A PLETHORA OF INERTIAL PRODUCTS

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Abstract. For a smooth Deligne-Mumford stack \( X \), we describe a large number of inertial products on \( K(I^2 X) \) and \( A^\star(I^2 X) \) and inertial Chern characters. We do this by developing a theory of inertial pairs. Each inertial pair determines an inertial product on \( K(I^2 X) \) and an inertial product on \( A^\star(I^2 X) \) and Chern character ring homomorphisms between them. We show that there are many inertial pairs; indeed, every vector bundle \( V \) on \( X \) defines two new inertial pairs. We recover, as special cases, both the orbifold products of [CR04, AGV02, FG03, JKK07, EJK10] and the virtual product of [GLS+07].

We also introduce an entirely new product we call the localized orbifold product, which is defined on \( K(I^2 X) \otimes \mathbb{C} \).

The inertial products developed in this paper are used in a subsequent paper [EJK12] to describe a theory of inertial Chern classes and power operations in inertial K-theory. These constructions provide new manifestations of mirror symmetry, in the spirit of the Hyper-Kähler Resolution Conjecture.

1. Introduction

The purpose of this note is to describe a large number of inertial products and Chern characters by developing a formalism of inertial pairs. An inertial pair for a Deligne-Mumford stack \( X \) is a pair \((\mathcal{R}, \mathcal{I})\), where \( \mathcal{R} \) is a vector bundle on the double inertia stack \( I^2 X \) and \( \mathcal{I} \) is a non-negative, rational K-theory class on the inertia stack \( I X \) satisfying certain compatibility conditions. For stacks with finite stabilizer, an inertial pair determines inertial products on cohomology, Chow groups, and K-theory of \( I^2 X \). In Chow and cohomology, this product respects an orbifold grading equal to the ordinary grading corrected by the virtual rank of \( \mathcal{I} \) (or age). An inertial pair also allows us to define an inertial Chern character, which is a ring homomorphism for the new inertial products.

The motivating example of an inertial pair is the orbifold pair \((\mathcal{R}, \mathcal{I})\), where \( \mathcal{R} \) is the obstruction bundle coming from orbifold Gromov-Witten theory, and \( \mathcal{I} \) is the class defined in [JKK07]. The corresponding product is the Chen-Ruan orbifold product, and the Chern character is the one defined in [JKK07]. One of the results of this paper is that every vector bundle \( V \) on a Deligne-Mumford stack determines two inertial pairs \((\mathcal{R}^+ V, \mathcal{I}^+ V)\) and \((\mathcal{R}^- V, \mathcal{I}^- V)\). The + product corresponds to the orbifold product on the total space of the bundle \( V \), but the − product is twisted by an isomorphism and does not directly correspond to an orbifold product on a bundle. However, we prove (Theorem 4.2.2) that there is an automorphism of the total Chow group \( A^\star(I^2 X) \otimes \mathbb{C} \) which induces a ring isomorphism between...
the − product for V and the + product for V∗. A similar result also holds for cohomology.

When V = T is the tangent bundle of ∃, we show that the virtual product considered by [GLS +07] is the product associated to the inertial pair (∃− T, ∃− T). It follows, after tensoring with C, that the virtual orbifold Chow ring is isomorphic (but not equal) to the Chen-Ruan orbifold Chow ring of the cotangent bundle T∗. Our result also implies that there is a corresponding Chern character ring homomorphism for the virtual product.

In the final section we show that in certain cases, even if V is not a vector bundle but just an element of K-theory, we can still determine a product in localized K-theory. This allows us to define a new product on K(I ∃) ⊗ C, which we call the localized product.

In a subsequent paper [EJK12] we will show that for Gorenstein inertial pairs (such as the one determining the virtual product) there is a theory of Chern classes and compatible power operations on inertial K-theory. This will be used to give further manifestations of mirror symmetry on hyper-Kähler Deligne-Mumford stacks.

Review of Previous Related Work. Because there has been much work in this area by many authors from different areas of mathematics, we give a brief overview here of previous work to help put the current paper in context.

In 2000, inspired by physicists [DHVW85, DHVW86], Chen-Ruan [CR02] developed a new product on the the cohomology of the inertia I ∃ of an almost complex orbifold ∃. In 2001, Fantechi-Göttsche [FG03] showed that when the orbifold ∃ was a global quotient [X/G] by a finite group, the Chen-Ruan orbifold cohomology ring HCR(∃) was the G-invariant subring of HFG(X, G), the cohomology of the inertia manifold IX endowed with a certain noncommutative product. It followed that if X is the symmetric product of a surface with trivial canonical class, then the orbifold cohomology of ∃ is isomorphic to the cohomology ring of the Hilbert scheme, as predicted by the Hyper-Kähler Resolution Conjecture [Rua06].

About the same time Kaufmann presented an axiomatic approach to orbifolding Frobenius algebras [Kau02b, Kau03], and described how the Fantechi-Göttsche construction fit into this framework [Kau02a].

Adem-Ruan [AR03] then studied the K-theory K(∃) of a global quotient orbifold ∃ = [X/G], where G is a Lie group and they also studied the twisted K-theory of [X/G]. They did not construct a new “orbifold” product on K(∃), but they did show that there is a Chern character that gives a vector space isomorphism from K(∃) to HCR(∃). This Chern character is not a ring homomorphism. Tu-Xu [TX06] later extended this result to more general twistings and orbifolds.

Abramovich-Graber-Vistoli [AGV02] constructed an algebraic version AAGV(∃) of the Chen-Ruan cohomology, producing the corresponding product on the Chow group AAGV(∃) = A∗(I ∃) of the inertial stack I ∃ of a (smooth) Deligne-Mumford stack with projective coarse moduli space.

In all of these constructions the basic idea is to use an analogue of the moduli space M0,3(X, 0) of genus-zero, three-pointed, orbifold (or G-equivariant) stable maps into ∃. This space has three evaluation maps ei : M0,3(∃, 0) −→ I ∃, and the structure constants (α1, α2, α3) for the new product on I ∃ are given by computing ∫M0,3(∃, 0) Πi=1 ei∗(αi) · eu(∃), where eu(∃) is the top Chern class of
an obstruction bundle on $\mathcal{M}_{0,3}(\mathcal{X}, 0)$. The main difficulty in computing the new product was computing the obstruction bundle $\mathcal{R}$ and its top Chern class.

In 2004, Chen-Hu [CH06] produced a formula for the obstruction class in the case of Abelian orbifolds and used it to describe a deRham model for the Chen-Ruan product. In 2005, Jarvis-Kaufmann-Kimura [JKK07] proved a simple, intrinsic formula for the obstruction bundle $\mathcal{R}$ for general (not just Abelian) orbifolds, requiring no mention of stable curves or moduli spaces of maps. In the Abelian case this formula reduces to Chen-Hu’s result. In [JKK07], that formula is used to do several things:

1. Create Chow- (respectively, K-) theoretic analogues of the Fantechi-Göettsche ring $H_{FG}(X, G)$ whose rings of invariants is the AGV ring $A_{AGV}(\mathcal{X})$ (respectively, a ring whose underlying vector space is $K(\mathcal{X})$ of Adem-Ruan). Corresponding products twisted by discrete torsion were also introduced.
2. Define a new (orbifold) product on the K-theory $K_{orb}(\mathcal{X})$ of the inertia $I\mathcal{X}$, for any smooth Deligne-Mumford stack $\mathcal{X}$.
3. Define an orbifold Chern character ring homomorphism from the new orbifold K-theory rings to the corresponding Chow or cohomology rings. This new Chern character is a deformation of the ordinary Chern character, as the latter fails to preserve the orbifold multiplications.
4. Outline how the same formula and formalism may be used to give analogous results in other categories, e.g. equivariant structures on almost complex manifolds with a Lie group action.

