NOTES ON THE CONSTRUCTION OF THE MODULI SPACE OF CURVES

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The purpose of these notes is to discuss the problem of moduli for curves of genus $g \geq 3$ [1] and outline the construction of the (coarse) moduli scheme of stable curves due to Gieseker. The notes are broken into 4 parts.

In Section 1 we discuss the general problem of constructing a moduli "space" of curves. We will also state results about its properties, some of which will be discussed in the sequel.

We begin Section 2 by recalling from [DM] (see also [Vi]) the definition of a groupoid, and define the moduli groupoid of curves, as well as the quotient groupoid of a scheme by a group. We then discuss morphisms and fiber products in the 2-category of groupoids. Once this is in place we state the geometric conditions required for a groupoid to be a stack, and prove that the quotient groupoid of a scheme by a group is a stack. After discussing properties of morphisms of stacks, we define a Deligne-Mumford stack and prove that if a group acts on a scheme so that the stabilizers of geometric points are finite and reduced then the quotient stack is Deligne-Mumford. In the last part of Section 2 we talk about some basic algebro-geometric properties of Deligne-Mumford stacks.

In Section 3 the notion of a stable curve is introduced, and we define the groupoid of stable curves. The groupoid of smooth curves is a subgroupoid. We then prove that the groupoid of stable curves of genus $g \geq 3$ is equivalent to the quotient groupoid of a Hilbert scheme by the action of the projective linear group with finite, reduced stabilizers at geometric points. By the results of the previous section we can conclude the the groupoid of stable curves is a Deligne-Mumford stack defined over $\operatorname{Spec} \mathbb{Z}$ (as is the groupoid of smooth curves). We also discuss the results of [DM] on the irreducibility of the moduli stack.

We begin Section 4 by defining the moduli space of a Deligne-Mumford stack, proving that a geometric quotient of a scheme by a group is the

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[1]Because every curve of genus 1 and 2 has non-trivial automorphisms, the problem of moduli is more subtle in this case than for curves of higher genus

moduli space of the quotient stack. We then discuss the method of geometric invariant theory for constructing geometric quotients for actions of reductive groups. Finally, we briefly outline Gieseker's approach to constructing the coarse moduli space over an algebraically closed field as the quotient of Hilbert schemes of pluricanonically embedded stable curves.

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1. Basics

Definition 1.1. Let S be a scheme. By a smooth curve of genus g over S we mean a proper, smooth family $C \to S$ whose geometric fibers are smooth, connected 1-dimensional schemes of genus g.

Remark 1.1. By the genus of a smooth, connected curve C over an algebraically closed field, we mean $\dim H^0(C, \omega_C) = \dim H^1(C, \mathcal{O}_C) = g$, where ω_C is the sheaf of regular 1-forms on C. If the ground field is $\mathbb{C}$, then C is a smooth, compact Riemann surface, and the algebraic definition of genus is the same as the topological one.

The basic problem of moduli is to classify curves of genus g. As a start, it is desirable to construct a space $\mathcal{M}_g$ whose geometric points represent all possible isomorphism classes of smooth curves. In the language of complex varieties, we are looking for a space that parametrizes all possible complex structures we can put on a fixed surface of genus g.

However, as modern (post-Grothendieck) algebraic geometers we would like $\mathcal{M}_g$ to have further functorial properties. In particular given a scheme S, a curve $C \to S$ should correspond to a morphism of S to $\mathcal{M}_g$ (when S is the spectrum of an algebraically closed field this is exactly the condition of the previous paragraph).

In the language of functors, we are trying to find a scheme $\mathcal{M}_g$ which represents the functor $\mathcal{F}_{\mathcal{M}_g}$: Schemes $\to$ Sets which assigns to a scheme S the set of isomorphism classes of smooth curves of genus g over S.

Unfortunately, such a moduli space can not exist because some curves have non-trivial automorphisms. As a result, it is possible to construct non-trivial families $C \to B$ where each fiber has the same isomorphism class. Since the image of B under the corresponding map to the moduli space is a point, if the moduli space represented the functor $\mathcal{F}_{\mathcal{M}_g}$ then $C \to B$ would be isomorphic to the trivial product family.

Given a curve X and a non-trivial (finite) group G of automorphisms of X we construct a non-constant family $C \to B$ where each fiber is isomorphic to X as follows. Let B' be a scheme with a free G action, and let $B = B'/G$ be the quotient. Let $C' = B' \times X$. Then G acts on C' by acting as it does on B' on the first factor, and by automorphism on the second factor. The quotient C'/G is a family of curves over B. Each fiber is still isomorphic to X, but C will not in general be isomorphic to $X \times B$.

There is however, a *coarse* moduli scheme of smooth curves. By this we mean:

Definition 1.2. ([GIT, Definition 5.6, p.99]) There is a scheme $\mathcal{M}_g$ and a natural transformation of functors $\phi : \mathcal{F}_{\mathcal{M}_g} \to \mathrm{Hom}(\cdot, \mathcal{M}_g)$ such that

1. For any algebraically closed field k, the map $\phi : \mathcal{F}_{\mathcal{M}_g}(\Omega) \to \mathrm{Hom}(\Omega, \mathcal{M}_g)$ is a bijection, where $\Omega = \operatorname{Spec} k$.
2. Given any scheme M and a transformation $\psi : \mathcal{F}_{\mathcal{M}_g} \to \mathrm{Hom}(\cdot, M)$ there is a unique transformation $\chi : \mathrm{Hom}(\cdot, \mathcal{M}_g) \to \mathrm{Hom}(\cdot, M)$ such that $\psi = \chi \circ \phi$.

The existence of ϕ means that given a family of curves $C \to B$ there is an induced map to $\mathcal{M}_g$. We do not require however, that a map to moduli gives a family of curves (as we have already seen a non-constant family with iso-trivial fibers). However, condition 1 says that giving a curve over an algebraically closed field is equivalent to giving a map of that field into $\mathcal{M}_g$.

Condition 2 is imposed so that the moduli space is a universal object.

In his book on geometric invariant theory Mumford proved the following theorem.

Theorem 1.1. *(*[GIT, Chapter 5]*) Given an algebraically closed field k there is a coarse moduli scheme $\mathcal{M}_g$ of dimension $3g-3$ defined over* $\operatorname{Spec} k$*, which is quasi-projective and irreducible.*

The proof of this theorem will be subsumed in our general discussion of the construction of $\overline{\mathcal{M}}_g$, the moduli space of stable curves.

A natural question to ask at this point, is whether $\mathcal{M}_g$ is complete (and thus projective). The answer is no. It is quite easy to construct curves $C \to \operatorname{Spec} \mathcal{O}$ with $\mathcal{O}$ a D.V.R. which has function field K, where C is smooth, but the fiber over the residue field is singular. Since C is smooth, the restriction $C_K \to \operatorname{Spec} K$ is a smooth curve over $\operatorname{Spec} K$, so there is a map $\operatorname{Spec} K \to \mathcal{M}_g$. The existence of such a family does not prove anything, since we must show that we can not replace the special fiber by a smooth curve. The total space $\tilde{C}$ of a family with modified special fiber is birational to C. Since C and $\tilde{C}$ are surfaces (being curves over 1-dimensional rings) there must be a sequence of birational transformations (centered in the special fiber) taking one to the other. It appears therefore, that it suffices to construct a family $C \to \operatorname{Spec} \mathcal{O}$ such that no birational modification of C centered in the special fiber will make it smooth. Unfortunately, because $\mathcal{M}_g$ is only a coarse moduli scheme, the existence of such a family does not prove that $\mathcal{M}_g$ is incomplete. The reason is that there may be a map $\operatorname{Spec} \mathcal{O} \to \mathcal{M}_g$ extending the original map $\operatorname{Spec} K \to \mathcal{M}_g$ without there being a family of smooth curves $\tilde{C} \to \operatorname{Spec} \mathcal{O}$ extending $C_K \to \operatorname{Spec} K$. However, we will see when we discuss the valuative criterion of properness for Deligne-Mumford stacks that it suffices to show that for every finite extension $K \subset K'$ we can not complete the induced family of smooth curves $C_{K'} \to \operatorname{Spec} K'$ to a family of smooth curves $C' \to \operatorname{Spec} \mathcal{O}'$, where $\mathcal{O}'$ is the integral closure of $\mathcal{O}$ in K'.

Example 1.1. The following family shows that $\mathcal{M}_3$ is not complete. It can be easily generalized to higher genera. Consider the family $x^4 + xyz^2 + y^4 + t(z^4 + z^3x + z^3y + z^2y^2)$ of quartics in $\mathbf{P}^2 \times \operatorname{Spec} \mathcal{O}$ where $\mathcal{O}$ is a D.V.R. with uniformizing parameter t. The total space of this family is smooth, but over the closed point the fiber is a quartic with a node at the point $(0 : 0 : 1) \in \mathbf{P}^2$. Moreover, even after base change, any blow-up centered at the singular point of the special fiber contains a (-2) curve, so there is no modification that gives us a family of smooth curves.

Since $\mathcal{M}_g$ is not complete, a natural question is to ask whether it is affine. The answer again is no. This follows from the fact that $\mathcal{M}_g$ has a projective compactification in the moduli space of abelian varieties, such that boundary has codimension 2. In particular, this means that there are complete curves in $\mathcal{M}_g$. On the other hand, Diaz proved the following theorem ([Di]).

Theorem 1.2. *Any complete subvariety of $\mathcal{M}_g$ has dimension less than $g - 1$.*

It is not known how close this bound is to being sharp.

Finally we state a spectacular theorem proved by Harris-Mumford, Harris and Eisenbud-Harris.

Theorem 1.3. *For $g > 23$, $\mathcal{M}_g$ is of general type.*

Remark 1.2. The importance of this theorem is that until its proof, there was some belief that $\mathcal{M}_g$ was rational, or at least unirational.

The reason is that for $g = 3, 4, 5$ it is very easy to show that $\mathcal{M}_g$ is unirational and not much more difficult to show that is in fact rational. For $g \leq 10$, the unirationality of $\mathcal{M}_g$ had been affirmed by the Italian school. (For a nice discussion of the rationality of moduli spaces of curves of low genus, see Dolgachev's article [Do].)

2. Stacks

Let S be a scheme, and let $\mathcal{S} = (Sch/S)$ be the category of schemes over S.

2.1. **Groupoids.**

Definition 2.1. A category over S is a category F together with a covariant functor $p_F : F \to \mathcal{S}$. If B is an object of $\mathcal{S}$ we say X lies over B if $p_F(X) = B$.

