R, U)$, is a bijection. Thus, for some suitable choice of $\lambda \in L$ there are morphisms $s_\lambda, t_\lambda \colon R_\lambda \to U_\lambda$, such that s, t are identified with $s_\lambda \times 1_K$ and $t_\lambda \times 1_K$ via the isomorphisms $U \simeq U_\lambda \times_{K_\lambda} K$ and $R \simeq R_\lambda \times_{K_\lambda} K$. By [**EGA4**, Proposition 17.7.8(ii)] we can choose λ such that s_λ and t_λ are smooth and [**EGA4**, Theorem 8.10.5(vi)] says that we can take s_λ to be surjective as well. Finally if we identify $U \times_K U = \lim_{\to} U_\lambda \times_{K_\lambda} U_\lambda$ we can [**EGA4**, Proposition 17.7.8(ii)] ensure that (s_λ, t_λ) is also unramified.

By base change, it suffices to prove the proposition for the pair of morphisms $s_\lambda, t_\lambda \colon R_\lambda \to U_\lambda$. Therefore, to prove the proposition we may assume that R and U are of finite type over $\operatorname{Spec} \mathbb{Z}$ and that $K = \operatorname{Spec} A$ where A is a finitely generated $\mathbb{Z}$-algebra.

Step 2. Construction of V.

By Lemma 0.3 for every closed point, $y \in U$ there is a scheme V_y and a morphism $V_y \xrightarrow{f_y} U$ such that $g_y \colon (V_y)_U \to U$ is étale at some point $z \in g^{-1}(y)$. (Here $(V_y)_U$ is defined to be $R \times_{t,U,f_y} V_y$ and $g_y \colon (V_y)_U \to U$ is the composition of $s \colon R \to U$ with the projection $(V_y)_U \to R$.) Since the set of points where a morphism is étale is open, there is an open set $B_y \subset (V_y)_U$ containing z such that $g_y \colon B_y \to U$ is étale. Replacing V_y by the open set $t_{V_y}(B_y)$ we may assume that $t_{V_y}(B_y) = V_y$. Now let $V = \coprod_{y \in Y} V_y$ and $B = \coprod_{y \in Y} B_y$. (Since U is assumed to be quasi-compact we can in fact let V be the disjoint union of the finite number of V_y's whose images under the g_y's cover U.) □

1.2. Proof of Theorem 0.1.

LEMMA 0.4. *Let F be a stack with representable diagonal and suppose that there is a smooth surjective morphism $U \xrightarrow{p} F$. Given a morphism from a scheme $V \xrightarrow{v} F$ set $V_U = U \times_F V$ and let v_U, p_V be be the projections to the first and second factors respectively. Then the representable morphism $V \xrightarrow{v} F$ is étale if there is an open set $B \subset V_U$ such that:*
i) $v_U \colon B \to U$ is étale.
ii) $p_V(B) = V$.

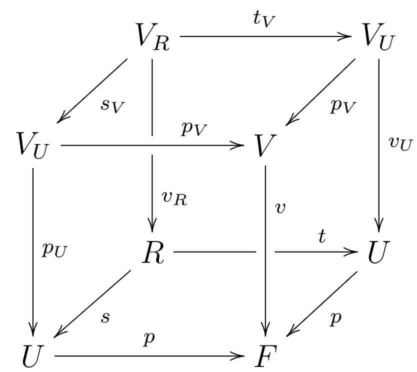
PROOF OF LEMMA 0.4. Since $U \xrightarrow{p} F$ is a smooth surjective morphism, the representable morphism $V \xrightarrow{v} F$ is étale if $v_U \colon V_U \to U$ is étale, which is what we will prove.

Let $R = U \times_F U$ and let $s, t : R \to U$ be the two projections. The square

(2)
$$\begin{array}{ccc} R & \xrightarrow{t} & U \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{p} & F \end{array}$$

is 2-cartesian, so the maps $p \circ s$ and $p \circ t$ are 2-isomorphic. Set $V_R = R \times_F V$ where the map $R \to F$ is taken to be either of the maps $p \circ s$ or $p \circ t$ and let $v_R \colon V_R \to R$ be projection to the first factor. Finally let $s_V, t_V \colon V_R \to V_U$ be the two morphisms obtained by base change from s and t. Then in the following cube is commutative and its faces are (2)-cartesian.

(3)



The top face of (3) is the square

(4)
$$\begin{array}{ccc} V_R & \xrightarrow{t_V} & V_U \\ \downarrow{\scriptstyle s_V} & & \downarrow{\scriptstyle p_V} \\ V_U & \xrightarrow{p_V} & V \end{array}$$

Thus, by base change, $s_V(t_V^{-1}(B)) = V_U$. Next consider the squares

(5)
$$\begin{array}{ccc} V_R & \xrightarrow{t_V} & V_U \\ \downarrow{\scriptstyle v_R} & & \downarrow{\scriptstyle v_U} \\ R & \xrightarrow{t} & U \end{array}$$

and

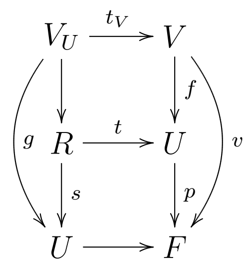
(6)
$$\begin{array}{ccc} V_R & \xrightarrow{s_V} & V_U \\ \downarrow{\scriptstyle v_R} & & \downarrow{\scriptstyle v_U} \\ R & \xrightarrow{s} & U \end{array}$$

Applying base change to (5) we see that $v_R\colon (t_V^{-1}(B)) \to R$ is étale. Since $s_V(t_V^{-1}(B)) = V_U$, we see from (6) that for every point in $x \in V_U$ there is a point in $x' \in V_R$ such that $s_V(x') = x$ and that v_R is étale at x'. Since the property of being étale is preserved by flat descent, it follows [**EGA4**, Theorem 17.7.1(ii)] that v_U is étale at every point of V_U. $\square$

Proof of Theorem 0.1. The only thing to prove is that there is an étale surjective map from a scheme to F.

Let $R = U \times_F U$ and let $s, t\colon R \to U$ be the two projections. By hypothesis U is of finite presentation over K. The morphism $R \overset{(s,t)}{\to} U \times_K U$, which obtained by base change from the diagonal $F \to F \times_K F$ is by hypothesis separated, quasi-compact and unramified. Thus R is of finite presentation[4] over K. By Theorem 0.2 applied to the pair of smooth surjective morphisms $s, t\colon R \to U$ there is a morphism $f\colon V \to U$ and an open set $B \subset V_U$ such that $t_V(B) = V$ and $g\colon B \to U$ is étale and surjective.

Let $v = p \circ f$. Since the squares in commutative diagram



are (2)-cartesian, we can identify $g\colon V_U \to U$ with the morphism $v_U\colon V_U \to U$ in Lemma 0.4 and conclude that $v\colon V \to F$ is étale. The morphism v is also surjective because its pullback by the surjective morphism $U \to F$ is surjective. $\square$

---

[4]First note that R is locally of finite presentation over K since the structure morphism $R \to K$ is the composition of the finitely presented morphism $U \to K$ with the smooth (hence locally of finite presentation) morphism $R \to U$. Since $R \to U \times_K U$ is assumed quasi-compact and separated and $U \to K$ is quasi-compact and quasi-separated (because a morphism of finite presentation is defined to be one that is locally of finite presentation, quasi-compact and quasi-separated) the composite morphism $R \to U \times_K U \to U \to K$ is quasi-compact and quasi-separated.

Bibliography

[EGA4] Grothendieck, A. and Dieudonné, J., *Élements de Géométrie Algébrique IV.*