About the same time, Adem-Ruan-Zhang [ARZ08] independently defined an orbifold product on twisted $K_{orb}(\mathcal{X})$, and in the case of a global quotient by a finite group, Kaufmann-Pham [KP09] connected this to the twisted Drinfel’d double of the group ring.

Becerra-Uribe in [BU09] extended these results to the equivariant setting for global quotients by infinite Abelian groups, and in [EJK10], we extended these results to an equivariant setting for global quotients by general (non-Abelian) infinite groups by introducing a variant of the formula for the obstruction bundle in [JKK07].

A recent paper [HW11] repeats the description of the orbifold product of [JKK07, ARZ08], the formula of [JKK07, EJK10] for the obstruction class, and the Chern character ring homomorphism of [JKK07, EJK10] in the almost-complex setting, as originally described in Section 10 of [JKK07].

In [BGNX07, BGNX12, GLS+07, LUX08], a different product in inertial Chow and inertial cohomology theory, analogous to the Chas-Sullivan product [CS99] on loop spaces, was introduced. This so-called virtual (orbifold) product is a special case of the constructions of this paper, cf. Section 4.3. Surprisingly, it is not equal to the orbifold product for the cotangent bundle. We note, however, that after tensoring with $\mathbb{C}$, both the orbifold Chow and orbifold cohomology (but not orbifold K-theory) of the cotangent bundle are isomorphic to their virtual counterparts.

Kaufmann has also studied the idea of stringy products on Frobenius algebras in settings involving functors other than just K-theory, cohomology, and Chow theory (see [Kau02b, Kau03, Kau10]). He treated this primarily as an algebraic question and reformulated the problem of constructing a stringy product in terms of certain cocycles. It is not a priori clear that there should always exist a stringy product, but [Kau10] shows how to extend the ideas of [JKK07] to prove existence of at least
one stringy product for his more general setting. In some cases he can also show uniqueness of the product [Kau04].

Finally, we note that Pflaum-Postuma-Tang-Tseng [PPTT07] have shown that the Hochschild cohomology of a certain algebra attached to a groupoid presentation of a symplectic orbifold is isomorphic to the cohomology of the inertia orbifold as vector spaces. The product in Hochschild cohomology induces a product on the cohomology of the inertia orbifold. It would be interesting to understand the relation of that product to the products described in this paper.

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2. Background material from [EJK10]

To make this paper self-contained, we recall some background material from the paper [EJK10].

2.1. Background notation. We work entirely in the complex algebraic category. We will work exclusively with smooth Deligne-Mumford stacks \( \mathcal{X} \) which have finite stabilizer, by which we mean the inertia map \( I_\mathcal{X} \to \mathcal{X} \) is finite. We will also assume that every stack \( \mathcal{X} \) has the resolution property. This means that every coherent sheaf is the quotient of a locally free sheaf. This assumption has two consequences. The first is that the natural map \( K(\mathcal{X}) \to G(\mathcal{X}) \) is an isomorphism, where \( K(\mathcal{X}) \) is the Grothendieck ring of vector bundles and \( G(\mathcal{X}) \) is the Grothendieck group of coherent sheaves. The second consequence is that \( \mathcal{X} \) is a quotient stack [EHKV01]. This means that \( \mathcal{X} = [X/G] \), where \( G \) is a linear algebraic group acting on a scheme or algebraic space \( X \).

If \( \mathcal{X} \) is a smooth Deligne-Mumford stack, then we will implicitly choose a presentation \( \mathcal{X} = [X/G] \). This allows us to identify the Grothendieck ring \( K(\mathcal{X}) \) with the equivariant Grothendieck ring \( K_G(X) \), and the Chow ring \( A^*(\mathcal{X}) \) with the equivariant Chow ring \( A^*_G(X) \). We will use the notation \( K(\mathcal{X}) \) and \( K_G(X) \) (resp. \( A^*(\mathcal{X}) \) and \( A^*_G(X) \)) interchangeably.

Definition 2.1.1. Let \( G \) be an algebraic group acting on a scheme or algebraic space \( X \). We define the inertia space

\[
I_G X := \{(g, x) \mid gx = x\} \subset G \times X.
\]

There is an induced action of \( G \) on \( I_G X \) given by \( g \cdot (m, x) = (gm^{-1}, gx) \). The quotient stack \( I_\mathcal{X} := [I_G X/G] \) is the inertia stack of the quotient stack \( \mathcal{X} := [X/G] \).

More generally, we define the higher inertia spaces to be the \( k \)-fold fiber products

\[
I^k_G X = I_G X \times_X \ldots \times_X I_G X = \{(m_1, \ldots, m_k, x) \mid m_i x = x, \forall i = 1, \ldots, k\} \subset G^k \times X.
\]

The quotient stack \( I^k \mathcal{X} := [I^k_G X/G] \) is the corresponding higher inertia stack.
The assumption that $\mathcal{X}$ has finite stabilizer means that the projection $I_G X \longrightarrow X$ is a finite morphism. The composition $\mu: G \times G \longrightarrow G$ induces a composition $\mu: I_G X \times_X I_G X \longrightarrow I_G X$. This composition makes $I_G X$ into an $X$-group with identity section $X \longrightarrow I_G X$ given by $x \mapsto (1, x)$.

**Definition 2.1.2.** Let $G^\ell$ be a $G$-space with the diagonal conjugation action. A diagonal conjugacy class is a $G$-orbit $\Phi \subset G^\ell$.

**Definition 2.1.3.** For all $m$ in $G$, let $X^m = \{(m, x) \in I_G X\}$. For all $(m_1, \ldots, m_\ell)$ in $G\ell$, let $X^{m_1, \ldots, m_\ell} = \{(m_1, \ldots, m_\ell, x) \in I_G X\}$. For all conjugacy classes $\Psi \subset G$, let $I(\Psi) = \{(m, x) \in I_G X \mid m \in \Psi\}$. More generally, for all diagonal conjugacy classes $\Phi \subset G^\ell$, let $I^\ell(\Phi) = \{(m_1, \ldots, m_\ell, x) \in I_G X \mid (m_1, \ldots, m_\ell) \in \Phi\}$.

By definition, $I(\Psi)$ and $I^\ell(\Phi)$ are $G$-invariant subsets of $I_G X$ and $I_G X$, respectively. If $G$ acts with finite stabilizer on $X$, then $I(\Psi)$ is empty unless $\Psi$ consists of elements of finite order. Likewise, $I^\ell(\Phi)$ is empty unless every $\ell$-tuple $(m_1, \ldots, m_\ell) \in \Phi$ generates a finite group. Since conjugacy classes of elements of finite order are closed, $I(\Psi)$ and $I^\ell(\Phi)$ are closed.

**Proposition 2.1.4.** ([EJK10] Prop. 2.11, Prop. 2.17) If $G$ acts properly on $X$, then $I(\Psi) = \emptyset$ for all but finitely many conjugacy classes $\Psi$ and the $I(\Psi)$ are unions of connected components of $I_G X$. Likewise, $I^\ell(\Phi)$ is empty for all but finitely many diagonal conjugacy classes $\Phi \subset G^\ell$ and each $I^\ell(\Phi)$ is a union of connected components of $I_G X$.

We frequently work with a group $G$ acting on a space $X$ where the quotient stack $[X/G]$ is not connected. As a consequence, some care is required in the definition of the rank and Euler class of a vector bundle. Note that for any $X$, the group $A^0_G(X)$ satisfies $A^0_G(X) = \mathbb{Z}^\ell$, where $\ell$ is the number of connected components of the quotient stack $\mathcal{X} = [X/G]$.