Definition 2.2. (see also [Vi, Definition 7.1]) If (F, p_F) is a category over S, then it is a groupoid over S if the following conditions hold:

(1) If $f : B' \to B$ is a morphism in $\mathcal{S}$, and X is an object of F lying over B, then there is an object X' over B' and a morphism $\phi : X' \to X$ such that $p_F(\phi) = f$.

(2) Let X, X', X'' be objects of F lying over B, B', B'' respectively. If $\phi : X' \to X$ and $\psi : X'' \to X$ are morphisms in F, and $h : B' \to B''$ is a morphism such that $p_F(\psi) \circ h = p_F(\phi)$ then there is a unique morphism $\chi : X' \to X''$ such that $\psi \circ \chi = \phi$ and $p_F(\chi) = h$.

Remark 2.1. Condition (2) implies that a morphism $\phi : X' \to X$ of objects over B' and B respectively is an isomorphism if and only if $p_F(\phi) : B' \to B$ is an isomorphism. (To see that $p_F(\phi)$ being an isomorphism is sufficient to ensure that ϕ is an isomorphism, apply condition (2) where one of the maps is $p_F(\phi)$ and the other the identity, and lift $p_F(\phi)^{-1} : B \to B'$ to $\phi^{-1} : X \to X'$. The other direction is trivial.) Define $F(B)$ to be the subcategory consisting of all objects X such that $p_F(X) = B$ and morphisms f such $p_F(f) = id_B$. Then $F(B)$ is a groupoid; i.e. a category where all morphisms are isomorphisms. This is the reason we say that F is a groupoid over S.

Condition (2) also implies that the object X' over B' in condition (1) is unique up to canonical isomorphism. This object will be called the pull-back of X via f and denoted f^*X. Moreover, if $X \xrightarrow{s} X'$ is a morphism in $F(B)$ then there is a canonical morphism $f^*X \xrightarrow{f^*s} f^*X'$. In other words, given a morphism $B' \xrightarrow{f} B$ of S-schemes, there is an induced functor $f^* : F(B) \to F(B')$. Note that f^* is actually a *covariant* functor.

Example 2.1. If $F : \mathcal{S} \to \text{Sets}$ is a contravariant functor, then we can associate a groupoid (also called F) whose objects are pairs (B, β) where B is an object of $\mathcal{S}$ and $\beta \in F(B)$. A morphism $(B', \beta') \to (B, \beta)$ is an S-morphism $f : B' \to B$ such that $F(f)(\beta) = \beta'$. In this case $F(B)$ in the groupoid sense is just the set $F(B)$ in the functor sense; i.e. all morphisms in the groupoid $F(B)$ are the identities.

In particular, if X is any S-scheme then its functor of points gives a groupoid $\underline{X}$. Objects are X-schemes, i.e. morphisms $B \xrightarrow{\phi} X$, and a morphism from $B' \xrightarrow{\phi'} X'$ to $B \xrightarrow{\phi} X$ is a morphism $B' \xrightarrow{f} B'$ such that $\psi f = \psi'$. The functor $p_{\underline{X}}$ simply forgets the X-structure, and views schemes and morphisms as being over S.

Example 2.2. If X/S is a scheme and G/S is a flat group scheme (of finite type) acting on the left on X then we define the quotient groupoid $[X/G]$ as follows. The sections (i.e. objects) of $[X/G]$ over B are G-principal bundles $E \to B$ together with a G-equivariant map $f : E \to X$. A morphism from $E' \to B'$ with equivariant map $f' : E' \to B'$ to $E \to B$ is a cartesian diagram

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ \downarrow & & \downarrow \\ B' & \to & B \end{array}$$

where g is an equivariant map such that $gf = f'$. If the action is free and a quotient scheme X/G exists, then there is an equivalence of categories between $[X/G]$ and the groupoid $\underline{X/G}$.

Example 2.3. The central example in these notes is the moduli groupoid $F_{\mathcal{M}_g}$ defined over $\operatorname{Spec}\mathbb{Z}$. The objects of $F_{\mathcal{M}_g}$ are smooth curves as defined in part 1. A morphism from $X' \to B'$ to $X \to B$ is a cartesian diagram

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ B' & \to & B \end{array}$$

The functor $F_{\mathcal{M}_g} \to \operatorname{Sch}\mathbb{Z}$ sends $X \to B$ to B. We will eventually prove that $F_{\mathcal{M}_g}$ is a quotient groupoid as in the previous example.

A related groupoid is the universal curve $F_{\mathcal{C}_g}$, also defined over $\operatorname{Spec}\mathbb{Z}$. Objects of $F_{\mathcal{C}_g}$ are smooth curves $X \to B$ together with a section $\sigma : B \to X$. A morphism is a cartesian diagram which is compatible with the sections.

Remark 2.2 (**Warning**). The groupoid we have just defined is not the groupoid associated to the moduli functor we defined in Part 1. The groupoid here is not a functor, since if X/B is a curve with non-trivial automorphisms $F(B)$ will not be a set because there are morphisms which are not identities. (A set is a groupoid where all the morphisms are identities.)

2.2. **Morphisms of groupoids.**

Definition 2.3. (a) If (F_1, p_{F_1}) and (F_2, p_{F_2}) are groupoids over S then a morphism $F_1 \to F_2$ is a functor $p : F_1 \to F_2$ such that $pp_{F_2} = p_{F_1}$.

(b) A morphism p is called an isomorphism if it is an equivalence of categories.

Example 2.4. The functor $F_{\mathcal{C}_g} \to F_{\mathcal{M}_g}$ defined by forgetting the section is a morphism of groupoids.

Here are some more subtle examples:

Example 2.5. If $f : X \to Y$ is an morphism of schemes then it induces a morphism of groupoids $\underline{X} \xrightarrow{p} \underline{Y}$ in a fairly obvious way: Objects of $\underline{X}$ are X-schemes, which, via the morphism f, can be viewed as Y-schemes, i.e. objects of $\underline{Y}$. Thus, $p(B \xrightarrow{u} X) = B \xrightarrow{fu} Y$. If $s : B \to B'$ is a morphism of X-schemes, then $p(s)$ is the morphism s viewed now as a morphism of Y-schemes.

Conversely, if $p : \underline{X} \to \underline{Y}$ is a morphism of groupoids over $\mathcal{S}$ then, because p preserves the projection to $\mathcal{S}$, $p(X \xrightarrow{id} X) = X \xrightarrow{f} Y$ for some morphism f. Yoneda's lemma implies that p is induced by the morphism of schemes $f : X \to Y$.

More generally, if F is a groupoid and B is a scheme then giving a morphism $\underline{B} \xrightarrow{p} F$ is equivalent to giving an object X in $F(B)$. The object X is simply $p(B \xrightarrow{id} B)$.

Example 2.6. We can view the category $\mathcal{S}$ of S-schemes, (which is trivially a groupoid over $\mathcal{S}$) as the groupoid $\underline{S}$. If F is a groupoid over $\mathcal{S}$ then functor p_F is then a morphism of groupoids $F \to \underline{S}$.

Example 2.7. Let X/S be a scheme and G/S a group scheme acting on the left on X. Then we can define a morphism $p : \underline{X} \to [X/G]$ as follows: If $B \xrightarrow{s} X$ is an object of $\underline{X}(B)$ then $p(s)$ is the bundle $G \times B \to B$ where G acts by left translation on G and trivially on B. The equivariant map $G \times B \to X$ is given by the formula $(g, b) \mapsto gs(b)$. If $f : B' \to B$ is a morphism in $\underline{X}$ then $p(f)$ is the cartesian diagram

$$\begin{array}{ccc} G \times B' & \xrightarrow{id \times f} & G \times B \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

Remark 2.3. Because isomorphisms of stacks are defined as equivalences of categories some subtleties arise. In particular an isomorphism $p : F_1 \to F_2$ need not have an inverse; i.e. there need not be a functor $q : F_2 \to F_1$ such that $pq = id_{F_2}$ and $qp = id_{F_1}$. However, any equivalence of categories has a quasi-inverse; that is a functor $q : F_2 \to F_1$ such that pq (resp. qp) are naturally isomorphic to the identity functors 1_{F_2} (resp. 1_{F_2}). Moreover, if $p : F_2 \to F_1$ is a equivalence which is a morphism of $\mathcal{S}$-groupoids, then then there is a quasi-inverse which is a morphism of groupoids.

The usual way to describe this point is to say that groupoids over $\mathcal{S}$ form a 2-category. This 2-category has objects, which are the groupoids; 1-morphisms, which are functors, and 2-morphisms, which are natural isomorphisms of functors. In other words, the category of groupoids contains extra information about *isomorphisms* between morphisms.

For the most part, the fact that groupoids (and thus stacks) form a 2-category will not require too much thought but there a few situations where it is relevant. For example, as we discuss below, a cartesian diagram commutes only up to homotopy. A more geometric situation is the valuative criterion, where (see Theorem 2.2) we require two extensions of a morpisms be isomorphic.

The following proposition shows that the notion of isomorphism of groupoids is an extension of our notion of isomorphism of schemes.

Proposition 2.1. *Let X and Y be S-schemes. Then there is an isomorphism $X \stackrel{f}{\to} Y$ as S-schemes if and only if there is an equivalence of $\mathcal{S}$ groupoids $\underline{X} \stackrel{p}{\to} \underline{Y}$*

Proof. If f is an isomorphism, then the induced functor $\underline{X} \stackrel{p}{\to} \underline{Y}$ is in fact a strong equivalence; i.e. the functor $\underline{Y} \stackrel{q}{\to} \underline{X}$ induced by f^{-1} has the property that that $pq = id_Y$ and $qp = id_X$.

Conversely, suppose that $\underline{X} \stackrel{p}{\to} \underline{Y}$ is an isomorphism of groupoids with quasi-inverse $\underline{Y} \stackrel{q}{\to} \underline{X}$. As we saw in Example 2.5 the functor p (resp. q) is induced by a morphism $f : X \to Y$ (resp. $g : Y \to X$). Then $qp(X \stackrel{id}{\to} X) = X \stackrel{gf}{\to} X$. Since q and p are equivalences $X \stackrel{id}{\to} X$ and $X \stackrel{gf}{\to} X$ are isomorphic as X-schemes. Hence $gf : X \to X$ is an automorphism. Likewise, $fg : Y \to Y$ is also an automorphism. Therefore, $f : X \to Y$ must be an isomorphism. □

Remark 2.4. If B is a scheme then from now on we will use the simpler notation $B \to F$ (resp. $F \to B$) to refer to a morphism $\underline{B} \to F$ (resp. $F \to \underline{B}$).