**Definition 2.1.5.** If $E$ is an equivariant vector bundle on $X$, then we define the rank of $E$ to be $\text{rk}(E) := \text{Ch}^0(E) \in \mathbb{Z}^\ell = A^0_G(X)$. Note that the rank of $E$ lies in the semi-group $\mathbb{N}^\ell$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. If $E_1, \ldots, E_n$ are vector bundles, then the virtual rank (or augmentation) of the element $\sum_{i=1}^n n_i[E_i] \in K_G(X)$ is the weighted sum $\sum_i n_i \text{rk}(E_i) \in \mathbb{Z}^\ell$.

If $E$ is a $G$-equivariant vector bundle on $X$, then the rank of $E$ on the connected components of $\mathcal{X} = [X/G]$ is bounded (since we assume that $\mathcal{X}$ has finite type).

**Definition 2.1.6.** If $E$ is a $G$-equivariant vector bundle on $X$, we call the element $\lambda_{-1}(E^*) = \sum_{i=0}^\infty (-1)^i[\Lambda^i E^*] \in K_G(X)$ the $K$-theoretic Euler class of $E$. (Note that this sum is finite.)

Likewise, we define the element $\epsilon_{\text{top}}(E) \in A^*_G(X)$, corresponding to the sum of the top Chern classes of $E$ on each connected component of $[X/G]$, to be the Chow-theoretic Euler class of $E$. These definitions can be extended to any non-negative element by multiplicativity. It will be convenient to use the symbol $\text{eu}(\mathcal{F})$ to denote both of these Euler classes for a non-negative element $\mathcal{F} \in K_G(X)$.

2.2. **The logarithmic restriction and twisted pullback.** We recall a construction from [EJK10] that will be used several times throughout the paper. However, to improve clarity, we use slightly different notation than in [EJK10].
Definition 2.2.1. \cite{EJK10} Let \( X \) be an algebraic space with an action of an algebraic group \( Z \). Let \( E \) be a rank-\( n \) vector bundle on \( X \) and let \( g \) be a unitary automorphism of the fibers of \( E \) \( \to \) \( X \). If we assume that the action of \( g \) commutes with the action of \( Z \) on \( E \), the eigenbundles for the action of \( g \) are all \( Z \)-subbundles. Let \( \exp(2\pi \sqrt{-1} \alpha_1), \ldots, \exp(2\pi \sqrt{-1} \alpha_r) \) be the distinct eigenvalues of \( g \) acting on \( E \), with \( 0 \leq \alpha_k < 1 \) for all \( k \in \{1, \ldots, r\} \), and let \( E_1, \ldots, E_r \) be the corresponding eigenbundles.

We define the \textit{logarithmic trace of} \( E \) by the formula
\[
L(g)(E) = \sum_{k=1}^{r} \alpha_k E_k \in K_Z(X) \otimes \mathbb{R}
\]
on each connected component of \( X \).

The following key fact about the logarithmic trace was proved in \cite{EJK10}.

Proposition 2.2.2. \cite{EJK10} Prop. 4.6] Let \( g = (g_1, \ldots, g_{\ell}) \) be an \( \ell \)-tuple of elements of a compact subgroup of a reductive group \( H \), satisfying \( \prod_{i=1}^{\ell} g_i = 1 \). Let \( X \) be an algebraic space with an action of an algebraic group \( Z \), and let \( V \) be a \((Z \times H)\)-equivariant bundle on \( X \), where \( H \) is assumed to act trivially on \( X \). The element
\[
\sum_{i=1}^{\ell} L(g_i)(V) - V + V^g
\]
in \( K_Z(X) \) is represented by a \( Z \)-equivariant vector bundle.

Using Proposition 2.2.2 we make the following definition.

Definition 2.2.3. Let \( G \) be an algebraic group acting quasi-freely on an algebraic space, and let \( V \) be a \( G \)-equivariant vector bundle on \( X \). Given \( g = (g_1, \ldots, g_{\ell}) \in G^\ell \), if the \( g_i \) all lie in a common compact subgroup and satisfy \( \prod_{i=1}^{\ell} g_i = 1 \), then set
\[
V(g) = \sum_{i=1}^{\ell} L(g_i)(V|_{X^*}) - V|_{X^*} + V^g|_{X^*}.
\]

We wish to extend this definition to give a map from \( K_G(X) \) to \( K_G(I_G X) \), but we must first understand the decomposition of \( K_G(I_G X) \) and \( A_G^*(I_G X) \) by conjugacy classes.

As a consequence of Proposition 2.1.4, we see that \( K_G(I_G X) \) (resp. \( A_G^*(I_G X) \)) is a direct sum of the of \( K_G(I(\Psi)) \) (resp. \( A_G^*(I(\Psi)) \)) as \( \Psi \) runs over conjugacy classes of elements of finite order in \( G \). A similar statement holds for the equivariant K-theory and Chow groups of the higher inertia spaces as well.

Using Morita equivalence we can give a more precise description of \( K_G(I(\Psi)) \). If \( m \in \Psi \) is any element and \( Z = Z_G(m) \) is the centralizer of \( m \) in \( G \), then \( K_G(I(\Psi)) = K_Z(X^m) \) and \( A_G^*(I(\Psi)) = A_Z^*(X^m) \). Similarly, if \( \Phi \subseteq G^\ell \) is a diagonal conjugacy class and \( (m_1, \ldots, m_{\ell}) \in \Phi \) and \( Z = \bigcap_{i=1}^{\ell} Z_G(m_i) \), then \( K_G(I(\Phi)) = K_Z(X^{m_1,\ldots,m_{\ell}}) \) and \( A_G^*(I(\Phi)) = A_Z^*(X^{m_1,\ldots,m_{\ell}}) \).

Definition 2.2.4. Define a map \( L : K_G(X) \to K_G(I_G X) \otimes \mathbb{Q} \), as follows. For each conjugacy class \( \Psi \subseteq G \) and each \( V \in K_G(X) \) let \( L(\Psi)(V) \) be the class in \( K_Z(I(\Psi)) \) which is Morita equivalent to \( L(g)(V|_{X^*}) \in K_Z(X^g) \). Here \( g \) is any element of \( \Psi \), and \( Z = Z_G(g) \) is the centralizer of \( g \in G \). The class \( L(V) \) is the class whose restriction to \( I(\Psi) \) is \( L(\Psi)(V) \).
The proof of [EJK10, Lm. 5.4] shows that \( L(\Psi)(V) \) (and thus \( L(V) \)) is independent of the choice of \( g \in \Psi \).

**Definition 2.2.5.** If the diagonal conjugacy class \( \Phi \subset G^\ell \) is represented by \((g_1, \ldots, g_\ell)\) such that \( \prod_{i=1}^\ell g_i = 1 \), then we define \( V(\Phi) \) to be the class in \( K_G(\tilde{\mathfrak{g}}_G\mathcal{X}) \) which is Morita equivalent to \( V(g) \), where \( g = (g_1, \ldots, g_\ell) \) is any element of \( \Phi \). Again \( V(\Phi) \) is independent of the choice of representative \( g \in \Phi \).

**Definition 2.2.6.** Identify \( \tilde{\mathfrak{g}}_G\mathcal{X} \) with the closed and open subset of \( \tilde{\mathfrak{g}}_G^{\ell+1}\mathcal{X} \) consisting of tuples \( \{(g_1, \ldots, g_{\ell+1}, x)|g_1g_2\cdots g_{\ell+1} = 1\} \). If \( V \in K_G(X) \), let \( LR(V) \in K_G(\tilde{\mathfrak{g}}_G\mathcal{X}) \) be the class whose restriction to \( \tilde{\mathfrak{g}}_G^{\ell+1}(\Phi) \) is \( V(\Phi) \), where the diagonal conjugacy class \( \Phi \in G^{\ell+1} \) is represented by a tuple \((g_1, \ldots, g_{\ell+1})\) satisfying \( g_1 \cdots g_{\ell+1} = 1 \).

### 2.3. Orbifold products and the orbifold Chern character.

In this section we briefly review the construction and properties of orbifold products and orbifold Chern characters because they serve as a model for what we will do later.

**Definition 2.3.1.** For \( i \in \{1, 2, 3\} \), let \( e_i : \tilde{\mathfrak{g}}_G^{2}\mathcal{X} \rightarrow I_G\mathcal{X} \) be the evaluation morphism taking \((m_1, m_2, m_3, x) \mapsto (m_i, x)\) and let \( \mu : \tilde{\mathfrak{g}}_G^{2}\mathcal{X} \rightarrow I_G\mathcal{X} \) be the morphism taking \((m_1, m_2, m_3, x) \mapsto (m_1m_2, x) = (m_3^{-1}, x)\).

**Definition 2.3.2.** Let \( \mathcal{T} \) be the equivariant bundle on \( X \) corresponding to the tangent bundle of \( \mathcal{X} \), which satisfies \( \mathcal{T} = TX - \mathfrak{g} \) in \( K_G(X) \), where \( \mathfrak{g} \) is the Lie algebra of \( G \).

**Definition 2.3.3 (EJK10, JKK07).** The orbifold product on \( K_G(I_G\mathcal{X}) \) and \( A_G^*(I_G\mathcal{X}) \) is defined as
\[
x \star y := \mu_*(e_1^*x \cdot e_2^*y \cdot eu(LR(\mathcal{T}))),
\]
both for \( x, y \in K_G(I_G\mathcal{X}) \) and for \( x, y \in A_G^*(I_G\mathcal{X}) \).

**Definition 2.3.4.** We define the element \( \mathcal{S} := L(\mathcal{T}) \) in \( K_G(I_G\mathcal{X})_\mathbb{Q} \) to be the logarithmic trace of \( \mathcal{T} \), that is, for each \( m \) in \( G \), we define \( \mathcal{S}_m \) in \( K_{Z_G(m)}(X^m)_\mathbb{Q} \) by
\[
\mathcal{S}_m := L(m)(\mathcal{T}).
\]
The rank of \( \mathcal{S} \) is a \( \mathbb{Q} \)-valued, locally constant function on \( I\mathcal{X} = [I_G\mathcal{X}/G] \) called the age.

**Remark 2.3.5.** If the age of a connected component \([U/G]\) of \( I\mathcal{X} \) is zero, then \([U/G]\) must be a connected component of \( \mathcal{X} \subset I\mathcal{X} \).

**Remark 2.3.6.** In [Kau10], it is pointed out that the classes \( \mathcal{S} \) are manifestations of what physicists call twist fields.

**Definition 2.3.7.** Given an element \( x \) in \( A_G^*(I_G\mathcal{X}) \) with ordinary Chow grading \( \deg x \), the orbifold degree (or grading) of \( x \) is, like the ordinary Chow grading, constant on each component \( U \) of \( I_G\mathcal{X} \) corresponding to a connected component of \([U/G]\) of \([I_G\mathcal{X}/G]\). On such a component \( U \) we define it to be the non-negative rational number
\[
\deg_{\text{orb}} x|_U = \deg x|_U + \text{age}([U/G]),
\]
The induced grading on the group \( A_G^*(I_G\mathcal{X}) \) consists of summands \( A_G^{(q)}(I_G\mathcal{X}) \) of all elements with orbifold degree \( q \).
Theorem 2.3.8 ([JKK07, EJK10]). The equivariant Chow group \( (A^*_G(I_GX), \ast, \deg_{orb}) \) is a \( \mathbb{Q}^C \)-graded, commutative ring with unity 1, where 1 is the identity element in \( A^*_G(X) = A^*_G(X^1) \subseteq A^*_G(I_GX) \) and \( C \) is the number of connected components of \([I_GX/G]\).

Equivariant K-theory \((K_G(I_GX), \ast)\) is a commutative ring with unity 1, where 1 := \( \theta_X \) is the structure sheaf of \( X = X^1 \subseteq I_GX \).

Definition 2.3.9. The orbifold Chern character \( \mathcal{Ch} : K_G(I_GX) \longrightarrow A^*_G(I_GX)_\mathbb{Q} \) is defined by the equation

\[
\mathcal{Ch}(F) := \text{Ch}(F) \cdot \text{Td}(-\mathcal{F})
\]

for all \( F \in K_G(I_GX) \), where Td is the usual Todd class. Moreover, for all \( \alpha \in \mathbb{Q} \) we define \( \mathcal{Ch}^\alpha(\mathcal{F}) \) by the equation

\[
\mathcal{Ch}(\mathcal{F}) = \sum_{\alpha \in \mathbb{Q}} \mathcal{Ch}^\alpha(\mathcal{F}),
\]

where each \( \mathcal{Ch}^\alpha(\mathcal{F}) \) belongs to \( A^\alpha_G(I_GX) \).

The orbifold virtual rank (or orbifold augmentation) is \( \mathcal{Ch}^0 : K_G(I_GX) \longrightarrow A^0_G(I_GX)_\mathbb{Q} \).

Theorem 2.3.10 ([EJK10, JKK07]). The orbifold Chern character

\[
\mathcal{Ch} : (K_G(I_GX), \ast) \longrightarrow (A^*_G(I_GX)_\mathbb{Q}, \ast)
\]

is a ring homomorphism.

In particular, if \([U/G]\) is a connected component of \([I_GX/G]\), then the virtual rank homomorphism restricted to the component \([U/G]\) gives a homomorphism \( \mathcal{Ch}^0 : K_G(U) \longrightarrow A^0_G(U)_\mathbb{Q} = \mathbb{Q} \), satisfying

\[
\mathcal{Ch}^0(\mathcal{F}) = \begin{cases} 0 & \text{if age}([U/G]) > 0 \\ \text{Ch}^0(\mathcal{F}) & \text{if age}([U/G]) = 0. \end{cases}
\]

for any \( \mathcal{F} \in K_G(U) \).

3. Inertial products, Chern characters, and inertial pairs

In this section we generalize the ideas of orbifold cohomology, obstruction bundles, orbifold grading and the orbifold Chern character, by defining inertial products on \( K_G(I_GX) \) and \( A^*_G(I_GX) \) using inertial bundles on \( I^2_GX \). We further define a rational grading and a Chern character ring homomorphism via Chern-compatible classes of \( K_G(I_GX)_\mathbb{Q} \).

The original example of an associative bundle is the obstruction bundle \( \mathcal{R} = LR(T) \) of orbifold cohomology, and the original example of a Chern-compatible class is the logarithmic trace \( \mathcal{S} \) of \( T \), as described in Definition 2.3.4.

We show below that there are many inertial pairs of associative inertial bundles on \( I^2_GX \) with Chern-compatible elements on \( I_GX \), and hence there are many associative inertial products on \( K_G(I_GX) \) and \( A^*_G(I_GX) \) with rational gradings and Chern character ring homomorphisms.
3.1. Associative bundles and inertial products. We recall the following definition from \[EJK10\].

**Definition 3.1.1.** Given a class \(c \in \mathcal{A}_G^*(\mathbb{P}^2_G X)\) (resp. \(K_G(\mathbb{I}^2_G X)\)), we define the inertial product with respect to \(c\) to be
\[
x \star_c y := \mu_x(e_1^x \cdot e_2^y \cdot c), \tag{4}
\]
where \(x, y \in \mathcal{A}_G^*(I_G X)\) (resp. \(K_G(I_G X)\)).

Given a vector bundle \(\mathcal{R}\) on \(\mathbb{P}^2_G X\) we define inertial products on \(\mathcal{A}_G^*(I_G X)\) and \(K_G(I_G X)\) via formula \((4)\), where \(c = eu(\mathcal{R})\) is the Euler class of the bundle \(\mathcal{R}\).