2.3. **Fiber products and cartesian diagrams.** Let F and G be groupoids over S. If $f : F \to G$ and $h : H \to G$ are morphisms of groupoids, then we define the fiber product $F \times_G H$ to be the following S-groupoid. Objects are triples (x, y, ψ) where $(x, y) \in F(B) \times H(B)$ and $\psi : g(x) \to h(y)$ is an isomorphism in $G(B)$. Here B is a fixed scheme in $\mathcal{S}$.

Suppose (x', y', ψ') is another object with $(x', y') \in F(B')$ and $\psi' : g(x) \to h(y)$ an isomorphism in $G(B')$. A morphism from (x', y', ψ') to (x, y, ψ) is a pair of morphisms $x' \xrightarrow{\alpha} x$, $y' \xrightarrow{\beta} y$ lying over the same morphism $B' \to B$, such that $\psi \circ f(\alpha) = g(\beta) \circ \psi'$.

By construction there are obvious functors $p : F \times_G H \to F$ and $q : F \times_G H \to H$. Note however, that the diagram

$$\begin{array}{ccc} F \times_G H & \to & H \\ \downarrow & & \downarrow \\ F & \to & G \end{array}$$

does not commute, since $fp(x, y, \psi) = f(x)$ and $gq(x, y, \psi) = g(y)$. The objects $f(x)$ are $g(y)$ isomorphic but not necessarily equal. There is however a natural isomorphism between the functors fp and gq. Following the language of 2-categories we say that such a diagram is 2-commutative. More generally given a 2-commutative diagram of groupoids

$$\begin{array}{ccc} T & \to & H \\ \downarrow & & \downarrow \\ F & \to & G \end{array}$$

there is a morphism $T \to F \times_G H$ which is unique up to canonical isomorphism. If this morphism is an isomorphism then we say the diagram is *cartesian*.

Example 2.8. If X, Y, Z are schemes then $\underline{X} \times_{\underline{Z}} \underline{Y}$ is isomorphic $\underline{X \times_Y Z}$, so our notion of fiber product is an extension of the usual one for schemes.

Remark 2.5. Despite the subtleties, this notion is correct for considering base change. In particular, suppose that F is an S-groupoid and $T \to S$ a morphism of schemes. Then if $B \to T$ is a T-scheme, one can check that the groupoids $F(B)$ and $(F \times_S T)(B)$ are equivalent; i.e. $F \times_S T$ is the T-groupoid obtained by base change to T.

2.4. **Definition of a stack.** Let (F, p_F) be an S-groupoid. Let B be an S-scheme and let X and Y be any objects in $F(B)$. Define a contravariant functor $\underline{\mathrm{Iso}}_B(X, Y) : (\mathrm{Sch}/B) \to (\mathrm{Sets})$ by associating to any morphism $f : B' \to B$, the set of isomorphisms in

$F(B')$ between f^*X and f^*Y. Let $B' \xrightarrow{f} B$ and $B'' \xrightarrow{g} B$ be B-schemes If $h : B'' \to B'$ is a morphism of B-schemes (ie. $g = fh$) then by construction of the pullback, there are canonical isomorphisms $\psi_X : g^*X \to h^*f^*X$ and $\psi_Y : g^*Y \to h^*f^*Y$. We define a map $\underline{\mathrm{Iso}}_B(X,Y)(B') \to \underline{\mathrm{Iso}}_B(X,Y)(B'')$ as follows: If $f^*X \xrightarrow{\phi} f^*Y$ is an isomorphism then (since $h^* : F(B') \to F(B'')$ is a functor) we obtain an isomorphism $h^*f^*X \xrightarrow{h^*\phi} h^*f^*Y$. The composite, $\psi_Y^{-1} \circ h^*\phi \circ \phi_X$ is the image of ϕ in $\underline{\mathrm{Iso}}_B(X,Y)(B'')$.

If $X = Y$ then $Iso_B(X,X)$ is the functor whose sections over B' mapping to B are the automorphisms of the pull-back of X to B'.

In the case of curves of genus $g \geq 2$, Deligne and Mumford proved that $\underline{\mathrm{Iso}}_B(X,Y)$ is represented by a scheme $\mathbf{Iso}_B(X,Y)$, because X/B and Y/B have canonical polarizations ([DM, p.84]). When $X = Y$ then Deligne and Mumford prove directly that the $\mathbf{Iso}_B(X,X)$ is finite and unramified over B ([DM, Theorem 1.11]). Applying the theorem to $B = \operatorname{Spec} k$, where k is an algebraically closed field, this theorem proves that every curve has a finite automorphism group.

The scheme $\mathbf{Iso}_B(X,X)$ is naturally a group scheme over B. However, in general it will not be flat over B. For example, if X/B is a family of curves, the number of points in the fibers of $\mathbf{Iso}_B(X,X)$ over B will jump over the points $b \in B$ where the fiber X_b has non-trivial automorphisms.

Definition 2.4. A groupoid (F, p_F) over S is a stack if

(1) $\underline{\mathrm{Iso}}_B(X,Y)$ is a sheaf in the étale topology for all B, X and Y.

(2) If $\{B_i \to B\}$ is a covering of B in the étale topology, and X_i is a collection of objects in $F(B_i)$ with isomorphisms

$$\phi_{ij} : X_{j|B_i \times_B B_j} \to X_{i|B_i \times_B B_j}$$

in $F(B_i \times_B B_j)$ satisfying the cocycle condition. Then there is an object $X \in F(X)$ with isomorphisms $X_{|B_i} \xrightarrow{\simeq} X_i$ inducing the isomorphisms ϕ_{ij} above.

Note that F is a groupoid associated to a functor then conditions (1) and (2) just assert that the functor is a sheaf in the étale topology. A functor represented by a scheme will be a stack, since condition (1) is trivially satisfied and (2) is equivalent to saying that the functor of points is a sheaf in the étale topology. The moduli functor we defined in Part 1 is not a stack, since it doesn't satisfy condition (2). In particular, as noted above, given a curve C with automorphism group G and $B' \to B$ a Galois cover with group G, there are two ways to descend the family $C \times B'/B'$ to a family over B, so a section of F over B, is not determined by its pull-back to an étale cover.

However, the moduli groupoid defined above is a stack. We will not prove this here, but instead we will prove that the moduli groupoid is the quotient groupoid of a scheme by $\mathbf{PGL}(N+1)$.

Proposition 2.2. *(cf.* [Vi, Example 7.17]*) Let G/S be a smooth*[2] *affine group scheme over X, then the groupoid $[X/G]$ defined above is a stack.*

Proof. Let e, e' be sections of $[X/G](B)$ corresponding to principal bundles $E \to B$ and $E' \to B$ and G-maps $f : E \to X$ and $f' : E' \to X$. Then $\underline{\mathrm{Iso}}_B(e, e')$ is the étale sheaf which is the quotient of $(X \times_{X \times X} E \times_B E')$ by the free product action of G. Moreover, descent theory show that this sheaf is in fact a scheme.

When $E = E'$ and $f = f'$ then the isomorphisms correspond to elements of G which preserve f. In other words $\mathbf{Iso}_B(e, e)$ is the is the stabilizer of the $G-$map $f : E \to X$ (see [GIT, Definition 0.4] for the definition of stabilizer).

Since any principal bundle $E \to B$ is locally trivial in the étale topology it determines descent data as follows: Let $\{B_i \xrightarrow{p_i} B\}$ be an étale cover on which $E \to B$ is trivial. Then we have and equivariant isomorphisms $\phi_i : p_i^* E \to G \times B_i$. If ϕ_{ij} is the pullback of ϕ_i to $B_i \times_B B_j$, then the ϕ_{ij}'s satisfy the cocycle condition; i.e. $\phi_{ij}\phi_{ik} = \phi_{jk}$.

Descent theory gives us the opposite direction. Given principal bundles (not necessarily trivial) $E_i \to B_i$ and isomorphisms of $E_{i|B_i \times_B B_j} \to E_{j|B_i \times_B B_j}$ satisfying the cocycle condition, there is a principal bundle $E \to B$ such that $E_i = p_i^* E$. This is condition (2) in the definition of a stack. □

Example 2.9. If F, G, H are stacks over S then one can check that the fiber product $F \times_G H$ is also a stack.

2.5. **Representable morphisms.** Most of the material in this section is taken from [DM, Section 4].

Definition 2.5. A morphism $f : F \to G$ of stacks is said to be representable if for any map of a scheme $B \to G$ the fiber product $F \times_G B$ is isomorphic to a stack associated to a scheme.

Remark 2.6. The definition of representable morphism is the one given by Deligne-Mumford in [DM]. Artin, who considered a larger category of stacks, extended the definition to require that if $B \to F$ is a map of an algebraic space (which we define below) then the the fiber product

[2] In fact the proposition holds for flat affine group schemes as well.

$F \times_G B$ is also an algebraic space. For the stacks we consider here, the Deligne-Mumford definition suffices.

Example 2.10. Let G/S be a smooth affine group scheme acting on X. We saw that there is a projection morphism $X \to [X/G]$ which associates to any X-scheme $Z \xrightarrow{s} X$, the principal bundle $G \times Z$ with the equivariant map given by $(g, z) \mapsto gs(z)$. Suppose $B \to [X/G]$ is a morphism from a scheme corresponding to a principal bundle $E \to B$ with equivariant map $E \to X$. Let us a consider the category $B \times_{[X/G]} X$. Its objects are triples $\{B_1, B_2, \psi\}$ where $B_1 \to B$, $B_2 \to X$ and ψ is an isomorphism between the images in $[X/G]$. Since B_1 and B_2 lie over the same object in the base category, there is a scheme B' such that $B_1 = B_2 = B'$. If $B' \xrightarrow{f} B$ is a morphism then the image of B' is the principal bundle $f^*E \to B'$ with equivariant map $f^*E \to E \to X$. Thus objects of $B \times_{[X/G]} X$ are triples $(B' \xrightarrow{f} B, B' \xrightarrow{s} X, \psi)$ where $\psi : G \times B' \xrightarrow{\sim} f^*E$ is a trivialization. The section $b \in B' \mapsto (1_G, b')$ gives a morphism $B' \to f^*E \to E$. Conversely, suppose we are given a morphism $B' \to E$. Let $f : B' \to E \to B$ be the composite morphism. The diagonal gives is a canonical trivialization of the bundle $E \times_B E \to E$ and hence of $f^*E \to B'$. Thus we can factor the morphism $B' \to E$ as $B' \to f^*E \to E$. A similar analysis shows that morphisms in $B \times_{[X/G]} X$ correspond to morphisms of E-schemes, so $B \times_{[X/G]} X$ is isomorphic to $\underline{E}$. Therefore, the morphism $X \to [X/G]$ is representable.