**Definition 3.1.2.** We say that \(\mathcal{R}\) is an associative bundle on \(\mathbb{P}^2_G X\) if the \(*_\text{eu}(\mathcal{R})\) products on both \(\mathcal{A}_G^*(I_G X)\) and \(K_G(I_G X)\) are commutative and associative with identity \(1\), where \(1\) is the identity class in \(\mathcal{A}_G^*(X)\) (resp. \(K_G(X)\)), viewed as a summand in \(\mathcal{A}_G^*(I_G X)\) (resp. \(K_G(I_G X)\)).

**Proposition 3.1.3.** A sufficient condition for \(*_\text{eu}(\mathcal{R})\) to be commutative with identity \(1\) is that the following conditions be satisfied.
\[
\begin{align*}
(1) & \quad \mathcal{R}|_{\{1\}} = \emptyset, \\
(2) & \quad i^*\mathcal{R} = \mathcal{R},
\end{align*}
\]
for every conjugacy class \(\Phi \subset G \times G\) such that \(e_1(\Phi) = 1\) or \(e_2(\Phi) = 1\).

**Proof.** This is almost just a restatement of Propositions 3.7–3.9 in \[EJK10\]. However, we note that in Proposition 3.9 of \[EJK10\] there is a slight error—that proposition incorrectly stated that the map \(i: \mathbb{I}^2_G X \rightarrow \mathbb{I}^2_G X\) was the map induced by the naive involution \((m_1, m_2) \mapsto (m_2, m_1)\), rather than the correct “braiding map” \((m_1, m_2, x) \mapsto (m_1m_2m_1^{-1}, m_1, x)\).

A sufficient condition for associativity is also given in \[EJK10\]. To state the condition requires some notation which we recall from that paper. Let \((m_1, m_2, m_3) \in G^3\) such that \(m_1m_2m_3 = 1\), and let \(\Phi_{1,2,3} \subset G^3\) be its diagonal conjugacy class.

Let \(\Phi_{12,3}\) be the conjugacy class of \((m_1m_2, m_3)\) and \(\Phi_{1,23}\) the conjugacy class of \((m_1, m_2m_3)\). Let \(\Phi_{i,j}\) be the conjugacy class of the pair \((m_i, m_j)\) with \(i < j\). Finally let \(\Psi_{12}\) be the conjugacy class of \((m_1m_2m_3)\); let \(\Psi_{ij}\) be the conjugacy class of \((m_im_j)\); and let \(\Psi_i\) be the conjugacy class of \(m_i\). There are evaluation maps \(e_1: \Gamma^3(\Phi_{a,b}) \rightarrow I(\Psi_a), e_2: \Gamma^3(\Phi_{a,b}) \rightarrow I(\Psi_b), e_{i,j}: \Gamma^3(\Phi_{1,2,3}) \rightarrow \Gamma^2(\Phi_{i,j})\), and the composition maps \(\mu_{12,3}: \Gamma^3(\Phi_{1,2,3}) \rightarrow \Gamma^2(\Phi_{1,23})\), and \(\mu_{12,3}: \Gamma^3(\Phi_{1,2,3}) \rightarrow \Gamma^2(\Phi_{1,23})\).

The various maps we have defined are related by the following Cartesian diagrams of l.c.i. morphisms:
\[
\begin{array}{c}
\Gamma^3(\Phi_{1,2,3}) \xrightarrow{e_{1,2}} \Gamma^2(\Phi_{1,2}) \xrightarrow{e_{2,3}} \Gamma^2(\Phi_{2,3}) \\
\mu_{12,3} \downarrow \quad \mu \downarrow \quad \mu_{12,3} \downarrow \quad \mu \\
\Gamma^2(\Phi_{12,3}) \xrightarrow{e_1} I(\Psi_{12}) \xrightarrow{e_2} \Gamma^2(\Phi_{2,3}) \xrightarrow{e_3} \Gamma^2(\Phi_{2,3}) \xrightarrow{e_3} I(\Psi_{23})
\end{array}
\tag{7}
\]
Let \(E_{1,2}\) and \(E_{2,3}\) be the respective excess normal bundles of the two diagrams \((7)\).
Proposition 3.1.4. Let $\mathcal{R}$ be a vector bundle on $\mathbb{I}_G^2 X$ satisfying \((5)\) and \((6)\). A sufficient condition for $\mathcal{R}$ to be an associative bundle is if
\[
e_1^*\mathcal{R} + \mu_1^* \mathcal{R} + E_1^2 = e_2^* \mathcal{R} + \mu_1^* \mathcal{R} + E_2 \tag{8}
\]
in $K_G(\mathbb{I}^3 X)$

Proof. This follows from the proof of Proposition 3.12 [EJK10], since the Euler class takes a sum of bundles to a product of Euler classes. □

In practice, the only way we have to show that a bundle $\mathcal{R}$ is associative is to show that it satisfies the identity \((8)\). This leads to our next definition.

Definition 3.1.5. A bundle $\mathcal{R}$ is strongly associative if it satisfies the identities \((5), (6),\) and \((8)\).

3.2. Chern characters, age, and inertial pairs. In many cases one can define a Chern character $K_G(I_G X)_\mathbb{Q} \longrightarrow A_G^*(-I_G X)$ which is a ring homomorphism with respect to the inertial product. To do this, however, we need to define a Chern compatible class $\mathscr{I} \in K_G(I_G X)$. As an added bonus, such a class will also allow us to define a new grading on $A_G^*(I_G X)$ compatible with the inertial product and analogous to the orbifold grading of orbifold cohomology.

Definition 3.2.1. Let $\mathcal{R}$ be an associative bundle on $\mathbb{I}_G^2 X$. A non-negative class $S \in K_G(I_G X)_\mathbb{Q}$ is called $\mathcal{R}$-Chern compatible if the map
\[
\tilde{ch}: K_G(I_G X)_\mathbb{Q} \longrightarrow A_G^*(I_G X)_\mathbb{Q},
\]
defined by
\[
\tilde{ch}(V) = \text{Ch}(V) \cdot \text{Td}(-\mathscr{I})
\]
is a ring homomorphism with respect to the $\mathcal{R}$-inertial products on $K_G(I_G X)$ and $A_G^*(I_G X)$.

Remark 3.2.2. Again, the original example of a Chern compatible class is the class $\mathscr{I}$ defined in [JKK07], but we will see other examples below.

Proposition 3.2.3. If $\mathcal{R}$ is an associative vector bundle on $\mathbb{I}_G^2 X$, then a non-negative class $\mathscr{I} \in K_G(I_G X)_\mathbb{Q}$ is $\mathcal{R}$-Chern compatible if the following identity holds in $K_G(\mathbb{I}_G^2 X)$
\[
\mathcal{R} = e_1^* \mathscr{I} + e_2^* \mathscr{I} - \mu^* \mathscr{I} + T_\mu. \tag{9}
\]

Proof. This follows from the same formal argument used in the proof of [EJK10 Theorem 7.3]. □

Definition 3.2.4. A class $\mathscr{I} \in K_G(I_G X)_\mathbb{Q}$ is strongly $\mathcal{R}$-Chern compatible if it satisfies Equation \((9)\).

A pair $(\mathcal{R}, \mathscr{I})$ is an inertial pair if $\mathcal{R}$ is a strongly associative bundle and $\mathscr{I}$ is $\mathcal{R}$-strongly Chern compatible.

Definition 3.2.5. We define the $\mathscr{I}$-age on a connected component $[U/G]$ of $I \mathscr{X}$ to be the rational rank of $\mathscr{I}$ on the component $[U/G]$: 
\[
\text{age}_{\mathscr{I}}([U/G]) = \text{rk}(\mathscr{I})_{[U/G]}.
\]
We define the $\mathscr{I}$-degree of an element $x \in A_G^*(I_G X)$ on such a component $U$ of $I_G X$ to be
\[
\text{deg}_{\mathscr{I}} x|_U = \text{deg} x|_U + \text{age}_{\mathscr{I}}([U/G]),
\]
where $\deg x$ is the degree with respect to the usual grading by codimension on $A_G^*(I_G X)$. Similarly, if $\mathcal{F}$ in $K_G(I_G X)$ is an element supported on $U$, then its $\mathcal{F}$-degree is

$$\deg_\mathcal{F} \mathcal{F} = \text{age}_\mathcal{F}(U) \mod \mathbb{Z}.$$ 

This yields a $\mathbb{Q}/\mathbb{Z}$-grading of the group $K_G(I_G X)$.

**Proposition 3.2.6.** If $\mathcal{R}$ is an associative vector bundle on $I_G^2 X$ and $\mathcal{F} \in K_G(I_G X)_\mathbb{Q}$ is strongly $\mathcal{R}$-Chern compatible, then the $\mathcal{R}$-inertial products on $A_G^*(I_G X)$ and $K_G(I_G X)$ respect the $\mathcal{F}$-degrees. Furthermore, the inertial Chern character homomorphism $\widehat{\text{ch}} : K_G(I_G X) \to A_G^*(I_G X)_\mathbb{Q}$ preserves the $\mathcal{F}$-degree modulo $\mathbb{Z}$.

**Proof.** If $x, y \in A_G^*(I_G X)$, then the formula

$$x \ast_{\text{eu}(\mathcal{R})} y = \mu_+ (e_1^* x \cdot e_2^* y \cdot \text{eu}(\mathcal{R}))$$

implies that $\deg(x \ast_{\text{eu}(\mathcal{R})} y) = \deg x + \deg y + \text{rk}\mathcal{R} + \text{rk}T_\mu$. Since $\mathcal{F}$ is strongly $\mathcal{R}$-Chern compatible, we know that $\mathcal{R} = e_1^* \mathcal{F} + e_2^* \mathcal{F} - \mu^* \mathcal{F} + T_\mu$. Comparing ranks shows that the $\mathcal{F}$-degree of $x \ast_{\text{eu}(\mathcal{R})} y$ is the sum of the $\mathcal{F}$-degrees of $x$ and $y$. The proof for $K_G(I_G X)$ follows from the fact that $\text{rk}\mathcal{R}$ and $\text{rk}T_\mu$ are integers. Finally, $\widehat{\text{ch}}$ preserves the $\mathcal{F}$-degree mod $\mathbb{Z}$ since if $\mathcal{F}$ in $K_G(I_G X)$ is supported on $U$, where $[U/G]$ is a connected component of $[I_G X/G]$, then so is its inertial Chern character. □

**Definition 3.2.7.** Let $A_G^{[q]}(I_G X)$ be the subspace in $A_G^*(I_G X)$ of elements with an $\mathcal{F}$-degree of $q$.

**Definition 3.2.8.** Given a class $\mathcal{F} \in K_G(I_G X)_\mathbb{Q}$, the restricted homomorphism $\widehat{\text{ch}}^0 : K_G(I_G X) \to A_G^{[0]}(I_G X)$ is called the inertial virtual rank (or inertial augmentation) for $\mathcal{F}$.

**Definition 3.2.9.** An inertial pair $(\mathcal{R}, \mathcal{F})$ is called Gorenstein if $\mathcal{F}$ has integral virtual rank and strongly Gorenstein if $\mathcal{F}$ is represented by a vector bundle.

The Deligne-Mumford stack $\mathcal{X} = [X/G]$ is strongly Gorenstein if the inertial pair $(\mathcal{R} = LR(\mathbb{T}), \mathcal{F})$ associated to the orbifold product (as in Definitions [2.3.3] and [2.3.4]) is strongly Gorenstein.

### 4. INERTIAL PAIRS ASSOCIATED TO VECTOR BUNDLES

In this section we show how, for each choice of $G$-equivariant bundle $V$ on $X$, we can use the methods of [EJK10] to define two new inertial pairs $(\mathcal{R}^+ V, \mathcal{F}^+ V)$ and $(\mathcal{R}^- V, \mathcal{F}^- V)$. We thus obtain corresponding inertial products and Chern characters. We denote the corresponding products associated to a vector bundle $V$ as the $\ast_V^+$ and $\ast_V^-$ products. The $\ast_V^+$ product can be interpreted as an orbifold product on the total space of $V$ while the $\ast_V^-$ product on the Chow ring is a sign twist of the $\ast_{(V^+)^*}$ product. Moreover, the two products induce isomorphic ring structures on $A^*(I_X) \otimes \mathbb{C}$. We prove that if $V = \mathbb{T}$ is the tangent bundle to $\mathcal{X} = [X/G]$, then the $\ast_V^-$ product agrees with the virtual orbifold product defined by [GLS+07].

To define the inertial pairs associated to a vector bundle, we introduce a variant of the logarithmic restriction introduced in [EJK10]. We begin with a simple proposition.
Proposition 4.0.10. Let $G$ be an algebraic group acting on a variety $X$ and suppose that $g_1, g_2$ lie in a common compact subgroup. Let $Z = Z_G(g_1, g_2)$ be the centralizer of $g_1$ and $g_2$ in $G$.

The virtual bundles

$$ V^+(g_1, g_2) = L(g_1)(V|_{X_{g_1, g_2}}) + L(g_2)(V|_{X_{g_1, g_2}}) - L(g_1, g_2)(V|_{X_{g_1, g_2}}) \quad (10) $$

and

$$ V^-(g_1, g_2) = L(g_1)^{-1}(V|_{X_{g_1, g_2}}) + L(g_2)^{-1}(V|_{X_{g_1, g_2}}) - L(g_1, g_2)^{-1}(V|_{X_{g_1, g_2}}) \quad (11) $$

are represented by non-negative integral elements in $K_Z(X^{g_1, g_2})$.

Proof. Since $X^g = X^{g^{-1}}$ and $V^-(g_1, g_2) = V^+(g_2^{-1}, g_1^{-1})$, it suffices to show $V^+(g_1, g_2)$ is represented by a non-negative integral element of $K_Z(X^{g_1, g_2})$. Let $g_3 = (g_2, g_1)^{-1}$. The identity $L(g)(V) + L(g^{-1})(V) = V - V^g$ implies that we can rewrite (10) as

$$ V^+(g_1, g_2) = L(g_1)(V|_{X_{g_1, g_2}}) + L(g_2)(V|_{X_{g_1, g_2}}) + L(g_3)(V|_{X_{g_1, g_2}}) - V + V^{g_1 g_2} + V^{g_1 g_2} - V^{g_1, g_2}. $$

Since $g_1 g_2 g_3 = 1$, by Proposition 2.2.2 [EJK10] Prop 4.6 the sum

$$ L(g_1, g_2, g_3)(V) = L(g_1)(V|_{X_{g_1, g_2}}) + L(g_2)(V|_{X_{g_1, g_2}}) + L(g_3)(V|_{X_{g_1, g_2}}) - V + V^{g_1 g_2} $$

is represented by a non-negative integral element of $K_Z(X_{g_1, g_2})$. Hence

$$ V^+(g_1, g_2) = L(g_1, g_2, g_3)(V) + V^{g_1 g_2} - V^{g_1, g_2} $$

is represented by a non-negative integral element of $K_Z(X^{g_1, g_2})$. □

Let $\Phi \subset G \times G$ be a diagonal conjugacy class. As in [EJK10] we may identify $K_G(\ell^2(\Phi))$ with $K_{Z_G(g_1, g_2)}(X^{g_1, g_2})$ for any $(g_1, g_2) \in \Phi$. Thanks to Proposition 4.0.10 we can define non-negative classes $V^+(\Phi)$ and $V^-(\Phi)$ in $K_G(\ell^2(\Phi))$. The argument used in the proof of [EJK10] Lemma 5.4] shows that the definitions of $V^+(\Phi)$ and $V^-(\Phi)$ are independent of the choice of $(g_1, g_2) \in G^2$. Thus we can make the following definition.