Example 2.11. The morphism $F_{\mathcal{C}_g} \to F_{\mathcal{M}_g}$ forgetting the section is also representable. If $B \to F_{\mathcal{M}_g}$ corresponds to a smooth curve $C \to B$ then $B \times_{F_{\mathcal{M}_g}} F_{\mathcal{C}_g}$ is represented by the scheme C.

Let **P** be a property of morphisms of schemes which is stable under base change. Most properties of morphisms of schemes satisfy this property. For example, finite type, separated, proper, affine, flat, smooth (hence étale), etc.

Definition 2.6. ([DM, Definition 4.3] A representable morphism of stacks $f : F \to G$ has property **P** if for all maps of scheme $B \to G$ the corresponding morphism of schemes $F \times_G B \to B$ has property **P**.

Example 2.12. The projection morphism $X \to [X/G]$ is smooth since for any $B \to [X/G]$ the corresponding map $E \to B$ is smooth because E is a principal bundle over E.

2.6. **Definition of a Deligne-Mumford stack.**

Definition 2.7. A stack is Deligne-Mumford if

(1) The diagonal $\Delta_F : F \to F \times_S F$ is representable, quasi-compact and separated. If the diagonal is proper, then we say that the stack is *separated*.

(2) There is a scheme U and an étale surjective morphism $U \to F$. Such a morphism $U \to F$ is called an (étale) atlas.

Remark 2.7. The representability of the diagonal automatically implies that the functor $\underline{\mathrm{Iso}}_B(X, Y)$ is representable. The reason is that if X and Y are objects in $F(B)$, then $\underline{\mathrm{Iso}}_B(X, Y)$ is represented by the fiber product $(B \times B)_{F \times F} F$ where the map $B \times B \to F \times F$ is the product of the the maps $B \to F$ corresponding to the objects X and Y.

Remark 2.8. In [DM], such a stack is called an *algebraic* stack. To conform to current terminology we use the term Deligne-Mumford stack. A more general class of stacks was studied by Artin, and they are now called Artin stacks. The basic difference is that an Artin stack need only have a *smooth* atlas. An *algebraic space* is defined as an étale sheaf with an étale cover by a scheme. This the same as Deligne-Mumford stack where the diagonal is an embedding. An algebraic space is separated if the diagonal is a closed embedding.

Remark 2.9. Condition (1) above is equivalent to the following condition:

(1') Every morphism $B \to F$ from a scheme is representable, so condition (2) makes sense.

(This fact is stated in [DM] and proved in [Vi, Prop 7.13]).

Condition (2) asserts the existence of a universal deformation space for deformations over Artin rings.

Remark 2.10 (Separated morphisms). The proof of the following lemma can be found in [L-MB].

Lemma 2.1. *Let $f : F \to G$ be a morphism of stacks satisfying property (1) above. Then the diagonal $\Delta_{F/G} : F \to F \times_G F$ is representable.*

As a consequence of the lemma we make the following extension of the notion of separated morphism of schemes.

Definition 2.8. A morphism of Deligne-Mumford stacks is separated if $\Delta_{F/G}$ is proper.

Vistoli also proves the following proposition:

Proposition 2.3. [Vi, Prop 7.15] *The diagonal of a Deligne-Mumford stack is unramified*

As a consequence of this proposition we can prove [Vi, p. 666]

Corollary 2.1. *If F is a Deligne-Mumford stack, B quasi-compact, and $X \in F(B)$ then X has only finitely many automorphisms.*

Remark 2.11. The are Artin stacks which are not Deligne-Mumford where each object has a finite automorphism group. In this case the diagonal is quasi-finite but ramified. Objects in the groupoid have *infinitesimal automorphisms.* This phenomenon only occurs in characteristic p, because all group schemes of finite type over a field of characteristic 0 are smooth.

Proof. Let $B \to F$ be map corresponding to X, and let $B \to F \times_S F$ be the composition with diagonal. The pullback $B \times_{F \times_S F} F$ can be identified with scheme $Iso_B(X, X)$. Since F is a Deligne-Mumford stack the map $Iso_B(X, X)$ is unramified over X. Furthermore, since B is quasi-compact, the map $Iso_B(X, X) \to X$ can have only finitely many sections. Therefore, X has only finitely many automorphisms over B. □

The following theorem is stated (but not proved) in [DM, Theorem 4.21]. We give the proof below with a slight additional assumption.

Theorem 2.1. *Let F be a stack over a Noetherian scheme S. Assume that*

(1) The diagonal is representable, quasi-compact, separated and unramified,

(2) There exists a scheme U of finite type over S and a smooth surjective S-morphism $U \to F$,

Then F is a Deligne-Mumford stack.

Remark 2.12. This theorem says that condition (1) and the existence of a versal deformation space (condition (2)) is actually equivalent to the existence of a universal deformation space.

Remark 2.13. We give the proof below under the additional assumption that the residue fields of the closed points of S are perfect. In particular we prove the theorem for stacks of finite type over $\operatorname{Spec} \mathbb{Z}$. Using the theorem we will prove that the stack of stable curves is a Deligne-Mumford stack of finite type over $\operatorname{Spec} \mathbb{Z}$. By assumption U is of finite type over S, so so it is relatively straightforward to reduce to the case that U is actually of finite type over $\operatorname{Spec} \mathbb{Z}$. Thus, the general statement can be reduced to the case we prove. However, we do not give the details here.

Proof. The only thing to prove is that F has an étale atlas of finite type over S. Let $u \in U$ be any closed point. Set $U_u = \delta^{-1}(u \times_S u) = u \times_F U$.

Let $z \in U_u$ be a closed point which is separable (i.e. étale) over u (The set of such closed points is dense in a smooth variety). Since U_u is smooth, the point z is cut out by a regular sequence in the local ring of U_u at z.

The diagonal $\delta : F \to F \times_S F$ is unramified. Thus, the map $U_u \to u \times_S U$ obtained by pulling back the morphisms $u \times_S U \to F \times_S F$ along the diagonal is unramified. We assume U is of finite type and that the residue fields of S are perfect. Thus, $k(u)$ is a finite, hence separable, extension of the residue field of its image in S. Hence the morphism $u \times_S U \to U$ is unramified and so is the composition $U_u \to u \times_S U \to U$.

Let x be the image of z in U. By [EGA4, 18.4.8] there are étale neighborhoods W' and U' of x and z respectively and a closed immersion $W' \hookrightarrow U'$ such that the diagram commutes

$$\begin{array}{ccc} z' \in W' & \hookrightarrow & U' \\ \text{étale} \downarrow & & \downarrow \text{étale} \\ z \in U_u & \to & x \in U \end{array}$$

Let z' be any point lying over z. Let Z_u be the closed subscheme of U' defined by lifts to $\mathcal{O}'_U$ of the local equations for $z' \in W'$. By construction, Z_u intersects U' transversally at z'. We will show that the induced morphism $Z_u \to F$ is étale in a neighborhood of z.

By definition, this means that for every map of a scheme $B \to F$, the induced map of schemes $B \times_F Z_u \to B$ is étale in a neighborhood of $z' \times_F Z_u$. Since $U \to F$ is smooth and surjective, it it suffices to check that the morphism is étale after base change to U.

By construction, $Z_u \subset U'$ is cut out by a regular sequence in a neighborhood of $z' \in U'$ (since z' is a smooth point of W'). Thus $Z_u \times_F U \to U' \times_F U$ is a regular embedding in a neighborhood of $z' \times_F U$. Since $U' \times_F U \to U'$ is smooth, we can apply [EGA4, Theorem 17.12.1], and conclude that $Z_u \times_F U' \to U'$ is smooth in a neighborhood of z'. Moreover, the relative dimension of this morphism is 0. Therefore, $Z_u \to F$ is étale in a neighborhood of z'.

Since U is of finite type over S, the Z_u's are as well. The union of the Z_u's cover F (since their pullbacks via the morphism $U \to F$ cover U). Also, U is Noetherian because it is of finite type over a Noetherian scheme. Thus a finite number of the Z_u's will cover the F. (To see this, we can pullback via the map $U \to F$. The pullback of the Z_u's form an étale cover of U which is Noetherian.) □

The theorem has a useful corollary.

Corollary 2.2. *Let X/S be a Noetherian scheme of finite type and let G/S be a smooth affine group scheme (also of finite type over S)*

acting on X such that the stabilizers of geometric points are finite and reduced.

(i) $[X/G]$ is a Deligne-Mumford stack. If the stabilizers are trivial, then $[X/G]$ is an algebraic space.

(ii) The stack is separated if and only if the action is proper.

Proof. The condition on the action ensures that $Iso_B(E, E)$ is unramified over E for any map $B \to [X/G]$ corresponding to the principal bundle $E \to B$. This in turn implies that the diagonal is also unramified, so condition (1) is satisfied. Furthermore, condition (2) is satisfied by the smooth map $X \to [X/G]$.

Suppose that $[X/G]$ is separated, i.e. the diagonal $[X/G] \to [X/G] \times [X/G]$ is proper and representable. One can check that

$$[X/G] \times_{[X/G]\times[X/G]} X \times X$$

is represented by the scheme $G \times X$ where the projection

$$[X/G] \times_{[X/G]\times[X/G]} X \times X \to X \times X$$

corresponds to the action map $G \times X \to X \times X$, so the action is proper.

Conversely, suppose that map $G \times X \to X \times X$ is proper. This implies that the diagonal is proper after base change to $X \times X$; i.e.,

$$[X/G] \times_{[X/G]\times[X/G]} X \times X \to X \times X$$

is proper. Let $Z \to [X/G] \times [X/G]$ be any scheme and set $W = [X/G] \times_{[X/G]\times[X/G]} [X/G]$. We will use descent to show that $W \to Z$ is proper.

If $Z' = Z \times_{[X/G]\times[X/G]} X \times X$ then the map $W' = W \times_Z Z' \to Z$ is proper. Since $X \to [X/G]$ is smooth and surjective (in particular it is faithfully flat) $Z' \to Z$ is as well. Descent theory for faithfully flat morphisms, implies that the map $W \to Z$ is proper. Therefore $[X/G]$ is separated. □

Example 2.13. In order for a group action to be proper it must have have finite stabilizers. However, it is not difficult to construct examples of group actions which, despite having finite stabilizers, are not proper. In [GIT, Example 0.4] there is an example an $\mathbf{SL}(2, \mathbb{C})$ action on a 4-dimensional variety X which has trivial stabilizers but is not proper. The quotient $[X/\mathbf{SL}(2, \mathbb{C})]$ is a non-separated algebraic space.

2.7. **Further properties of Deligne-Mumford stacks.** From now on the term *stack* will mean Deligne-Mumford stack, though we will often use the term Deligne-Mumford stack for emphasis.

Not all morphisms of stacks are representable, so we can not define algebro-geometric properties of these morphisms as we did for representable morphisms. However, if we consider morphisms of *Deligne-Mumford* stacks then we can define properties of morphisms as follows (see [DM, p. 100]).

Let **P** be a property of morphisms of schemes which at source and target is of a local nature for the étale topology. This means that for any family of commutative squares

$$\begin{array}{ccc} X_i & \xrightarrow{g_i} & X \\ f_i \downarrow & & f \downarrow \\ Y_i & \xrightarrow{h_i} & Y \end{array}$$

where the g_i (resp h_i) are étale and cover X (resp. Y), then f has property **P** if and only if f_i has property **P** for all i.

Examples of such properties are f flat, smooth, étale, unramified, locally of finite type, locally of finite presentation, etc.

Then if $f : F \to G$ is any morphism of Deligne-Mumford stacks we say that f has property **P** if there are étale atlases $U \to F$, $U' \to G$ and a compatible morphism $U \to U'$ with property **P**.

Likewise, if **P** is property of schemes which is local in the étale topology (for example regular, normal, locally Noetherian, of characteristic p, reduced, Cohen-Macaulay, etc.) then a Deligne-Mumford stack F has property **P** if for one (and hence every) étale atlas $U \to F$, the scheme U has property **P**.

A stack F is quasi-compact if it has an étale atlas which is quasi-compact. A morphism $f : F \to G$ of stacks is quasi-compact if for any map of scheme, $X \to G$ the fiber product $X \times_G F$ is a quasi-compact stack. We can now talk about morphisms of finite type; a morphism of finite type is a quasi-compact morphism which is locally of finite type. Similarly, a stack is Noetherian if it quasi-compact and locally Noetherian.

Definition 2.9. [DM, Definition 4.11] A morphism $f : F \to G$ is proper if it is separated, of finite type and locally over F there is a Deligne-Mumford stack $H \to F$ and a (representable) proper map $H \to G$ commuting with the projection to F and the original map $F \to G$.

$$\begin{array}{ccc} H & & \\ \downarrow & \searrow & \\ F & \to & G \end{array}$$

Remark 2.14. By a theorem of Vistoli [Vi, Prop. 2,6] and Laumon–Moret-Baily [L-MB, Theorem 10.1] every Noetherian stack has a finite cover by a scheme. Using this fact we can say that a morphism $F \to G$ is proper if there is a finite cover $X \to F$ by a scheme such that the composition $X \to F \to G$ is a proper representable morphism. (Recall that any morphism from a scheme to a stack is representable). Similarly we say that a morphism $f : F \to G$ of Noetherian stacks is (quasi)-finite if for any finite cover $X \to F$, the composition $X \to F \to G$ is representable and (quasi)-finite.

As is the case with schemes, there are valuative criteria for separation and properness ([DM, Theorem 4.18-4.19]). The valuative criterion for separation is equivalent to the criterion for schemes, but we only construct an *isomorphism* between two extensions.

Theorem 2.2. *A morphism $f : F \to G$ is separated iff the following condition holds:*

For any complete discrete valuation ring V and fraction field K and any morphism $f : \operatorname{Spec} V \to G$ with lifts $g_1, g_2 : \operatorname{Spec} V \to F$ which are isomorphic when restricted to $\operatorname{Spec} K$, then the isomorphism can be extended to an isomorphism between g_1 and g_2.

Theorem 2.3. *A separated morphism $f : F \to G$ is proper if and only if for any complete discrete valuation ring V with field of fractions K and any map $\operatorname{Spec} V \to G$ which lifts over $\operatorname{Spec} K$ to a map to F, there is a finite separable extension K' of K such that the lift extends to all of $\operatorname{Spec} V'$ where V' is the integral closure of V in K'*

Remark 2.15. When F is a scheme it is easy to show to that there is a lift $\operatorname{Spec} V' \to F$ if and only if there is a lift $\operatorname{Spec} V \to F$.

Example 2.14 (Angelo Vistoli). Here is an example showing the necessity of passing to a cover.

Let $G = \{\pm 1\}$ act on $X = \mathbf{A}^1_{\mathbb{C}}$ by by left multiplication and set $Y = \mathbf{A}^1$. The double cover $f : X \to Y$ given by $z \mapsto z^2$ is clearly G-invariant (and in fact is a geometric quotient in the sense of [GIT] -see Section 4.1). In particular if $E \to B$ is a principal G-bundle with equivariant map to X, then composition $E \to X \to Y$ is G-invariant. Hence, since $E \to B$ is a categorical quotient in the sense of [GIT] and there is a unique map $B \to Y$ making the diagram

$$\begin{array}{ccc} E & \to & X \\ \downarrow & & \downarrow \\ B & \to & Y \end{array}$$

commute. Moreover,

$$\begin{array}{ccc} E' & \to & E \\ \downarrow & & \downarrow \\ B' & \to & B \end{array}$$

is a cartesian diagram corresponding to a morphism in $[X/G]$, then the morphism $B' \to B$ is actually a morphism of Y-schemes. and we obtain a morphism $\psi : [X/G] \to Y$.

It is easy to check that the double cover $f : X \to Y$ factors as $X \xrightarrow{p} [X/G] \xrightarrow{\psi} Y$ where $p : X \to [X/G]$ is the map defined in Example 2.7. Since G is finite, p is a finite surjective (and representable) morphism, and since f is finite, it follows that the non-representable morphism $[X/G] \to Y = \mathbf{A}^1$ is finite. We wish to test the valuative criterion for this morphism.

Let $R = \mathbb{C}[[t]]$ be the complete local ring of Y at 0, and let $\operatorname{Spec} R \to Y$ be the obvious morphism. The restriction of the double cover $X \to Y$ to the generic point is a non-trivial degree 2 Galois covering corresponding the extension $\mathbb{C}(t) \subset \mathbb{C}(u)$ where $t = u^2$. We can view this cover as a principal G-bundle, giving us a lift $\operatorname{Spec} \mathbb{C}(t) \to [X/G]$. However, this lift can not be extended to all of $\operatorname{Spec} R$ since it has no non-trivial Galois covers. To obtain a lift we must first trivialize the bundle $\operatorname{Spec} \mathbb{C}(u) \to \operatorname{Spec} \mathbb{C}(t)$ by normalizing R in $\mathbb{C}(u)$.

2.8. **The topology of stacks.** Most of the topological properties of schemes make sense for Deligne-Mumford stacks. Thus in many ways we can think of them as spaces.

Remark 2.16 (Connectedness). If F_1 and F_2 are stacks over S, define the disjoint sum $F = F_1 \coprod F_2$ as follows: Objects are disjoint unions $X_1 \coprod X_2$ where X_1 is an object of F_1 and X_2 is an object of X_2. A morphism from $X_1' \coprod X_2' \to X_1 \coprod X_2$ is specified by giving morphisms $X_1' \to X_1$ $X_2' \to X_2$.

A stack is connected if it is not the disjoint sum of two non-void stacks.

Proposition 2.4. [DM, Proposition 4.14] *A Noetherian stack F over a field is connected if and only if there is a surjective morphism $X \to F$ from a connected scheme.*

Remark 2.17 (Open and closed substacks). An open substack $F \subset G$ is a full subcategory of G such that for any $x \in Obj(F)$, all objects in G isomorphic to x are also in F. Furthermore, the inclusion morphism $F \to G$ is represented by open immersions . In a similar way we can talk about closed (or locally closed) substacks.

In particular a stack is irreducible if it is not the disjoint union of two closed substacks.

A normal stack is irreducible if and only if it is connected. (Since a being normal is an étale local property of schemes, a stack is normal iff it has a normal étale atlas).

The following theorem is crucial to the proof of irreducibility of $F_{\overline{\mathcal{M}}_g}$ in arbitrary characteristic.

Theorem 2.4. [DM, Theorem 4.17(iii)] *Let $f : F \to S$ be a proper flat morphism with geometrically normal fibers. Then the number of connected components of the geometric fibers is constant.*

3. Stable curves

In this section we discuss stable curves and the compactification of the moduli of curves to the moduli of stable curves.

Definition 3.1. [DM, Definition 1.1] A Deligne-Mumford stable (resp. semi-stable) curve of genus g over a scheme S is a proper flat family $C \to S$ whose geometric fibers are reduced, connected, 1-dimensional schemes C_s such that:

(1) C_s has only ordinary double points as singularities.

(2) If E is a non-singular rational component of C, then E meets the other components of C_s in more than 2 points (resp. in at least 2 points).

(3) C_s has arithmetic genus g; i.e. $\dim H^1(\mathcal{O}_{C_s}) = g$.

Remark 3.1. Clearly, a smooth curve of genus g is stable. Condition (2) ensures that stable curves have finite automorphism groups, so that we will be able to form a Deligne-Mumford stack out of the category of stable curves. We will not use the notion of semi-stable curves until we discuss geometric invariant theory in Section 4.

Denote by $F_{\overline{\mathcal{M}}_g}$ the groupoid over $\operatorname{Spec} \mathbb{Z}$ whose sections over a scheme B are families of stable curves $X \to B$. As is the case with smooth curves, we define a morphism from $X' \to B'$ to $X \to B$ as a cartesian diagram

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ B' & \to & B \end{array} .$$

3.1. The stack of stable curves is a Deligne-Mumford stack.

Let $\pi : C \to S$ be a stable curve. Since π is flat and its geometric fibers are local complete intersections, the morphism is a local complete intersection morphism. It follows from the theory of duality that there is a canonical invertible dualizing sheaf $\omega_{C/S}$ on C. If C/S is smooth, then this sheaf is the relative cotangent bundle. The key fact we need about this sheaf is a theorem of Deligne and Mumford [DM, p. 78].