Definition 4.0.11. Define classes $R^+ V$ and $R^- V$ in $K_G(\ell^2_G X)$ by setting the component of $R^+ V$ (resp. $R^- V$) in $K_G(\ell^2_G(\Phi))$ to be $V^+(\Phi)$ (resp. $V^-(\Phi)$). Similarly, we define classes $S^\pm V \in K_G(I_G X)$ by setting the restriction of $S^\pm V$ to a summand $K_G(I(\Psi))$ of $K_G(I_G X)$ to be the class Morita equivalent to $L(g^{\pm 1})(V) \in K_{Z_G(g)}(X^g)$, where $g \in \Psi$ is any element.

Theorem 4.0.12. For any $G$-equivariant vector bundle $V$ on $X$, the pairs $(\mathcal{R}^+ V, \mathcal{J}^+ V) = (LR(T) + R^+ V, \mathcal{J}^+ T + S^+ V)$ and $(\mathcal{J}^- V, \mathcal{J}^+ V) = (LR(T) + R^+ V, \mathcal{J}^+ T + S^- V)$ are inertial pairs. Hence they define associative inertial products with a Chern character homomorphism.

Proof. Since $LR(T) = e_1^* \mathcal{J}^+ T + e_2^* \mathcal{J}^- T - \mu^* \mathcal{T} + T_\mu$ and $R^+ V = e_1^* S^+ V + e_2^* S^+ V - \mu^* S^+ V$, it follows that $\mathcal{J}^+ V$ is strongly $\mathcal{R}^+ V$-Chern compatible.

To complete the proof we must show that $LR(T) + R^+ V$ is a strongly associative bundle. From their definitions we know that $LR(T)$ and $R^+ V$ satisfy the identities (5) and (6). We also know that $LR(T)$ satisfies (8). Thus to prove that $LR(T) + R^+ V$, it suffices to show that $R^+ V$ satisfies the “cocycle” condition

$$ e_{1,2}^* R^+ V + \mu_{1,2,3}^* R^+ V = e_{2,3}^* R^+ V + \mu_{1,2,3}^* R^+ V. \quad (12) $$
Now (12) follows from the following identity of bundles restricted to $X^{m_1,m_2,m_3}$:

$$V^+(m_1,m_2) + V^+(m_1m_2,m_3) = V^+(m_2,m_3) + V^+(m_1,m_2m_3).$$

(13)

Equation (13) is a formal consequence of the definition of the bundles $V^+$. The result with $R^+V$ and $S^+V$ replaced by $R^-V$ and $S^-V$, respectively, is analogous. □

4.1. Geometric interpretation of the $\star_{V^+}$ product. The $\star_{V^+}$ has a relatively direct interpretation in terms of an orbifold product on the total space of the vector bundle $V \longrightarrow X$.

**Lemma 4.1.1.** Given a $G$-equivariant vector bundle $\pi : V \longrightarrow X$, the inertia space $I_GV$ is a vector bundle (of non-constant rank) on $I_GX$ with structure map $I\pi : I_GV \longrightarrow I_GX$.

**Proof.** Let $\Psi \subset G$ be a conjugacy class. Denote by $I_X(\Psi) \subset I_GX$ the component of $I_GX$ defined by $\{(g,x)|gx = x, g \in \Psi\}$. For any morphism $V \longrightarrow X$ and any conjugacy class $\Psi \subset G$, if $I_X(\Psi) = \emptyset$, then $I_V(\Psi)$ is also empty. Thus it suffices to show that $I_V(\Psi)$ is a vector bundle over $I_X(\Psi)$ for every conjugacy class $\Psi \subset G$ with $I_X(\Psi) \neq \emptyset$. Given $g \in \Psi$ the identification $I_X(\Psi) = G \times Z_G(g) \times X^g$ reduces the problem to showing that for $g \in G$ the fixed locus $V^g$ is a $Z_G(g)$-equivariant vector bundle over $X^g$. Since the map $V \longrightarrow X$ is $G$-equivariant, the map $V^g \longrightarrow X$ has image $X^g$. The fiber over a point $x \in X^g$ is just $(V_x)^g$, where $V_x$ is the fiber of $V \longrightarrow X$ at $x$.

Since $I_GV \longrightarrow I_GX$ is a vector bundle, the pullback maps

$$(I\pi)^*: K_G(I_GV) \longrightarrow K_G(I_GX) \quad \text{and} \quad (I\pi)^*: A^*_G(I_GV) \longrightarrow A^*_G(I_GX)$$

are isomorphisms. Both isomorphisms are compatible with the ordinary products on $K$-theory and equivariant Chow groups.

**Theorem 4.1.2.** For $x, y \in A^*_G(I_GX)$ or $x, y \in K_G(X)$, we have

$$x \star_{V^+} y = (I\pi)^* (((I\pi)^* x) \star ((I\pi)^* y)),$$

(14)

where $\star$ is the usual orbifold product on the total space of the $G$-equivariant vector bundle $V \longrightarrow X$ and $I\pi^*$ is the Gysin map which is inverse to $I\pi^*$.

**Proof.** We give the proof only in equivariant Chow theory—the proof in equivariant K-theory is essentially identical. We compare the two sides of (14). If $\Psi_1, \Psi_2, \Psi_3 \subset G$ are conjugacy classes and $x \in A^*_G(I_X(\Psi_1)), y \in A^*_G(I_X(\Psi_2))$, then the contribution of $x \star_{V^+} y$ to $A^*_G(I_X(\Psi_3))$ is

$$\sum_{\Phi_{1,2}} \mu_\Phi (e_1^*x \cdot e_2^*y \cdot eu(LR(T) + R^+V)),$$

(15)

where the sum is over all conjugacy classes $\Phi_{1,2} \subset G \times G$ satisfying

$$e_1(\Phi_{1,2}) = \Psi_1, e_2(\Phi_{1,2}) = \Psi_2, \mu(\Phi_{1,2}) = \Psi_3.$$

Since the class of tangent bundle of $V$ equals $TX + V$, the tangent bundle to the stack $[V/G]$ is $TX + V - g = T + V$. Thus, the contribution of the right-hand side of (14) is the sum

$$\sum_{\Phi_{1,2}} I\pi^* (\mu_{V^+}(I\pi)^* (e_1^*x \cdot e_2^*y \cdot eu(LR(T) + LR(V)))),$$

(16)
where the map $\mu_V$ in (16) is understood to be the multiplication map $I^2 V \to IGV$. If $\Phi$ is a conjugacy class in $G \times G$ with $\mu(\Phi) = \Psi$, then the multiplication map $\mu_V: I_V(\Phi) \to I_V(\Psi)$ factors through the inclusion

$$I_V(\Phi) \hookrightarrow \mu^* I_V(\Psi) \xrightarrow{I^{\pi,\mu}_V} I_V(\Psi),$$

and we have the following diagram, with a Cartesian square on the right.

\[ \begin{array}{ccc}
I_V(\Phi) & \xrightarrow{\mu^*} & I_V(\Psi) \\
\downarrow I & & \downarrow I \\
I(\Phi) & \xrightarrow{\mu} & I(\Psi)
\end{array} \]