Theorem 3.1. *Let $C \xrightarrow{\pi} S$ be a stable curve of genus $g \geq 2$. Then $\omega_{C/S}^{\otimes n}$ is relatively very ample for $n \geq 3$, and $\pi_*(\omega_{C/S}^{\otimes n})$ is locally free of rank $(2n-1)(g-1)$.*

Remark 3.2. When π is smooth, the theorem follows from the classical Riemann-Roch theorem for curves. The general case is proved by analyzing the locally free sheaf obtained by restricting $\omega_{C/S}$ to the geometric fibers of C/S. In particular, if $S = \operatorname{Spec} k$, with k algebraically closed, then $\omega_{C/S}$ can be described as follows. Let $f : C' \to C$ be the normalization of C (note C' need not be connected). Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be the points of C' such that the $z_i = f(x_i) = f(y_i)$ are the double points of C. Then $\omega_{C/S}$ can be identified with the sheaf of 1-forms η on C' regular except for simple poles at the x's and y's and with $Res_{x_i}(\eta) + Res_{y_i}(\eta) = 0$.

As a result, if $N = (2n-1)(g-1) - 1$, then every stable curve can be realized as a curve in $\mathbf{P}^N$ with Hilbert polynomial $P_{g,n}(t) = (2nt - 1)(g-1)$. There is a subscheme (defined over $\operatorname{Spec} \mathbb{Z}$) $\overline{H}_{g,n} \subset Hilb_{\mathbf{P}^N}^{P_{g,n}}$ of the Hilbert scheme corresponding to $n-$canonically embedded stable curves. Likewise there is a subscheme $H_{g,n} \subset \overline{H}_{g,n}$ corresponding to $n-$canonically embedded smooth curves. A map $S \to \overline{H}_{g,n}$ corresponds to a stable curve $C \xrightarrow{\pi} S$ of genus g and an isomorphism of $\mathbf{P}(\pi_*(\omega_{C/S}))$ with $\mathbf{P}^N \times S$.

Now, $\mathbf{PGL}(N+1)$ naturally acts on $H_{g,n}$ and $\overline{H}_{g,n}$.

Theorem 3.2. $F_{\mathcal{M}_g} \simeq [H_{g,n}/\mathbf{PGL}(N+1)]$ *and* $F_{\overline{\mathcal{M}}_g} \simeq [\overline{H}_{g,n}/\mathbf{PGL}(N+1)]$.

Note that the theorem asserts that the quotient is independent of n.

Proof. We construct a functor $p : F_{\overline{\mathcal{M}}_g} \to [\overline{H}_{g,n}/\mathbf{PGL}(N+1)]$ which takes $F_{\mathcal{M}_g}$ to $[H_{g,n}/\mathbf{PGL}(N+1)]$ as follows: Given a family of stable curves $C \xrightarrow{\pi} B$, let $E \to B$ be the principal $\mathbf{PGL}(N+1)$ bundle associated to the projective bundle $\mathbf{P}(\pi_*(\omega_{C/B}^{\otimes n}))$. Let $\pi' : C \times_B E \to E$ be the pullback family. The pullback of this projective bundle to E is trivial and is isomorphic to $\mathbf{P}(\pi'_*(\omega_{C \times_B E/E}))$, so there is a map $E \to \overline{H}_{g,n}$ which is clearly $\mathbf{PGL}(N+1)$ invariant. If

$$\begin{array}{ccc} C' & \to & C \\ \pi' \downarrow & & \pi \downarrow \\ B' & \xrightarrow{\phi} & B \end{array}$$

is a morphism in $F_{\overline{\mathcal{M}}_g}$, then $\pi'_*(\omega_{C'/B'}) \simeq \phi^*\pi_*(\omega_{C/B})$, so we obtain a morphism of associated $\mathbf{PGL}(N+1)$ bundles

$$\begin{array}{ccc} E' & \to & E \\ \downarrow & & \downarrow \\ B' & \to & B \end{array}.$$

If C is a stable curve defined over an algebraically closed field k then any non-trivial automorphism of C induces an automorphism of the projectivized n-canonical linear system $\mathbb{P}(H^0(\omega_C^{\otimes n}))$. This automorphism must be non-trivial because it acts non-trivially on the n-canonical embedding of C into $\mathbb{P}^N$. As a result our functor is faithful.

Conversely, any non-trivial element $\phi \in \mathbf{PGL}(N+1)$ which leaves the n-canonical curve $C \hookrightarrow \mathbb{P}^N$ invariant must act non-trivially on C. The reason is that the fixed locus of ϕ is necessarily a proper linear subspace of $\mathbb{P}^N$. However, the n-canonical embedding of C can not be contained in a proper linear subspace. This implies that the functor p is full.

Now if $E \to B$ is an object of $[\overline{H}_{g,n}/\mathbf{PGL}(N+1)]$ then there is a family $C_E \stackrel{\pi_E}{\to} E$ of curves of genus g together with an isomorphism $\mathbf{P}(\pi_{E,*}(\omega_{C_E/E})) \simeq \mathbf{P}^N_E$, where $\mathbf{PGL}(N+1)$ acts by changing the polarization. The morphism $C_E \to E$ has a $\mathbf{PGL}(N+1)$-linearized relatively ample line bundle and the quotient $B = E/\mathbf{PGL}(N+1)$ is a scheme. Descent theory says that in this case a quotient $C = C_E/\mathbf{PGL}(N+1)$ also exists as a scheme. Then $C_E \simeq C \times_B E$, so the object $E \to B$ is isomorphic to of $p(C \to B)$. Therefore p is an equivalence of categories. □

Corollary 3.1. *$F_{\mathcal{M}_g}$ and $F_{\overline{\mathcal{M}}_g}$ are separated Deligne-Mumford stacks.*

Proof. We have just shown that $F_{\mathcal{M}_g}$ and $F_{\overline{\mathcal{M}}_g}$ are quotients of a scheme by a smooth group scheme. Moreover, every stable curve defined over an algebraically closed field has a finite and reduced automorphism group, so the stabilizers of the geometric points are finite and reduced. Therefore, they are Deligne-Mumford stacks by Corollary 2.2.

The separatedness of $F_{\mathcal{M}_g}$ and $F_{\overline{\mathcal{M}}_g}$ follows from [DM, Theorem 1.11] which states that if C'/B and C/B are stable curves then the map $\mathbf{Iso}_B(C', C) \to B$ is finite. □

3.2. **Properness of $F_{\overline{\mathcal{M}}_g}$.** Given that $F_{\overline{\mathcal{M}}_g}$ is a Deligne-Mumford stack, the valuative criterion of properness and the following stable reduction theorem show that it is proper over $\operatorname{Spec} \mathbb{Z}$.

Theorem 3.3. *Let B be the the spectrum of a DVR with function field K, and let $X \to B$ be a family of curves such that its restriction*

$X_K \to \operatorname{Spec} K$ is a smooth curve. Then there is a finite extension K'/K and a unique stable family $X' \to B'$ where B' is the normalization of B in K' such that the restriction $X' \to \operatorname{Spec} K'$ is isomorphic to $X_K \times_K K'$.

Remark 3.3 (Remarks on the proof of Theorem 3.3)*.* The properness of $F_{\overline{\mathcal{M}}_g}$ follows from the observation: To check the properness of a morphism using the valuative criterion, it suffices to consider maps where the image of $\operatorname{Spec} K$ is contained in a fixed dense open substack (see the discussion after the statement of [DM, Theorem 4.19]).

This theorem was originally proved (but not published) in characteristic zero by Mumford and Mayer ([GIT, Appendix D]). There is a relatively straightforward algorithmic version of this theorem in characteristic 0 which I learned from Joe Harris. Blow up the singular points of the special fiber of X/B until the total space of the family is smooth and the special fiber has only nodes as singularities. The modified special fiber will have a number of components with positive multiplicity coming from the exceptional divisors in the blowups. Next, do a base change of degree equal to the l.c.m. of the multiple components. After base change all components of the special fiber will have multiplicity 1. Then contract all (-1) and (-2) rational components in the total space. (That this can be done follows from the existence of minimal models for surfaces.) The special fiber is now stable. Furthermore, the total space of the new family is a minimal model for the surface. Since minimal models of surfaces are unique, the stable limit curve is unique.

This algorithmic proof fails in characteristic $p > 0$, because after blowing up some components of the special fiber may have multiplicity divisible by p. In this case, it will not be possible to make the component become reduced after base change.

Deligne and Mumford proved the stable reduction theorem in arbitrary characteristic using Neron models of the Jacobians of the curves ([DM]). Later Artin and Winters [AW] gave a direct geometric proof using the theory of curves on surfaces.

3.3. **Irreducibility of $F_{\mathcal{M}_g}$ and $F_{\overline{\mathcal{M}}_g}$.** Using the description of the moduli stacks as quotients of H_g and $\overline{H}_g$ we can deduce properties of the stacks from the corresponding properties of the Hilbert scheme. In particular, deformation theory shows that H_g and $\overline{H}_g$ are smooth over $\operatorname{Spec} \mathbb{Z}$ ([DM, Cor 1.7]). Since the map $H_g \to F_{\mathcal{M}_g}$ (resp. $\overline{H}_g \to F_{\overline{\mathcal{M}}_g}$) is smooth we see that $F_{\overline{\mathcal{M}}_g}$ is smooth.

Further analysis [DM, Cor 1.9] shows that the scheme $\overline{H}_g - H_g$ representing polarized, singular, stable curves is a divisor with normal crossings in $\overline{H}_g$. This property descends to the moduli stacks.

Theorem 3.4. [DM, Thm 5.2] *$F_{\overline{\mathcal{M}}_g}$ is smooth and proper over* $\operatorname{Spec}\mathbb{Z}$. *The complement $F_{\overline{\mathcal{M}}_g} - F_{\mathcal{M}_g}$ is a divisor with normal crossings in $F_{\overline{\mathcal{M}}_g}$.*

The main result of [DM] is the following theorem:

Theorem 3.5. [DM] *$F_{\overline{\mathcal{M}}_g}$ has irreducible geometric fibers over* $\operatorname{Spec}\mathbb{Z}$.

Remark 3.4. Deligne and Mumford gave two proofs of this theorem. In both cases they deduce the result from the classical characteristic 0 result stated below. We outline below their second proof, which uses Deligne-Mumford stacks.

Proposition 3.1. $F_{\mathcal{M}_g} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C}$ *is irreducible.*

Proof. It was shown classically that there is a space $H_{k,b}$ parametrizing degree k covers of $\mathbf{P}^1$ simply branched over b points defined over the complex numbers. In [Fu], Fulton showed that the functor $F_{H_{k,b}}$ whose sections over a base B are families of smooth curves $C \to B$ together with a degree k map $C \to \mathbf{P}^1_B$ expressing each geometric fiber as a cover of $\mathbf{P}^1$ simply branched over b points is represented by a scheme which we also call $H_{k,b}$. In characteristic greater than k it is a finite étale cover of $P_b = (\mathbf{P}^1)^b - \Delta$, where Δ is the union of all diagonals (This fact was known classically over $\mathbb{C}$.)

Since P_b is smooth, $H_{k,b}$ is also smooth (in sufficiently high characteristic). Thus, $H_{k,b}$ is irreducible if and only if it is connected. Over $\operatorname{Spec}\mathbb{C}$, the connectedness of $H_{k,b}$ in the classical topology was demonstrated by Hurwitz who showed that the monodromy group of the covering $H_{k,b} \to P_b$ acts transitively on the fiber over a base point in P_b for all k, b. Thus, $H_{k,b}$ is irreducible in sufficiently high characteristic.

Since there is a universal family of branched covers $C_{k,b} \to H_{k,b}$ there is a map $H_{k,b} \to F_{\mathcal{M}_g}$ (where $g = b/2 - k + 1$). By the Riemann-Roch theorem for smooth curves, every curve of genus g can be expressed as a degree k cover of $\mathbf{P}^1$ with b simple branch points, as long as $k > g + 1$. Thus for k (and thus b) sufficiently large, the map is surjective. Therefore $F_{\mathcal{M}_g}$ is irreducible in characteristic greater than k, and thus $F_{\mathcal{M}_g} \times_{\mathbb{Z}} \mathbb{C}$ is irreducible. □

Proof. (Outline of the proof of Theorem 3.5) Since $F_{\overline{\mathcal{M}}_g} - F_{\mathcal{M}_g}$ is a divisor $F_{\overline{\mathcal{M}}_g}$ is irreducible if and only if $F_{\mathcal{M}_g}$ is as well. The fibres of $F_{\overline{\mathcal{M}}_g} \to \operatorname{Spec}\mathbb{Z}$ are smooth, so it suffices to show that they are connected. The morphism $F_{\overline{\mathcal{M}}_g} \to \operatorname{Spec}\mathbb{Z}$ is proper, flat and has smooth geometric fibers, so by Theorem 2.4 the number of connected components of the geometric fibers is constant. By the proposition the geometric fiber $F_{\overline{\mathcal{M}}_g} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C}$ is connected, so every geometric fiber is connected. □

Remark 3.5. In [HM] Harris and Mumford constructed a compactification of $H_{k,b}$ where the boundary represents stable curves expressed as branched covers of chains of $\mathbf{P}^1$'s. The existence of this compactification implies that every smooth curve admits degenerations to singular stable curves. Fulton [Fu82] used this fact to resurrect an argument of Severi giving a purely algebraic proof of the irreducibility of $F_{\mathcal{M}_g}$ in characteristic 0. This combined with the results of [DM] give a purely algebraic proof that $F_{\mathcal{M}_g}$ is irreducible in arbitrary characteristic.

4. Construction of the moduli scheme

As we have previously seen, the moduli stack is a quotient stack of a smooth scheme $\overline{H}_g$ by $\mathbf{PGL}(N+1)$. In this, the final section, we discuss the construction of a quotient scheme $\overline{H}_g/\mathbf{PGL}(N+1)$ over an algebraically closed field k. We first prove that such a scheme is unique and is the coarse moduli space for the quotient stack. We then briefly discuss Gieseker's GIT construction of a quotient scheme.

4.1. **Moduli schemes and geometric quotients.** This definition is completely analogous to Mumford's definition ([GIT, p. 99]) of a coarse moduli scheme mentioned above.

Definition 4.1. The moduli space of a Deligne-Mumford stack F is a scheme M together with a morphism $\pi : F \to M$, such that

(*) for any algebraically closed field k there is a bijection between the set of isomorphism classes of objects in the groupoid $F(\Omega)$ and $M(\Omega)$, where $\Omega = \operatorname{Spec} k$.

Furthermore, M is universal in the sense that if N is a scheme then any morphism $F \to N$ factors through a morphism $M \to N$.

Remark 4.1. The universal property guarantees that the moduli scheme is unique if it exists. Though it is sufficient for our purpose, there are two drawbacks to this definition.

(1) In characteristic p, the property of being a moduli space is not invariant under base change; i.e. if $M' \to M$ is a morphism then M' need not be the moduli space of $M' \times_M F$. As a result, Gillet [**?**] gave an alternative definition: Namely, M is a moduli space for F if the morphism $F \to M$ is proper and a bijection on geometric points. This notion is clearly preserved by base change but such a scheme is not

unique. However, two moduli spaces are universally homeomorphic as schemes.

(2) If F is a Deligne-Mumford stack, then F may not have a moduli scheme (in either sense). If one is willing to look in the category of algebraic spaces a theorem of Keel and Mori [KM] states that, under very mild assumptions, if F is a D-M stack[3] there is an algebraic space M such that M is a moduli space for F; i.e. there is a morphism $F \to M$ which is a bijection on geometric points and which is universal for maps of *algebraic spaces*. However, these two notions may differ. For example, $\mathbf{A}^1$ is the moduli scheme (in our sense) of the non-separated algebraic space $[X/\mathbf{SL}(2, \mathbb{C})]$ discussed in Example 2.13.

Definition 4.2. [GIT, Definitions 0.5, 0.6] Let X/S be a scheme and let G/S be a smooth affine group scheme acting on X. An S-scheme Y is a *geometric quotient* of X by G if there is a morphism $X \to Y$ such that

(1) f is G invariant.

(2) The geometric fibers of f are orbits. (In particular f is surjective).

(3) f is universally submersive, i.e. $U \subset Y$ is open iff $f^{-1}(U)$ is open, and this property is preserved by base change.

(4) $f_*(\mathcal{O}_X)^G = \mathcal{O}_Y$.

Remark 4.2. The purpose of the geometric invariant theory developed by Mumford is to construct geometric quotients for the action of a geometrically reductive group The definition of a geometrically reductive group is given in [GIT, Appendix A]. In characteristic 0 this notion is the same as the notion of *linear* reductivity; i.e. every representation decomposes as a direct sum of irreducibles. However, in characteristic p the only linear reductive groups are extensions of tori by finite groups of order prime to p. However, for the purpose of these notes, it suffices to know that $\mathbf{SL}(N+1, k)$ is reductive for a field k.

The following is a restatement of [GIT, Prop 0.1].

Proposition 4.1. *A geometric quotient is a categorical quotient. That is, if $X \xrightarrow{f} Y$ is a geometric quotient and if $X \xrightarrow{g} Z$ is a G invariant morphism, then there is a unique morphism $\phi : Y \to Z$ such that $g = \phi \circ f$.*

Note that the proposition implies that if a geometric quotient exists then it is unique.

[3]The theorem of Keel and Mori also applies to a certain class of Artin stacks.

Now let X be a scheme with a G action such that the stabilizers of geometric points are finite and reduced. We have seen that the groupoid $[X/G]$ is a Deligne-Mumford stack.

Proposition 4.2. *If $f : X \to Y$ is a geometric quotient of X by G then Y is the moduli space of the stack $[X/G]$.*

If in addition the action of G on X is proper then the morphism $[X/G] \to Y$ is proper.

Proof. If $\Omega = \operatorname{Spec} K$ where K is algebraically closed, then $Hom(\Omega, Y)$ is, by Condition (2) of the definition, the set of orbits of K-valued points of X. This is exactly set of isomorphism classes in $[X/G](\Omega)$. Therefore, condition (*) is satisfied.

Next suppose that $[X/G] \to N$ is a morphism to a scheme. It is easy to see that that the induced morphism $X \to N$ is G-invariant. By the universal mapping property of the quotient, the morphism $X \to N$ factors through Y. Thus, the morphism $[X/G] \to N$ also factors through Y, so Y is the moduli scheme for $[X/G]$

If the action is proper then $[X/G]$ is separated, so the morphism $[X/G] \to Y$ is also separated. Then the the universal submersiveness of $f : X \to Y$ implies that the morphism $[X/G] \to Y$ satisfies the valuative criterion of properness. The proof is given in [Vi, Proof of Prop. 2.11] □

4.2. **Construction of quotients by geometric invariant theory.** From now on we will assume that all schemes are defined over an algebraically closed field k.

In this paragraph we discuss the geometric invariant theory necessary to construct $\mathcal{M}_g$ and and $\overline{\mathcal{M}}_g$ as quotients of Hilbert schemes of n-canonically embedded (stable) curves. Our source is [Gi, Chapter 0]. For a full treatment of geometric invariant the classic reference is Mumford's [GIT].

Let $X \subset \mathbf{P}^N$ be a projective scheme, and let G be a reductive group acting on X via a representation $G \to \mathbf{GL}(N+1)$.

Definition 4.3. (1) A closed point $x \in X$ is called semi-stable if there exists a non-constant G-invariant homogeneous polynomial F such that $F(x) \neq 0$.

(2) $x \in X$ is called stable if: $\dim o(x) = \dim G$ (where $o(x)$ denotes the orbit of x) and there exists a non-constant G-invariant polynomial such that $F(x) \neq 0$ and for every y_0 in $X_F = \{y \in X | F(y) \neq 0\}$, $o(y_0)$ is closed in X_F.

Let X^{ss} denote the semi-stable points of X, and X^s denote the stable points. Then $X^s \subset X^{ss}$ are both open in X. However, they may be empty.

The following is the first main theorem of geometric invariant theory.

Theorem 4.1. *There exists a projective scheme Y and an affine, universally submersive morphism $f_{ss} : X^{ss} \to Y$ such that Y is a categorical quotient (such a morphism is often called a good quotient in the literature). Furthermore, there exists $U \subset Y$ open such that $f^{-1}(U) = X^s$ and $f_s : X^s \to U$ is a geometric quotient of X^s by G.*

Remark 4.3. Proposition 4.2 implies that U is the moduli space of $[X^s/G]$. Moreover, geometric invariant theory also says that G acts properly on X^s so the morphism $[X^s/G] \to U$ is proper.

4.3. **Criteria for stability.** Let $X \subset \mathbf{P}^N$ be a projective scheme, and let $\tilde{X} \subset \mathbf{A}^{N+1}$ be the affine cone over X. Assume as above, that a reductive group G acts on X via a representation $G \to \mathbf{GL}(N+1)$. Then G acts on $\tilde{X}$ as well. The stability of $x \in X$ can be rephrased in terms of the stability of the points $\tilde{x} \in \tilde{X}$ lying over x.

Proposition 4.3. [GIT, Chapter 1, Proposition 2.2 and Appendix B] *A geometric point $x \in X$ is semi-stable if for one (and thus for all) $\tilde{x} \in X$ lying over X, $0 \notin o(\tilde{x})$. The point x is stable if $o(\tilde{x})$ is closed in $\mathbf{A}^{N+1}$ and has dimension equal to the dimension of G.*

The second main theorem of geometric invariant theory is Mumford's numerical criterion for stability which we now discuss.

Definition 4.4. A 1-parameter subgroup of G is a homomorphism $\lambda : G_m \to G$. This will be abbreviated to "λ is a 1-PS of G".

Now if λ is a 1-PS of G, then the since λ is 1-dimensional, there is a basis $\{e_0, \ldots, e_N\}$ of $\mathbf{A}^{N+1}$ such that the action of λ is diagonalizable with respect to this basis; i.e. $\lambda(t)e_i = t^{r_i}e_i$ where $t \in G_m$ and $r_i \in \mathbb{Z}$. If $\tilde{x} = \sum x_i e_i \in \tilde{X}$, then the set of r_i such that x_i is non-zero is called the λ-weights of $\tilde{x}$. Note that if $x \in \mathbf{P}^N$ then the λ-weights are the same for all points in $\mathbf{A}^{N+1} - 0$ lying over x.

Definition 4.5. $x \in X$ is λ-semi-stable if for one (and thus for all) $\tilde{x} \in \tilde{X}$ lying over x, $\tilde{x}$ has a non-positive λ weight. A point x is λ-stable if $\tilde{x}$ has a negative λ-weight.

Theorem 4.2. [GIT] *A point $x \in X$ is (semi)stable if and only if x is λ-(semi)stable for all 1-PS $\lambda : G_m \to G$.*

Remark 4.4 (Remark on the Proof). It is easy to see that if x is unstable (i.e. not semi-stable) with respect to $\lambda : G_m \to G$ then x is unstable. The reason is that if all the weights of λ are positive then 0 will be in the closure of the G-orbit of $\tilde{x}$ in $\mathbf{A}^{N+1} - 0$. The converse is more difficult.

Example 4.1. (cf. [GIT, Proposition 4.1]). The set of homogeneous forms of degree 4 in two variables forms a 5-dimensional vector space V. We will view $\mathbf{P}(V)$ as the space parametrizing 4-tuples of (not necessarily) distinct points in $\mathbf{P}^1$. There is a natural action of $\mathbf{SL}(2)$ on V inducing an action on $\mathbf{P}(V)$. Let us use the numerical criterion to determine the stable and semi-stable locus in $\mathbf{P}(V)$.

If $v \in V$ is a form of degree 4 and λ is a 1-PS subgroup of $\mathbf{SL}(2)$, then we can write $v = a_4X_0^4 + a_3X_0^3X_1 + a_2X_0^2X_1^2 + a_1X_0X_1^3 + a_0X_1^4$, and λ acts by $\lambda(t)(X_0) = t^rX_0$, $\lambda(t)(X_1) = t^{-r}X_0$ and $r > 0$(the weight on X_1 must be the negative of the weight on X_0, since λ maps to $\mathbf{SL}(2)$). The possible weights of v are $\{4r, 2r, 0, -2r, -4r\}$. In order for v to be λ-stable one of a_1 or a_0 must be non-zero. It is λ-semi-stable if one of a_2, a_1 or a_0 is non-zero. On the other hand, we can consider the 1-PS, τ which acts by $\tau(t)X_0 = t^{-r}X_0$ and $\tau(t)X_1 = t^rX_1$. In order for v to be τ-stable one of a_4 and a_3 must be non-zero, while it is τ-semi-stable if a_2 is non-zero. Combining the conditions imposed by λ and τ we see that if v is stable, then one of a_0 or a_1 is non-zero and one of a_3 or a_4 is non-zero. This condition is equivalent to the condition that $(1 : 0)$ and $(0 : 1)$ are not multiple points of the subscheme of $\mathbf{P}^1$ cut out by the form v. Likewise, v is semi-stable if $(1 : 0)$ or $(0 : 1)$ is cut out with multiplicity no more than 2. Finally v is unstable if $(1 : 0)$ or $(0 : 1)$ is cut out with multiplicity more than 2.

From this analysis it is clear that if $v \in V$ cuts out 4 distinct points then it will be stable for every 1-PS. Likewise if v cuts out a subscheme of $\mathbf{P}^1$ with each point having multiplicity 2 or less then it is semi-stable for every 1-PS. Conversely, if v cuts a point of multiplicity 3 or more then $v = X_0^3(a_0X_0 + a_1X_1)$ for some choice of coordinates on $\mathbf{P}^1$. Then v will have strictly positive weights for a 1-PS λ acting diagonally by $\lambda(t)X_0 = t^rX_0$ for $r > 0$.

4.4. **Gieseker's construction of $\overline{\mathcal{M}}_g$.** Let $Hilb^{N+1}_{P(t)}$ be the Hilbert scheme of curves in $\mathbf{P}^N$ with Hilbert polynomial $P(t)$. Grothendieck's uniform m-lemma says that if $X \subset \mathbf{P}^N$ is a curve with Hilbert polynomial $P(t)$, then there exists $m >> 0$ (independent of X) such that the restriction map $H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(m)) \to H^0(X, \mathcal{O}_X(m))$ is surjective and $\dim H^0(X, \mathcal{O}_X(m)) = P(m)$. Taking the $P(m)$-th exterior power of ϕ_m we obtain a linear map $V^m = \bigwedge^{P(m)} H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(m)) \to$

$\bigwedge^{P(m)} H^0(X, \mathcal{O}_X(m)) = k$ unique up to scalars; i.e., an element of $\mathbf{P}(V^m)$. The corresponding point in $\mathbf{P}(V^m)$ is called the m-th Hilbert point of X and is denote $H_m(X)$. In this way we obtain a map $Hilb^{N+1}_{P(t)} \to \mathbf{P}(V^m)$. For m sufficiently large this map is an embedding.

Both $\mathbf{SL}(N+1)$ and $\mathbf{PGL}(N+1)$ act on $\mathbf{P}(V^m)$ via the m-th exterior power representation of $\mathbf{SL}(N+1) \to \mathbf{GL}(V^m)$. Now the action of $\mathbf{SL}(N+1)$ factors through the action of $\mathbf{PGL}(N+1)$ (the stabilizer of $\mathbf{SL}(N+1)$ at a geometric point is the group of $N+1$ roots of unity) so we have the following proposition.

Proposition 4.4. *If $X \subset \mathbf{P}(V^m)$ then $X \to Y$ is a geometric quotient by $\mathbf{SL}(N+1)$ if and only if it is a geometric quotient by $\mathbf{PGL}(N+1)$.*

Proof. If $X \to Y$ is a geometric quotient by $\mathbf{SL}(N+1)$ then the geometric fibers are $\mathbf{SL}(N+1)$ orbits. These orbits are the same as the $\mathbf{PGL}(N+1)$ orbits. Likewise, $\mathcal{O}_X^{\mathbf{SL}(N+1)} = \mathcal{O}_X^{\mathbf{PGL}(N+1)}$. Thus, $\mathcal{O}_Y \simeq f_*\mathcal{O}_X^{\mathbf{PGL}(N+1)}$. Finally if W and V are $\mathbf{PGL}(N+1)$ invariant, they are also $\mathbf{SL}(N+1)$ invariant. Thus if they are disjoint, then since $X \to Y$ is an $\mathbf{SL}(N+1)$ quotient, their images will be disjoint as well. Hence $X \to Y$ is a $\mathbf{PGL}(N+1)$ quotient. The converse is similar. $\square$

Let $g \geq 3$ and $d \geq 20(g-1)$ be integers. Consider the Hilbert scheme $H^{N+1}_{P(t)}$ of curves in $\mathbf{P}^{N=d-g}$ with Hilbert polynomial $P(t) = dt - g + 1$ (the curves parametrized necessarily have arithmetic genus g). The first step in Gieseker's construction is to prove the following theorem. The proof is 10 pages long and uses the numerical criterion.

Theorem 4.3. [Gi, Theorem 1.0.0] *There exists an integer $m_0 >> 0$ such that if X is smooth then $H_{m_0}(X)$ is $\mathbf{SL}(N+1)$ stable.*

Remark 4.5. The theorem is not necessarily true for arbitrary $m_0 >> 0$. However there are infinitely many m_0 for which the theorem is true ([Gi, Remark after Theorem 1.0.0]).

The next, and technically most difficult step is to prove the following theorem. The proof takes 50 pages!

Theorem 4.4. [Gi, Theorem 1.0.1] *For the same integer m_0, every point in $Hilb^{N+1}_{P(t)} \cap \mathbf{P}(V^{m_0})^{ss}$ parametrizes a Deligne-Mumford semi-stable curve.*

Let $U \subset Hilb^{N+1}_{P(t)}$ be the subscheme of semi-stable curves with respect to the m_0-th Hilbert embedding. Let $Z_U \subset \mathbf{P}^N_U$ be the restriction of the universal family of projective curves. As before, view a point

$h \in U$ as parametrizing a curve X_h and a very ample line bundle L_h of degree d on X_h. Set $U_c = \{h \in U | L_h \simeq \omega^n_{X_h}\}$. This is a locally closed subscheme of U which is empty unless $2g - 2$ divides d. Gieseker then proves that U_c is in fact closed in U. He also proves that U_c is smooth ([Gi, Theorem 2.0.1]) and parametrizes only all Deligne-Mumford stable curves; thus, $U_c \simeq \overline{H}_{g,n}$. Since U_c is closed in U there is a projective quotient $U_c/\mathbf{SL}(N+1)$. Finally note that $\mathbf{PGL}(N+1)$ (and thus $\mathbf{SL}(N+1)$) acts with finite stabilizers on points of U_c because the curves parametrized have finite automorphism groups. Hence the points of U_c are in fact $\mathbf{SL}(N+1)$ stable. Thus a geometric quotient $U_c/\mathbf{SL}(N+1)$ exists. Since this is isomorphic to a geometric quotient $U_c/\mathbf{PGL}(N+1) \simeq \overline{H}_{g,n}/\mathbf{PGL}(N+1)$ we have succeeded in constructing a coarse moduli scheme for the stack of stable curves. □

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