The normal bundle to the inclusion $I_V(\Phi) \hookrightarrow \mu^*(I_V(\Psi))$ is the pull-back of the bundle $V_\Phi/V_\Phi$ on $I^2 X(\Phi)$, where $V_\Phi \subset V|_{I_V(\Psi)}$ is the subbundle whose fiber over a point $(g, x)$ is the subspace $V^g$, and the fiber of $V_\Phi$ over a point $(g_1, g_2, x)$ is the subspace $V^{g_1 \cdot g_2} \subset V$. Using this information about the normal bundle we can rewrite (16) as

$$\mu_* (e^1 x \cdot e^2 y \cdot eu(LR(T + V) + V_\Phi - V_\Phi)).$$

Finally, expression (18) can be identified with (15) by observing that if $g_1, g_2 \in G$, then

$$L(g_1)(V) + L(g_2)(V) + L((g_1 g_2)^{-1}V) + V - V^{g_1 \cdot g_2} + V^{g_1 \cdot g_2} - V^{g_1 \cdot g_2} = L(g_1)(V) + L(g_2)(V) - L(g_1 g_2)(V).$$

\[ \square \]

4.2. Geometric interpretation of the $\ast_V$- product. The $\ast_V$- product does not generally correspond to an orbifold product on a bundle. However, we will show that, after tensoring with $\mathbb{C}$, the inertial Chow (or cohomology) ring with the $\ast_V$- product is isomorphic to the inertial Chow (or cohomology) ring coming from the total space of the dual bundle. The latter is isomorphic to the orbifold Chow (or cohomology) ring of the total space of the dual bundle.

**Definition 4.2.1.** Given a vector bundle $V$ on a quotient stack $X = [X/G]$, we define an automorphism $\Theta_V$ of $A^*(I X) \otimes \mathbb{C}$ as follows. If $x_\Psi$ is supported on a component $I(\Psi)$ of $I X$ corresponding to a conjugacy class $\Psi \subset G$ then we set $\Theta_V(x_\Psi) = e^{i\pi a_\Psi} x_\Psi$, where $a_\Psi$ is the virtual rank of the logarithmic trace $L(g^{-1})(V)$ for any representative element $g \in \Psi$. The same formula defines an automorphism of $H^*(I X, \mathbb{C})$.

**Theorem 4.2.2.** For $x, y \in A^*_G(I_G X)$ we have

$$x \ast_V y = \pm (Is)^* ((I\pi)^* x \ast (I\pi)^* y) = \pm x \ast_{V^+} y,$$

where $\ast$ is the usual orbifold product on the total space of the $G$-equivariant vector bundle $V^* \to X$, and $Is^*$ is the Gysin map which is inverse to to $I\pi^*$, and the sign $\pm$ is $(-1)^{a_{\Psi_1} + a_{\Psi_2} - a_{\Psi_{12}}}$ where $a_{\Psi_1} + a_{\Psi_2} - a_{\Psi_{12}}$ is a non-negative integer. Moreover, if we tensor with $\mathbb{C}$, then we have the identity

$$\Theta_V(x \ast_V y) = \Theta_V(x) \ast_{V^+} \Theta_V(y).$$
Proof. Observe that if \( g \in G \) acts on a representation \( V = \mathbb{Z}_G(g) \) with weights \( e^{i\theta_1}, \ldots, e^{i\theta_r} \), then \( g \) naturally acts on \( V^* \) with weights \( e^{-i\theta_1}, \ldots, e^{-i\theta_r} \), and the \( e^{i\theta_k} \) eigenspace of \( V \) is dual to the \( e^{-i\theta_k} \) eigenspace of \( V^* \). Hence \( L(g^{-1})(V) = L(g)(V^*) \) as elements of \( K(X^q) \otimes \mathbb{Q} \). Thus given a pair \( g_1, g_2 \in G \), we see that \( V^-(g_1, g_2) = ((V^*)^+)(g_1, g_2) \) as a \( \mathbb{Z}_G(g_1, g_2) \)-equivariant bundles on \( X^{g_1^{-1}g_2^{-1}} \). Hence, \( eu(R^1V) = (-1)^{rk}R^veu(R^1V^*) \), so (19) holds. If \( x \) is supported in the component \( I(\Psi_1) \) and \( y \) is supported in the component \( I(\Psi_2) \), then \( x \star_{V^-} y \) is supported at components \( I(\Psi_{12}) \), where \( \Psi_{12} \) is a conjugacy class of \( g_1g_2 \) for some \( g_1 \in \Psi_1 \) and \( g_2 \in \Psi_2 \).

Now we have

\[
\Theta_V(x \star_{V^-} y) = \sum_{\Psi_{12}} e^{i\pi a_{\Psi_{12}}}(\sum g^{-1}V^-(g_1, g_2))_x \star_{V^+} y,
\]

while

\[
\Theta_V(x \star_{V^+} \Theta_V(y) = \sum_{\Psi_{12}} e^{i\pi(a_{\Psi_1} + a_{\Psi_2})}_x \star_{V^+} y.
\]

Thus Equation (20) follows from the fact that \( \text{rk} V^-(g_1, g_2) = a_{\Psi_1} + a_{\Psi_2} - a_{\Psi_{12}} \). \( \square \)

4.3. The virtual orbifold product is the \( \ast_TX \) product. The virtual orbifold product was introduced in [GLS+07]. In our context it (or more precisely its algebraic analogue) can be defined as follows:

**Definition 4.3.1.** Let \( T^\text{virt} \) be the class in \( K_G(\mathbb{V}_2^G X) \) defined by the formula

\[
T|_{\mathbb{V}_2^G X} = T_{\mathbb{V}_2^G X} - e_1T_{IGX} - e_2T_{IGX},
\]

where \( T|_{\mathbb{V}_2^G X} \) refers to the pullback of the class \( T \) to \( \mathbb{V}_2^G X \) via any of the three natural maps \( \mathbb{V}_2^G X \to X \), where \( T_{IGX} \) denotes the tangent bundle to the stack \( I\mathcal{X} = [I_GX/G] \), and where \( T_{IGX} \) denotes the tangent bundle to the stack \( I^2\mathcal{X} \).

**Proposition 4.3.2.** The identity \( T^\text{virt} = LR(T) + R^{-1}T \) holds in \( K_G(\mathbb{V}_2^G X) \). In particular, \( T^\text{virt} \) is represented by a non-negative element of \( K_G(\mathbb{V}_2^G X) \) and the \( \ast_{\text{eu}} T^\text{virt} \)-product is commutative and associative. Moreover, \( \mathcal{S}_T + S^{-1}T = N \), where \( N \) is the normal bundle of the canonical morphism \( I_GX \to X \), so \( (T^\text{virt}, N) \) is a strongly Gorenstein inertial pair.

**Proof.** The proof follows from the identity \( L(g)(T) + L(g^{-1})(T) = T|_{X^q} - T_{X^q} = N|_{X^q} \). \( \square \)

**Definition 4.3.3.** Following [GLS+07], we define the virtual orbifold product to be the \( \ast_{\text{eu}(T^\text{virt})} \)-product.

**Corollary 4.3.4.** The virtual product \( \ast_{\text{virt}} \) on \( A_G^*(I_GX) \) agrees up to sign with the \( \ast_{T^+} \) inertial product on \( A_G^*(I_GX) \) induced by the cotangent bundle \( T^* \) of \( \mathcal{X} = [X/G] \), and there is an isomorphism of rings \( (A_G^*(I_GX)) \cong (A_G^*(I_GX)) \).

4.4. A\n
**Example with \( \mathbb{P}(1, 3, 3) \).** We illustrate the various inertial products in K-theory and Chow theory with the example of the weighted projective space \( \mathcal{X} = \mathbb{P}(1, 3, 3) = [(\mathbb{A}^3 \setminus \{0\})/C^+] \) where \( C^+ \) acts with weights \( (1, 3, 3) \). The inertia \( I\mathcal{X} \) has three sectors—the identity sector \( \mathcal{X}^1 = \mathcal{X} \) and two twisted sectors \( \mathcal{X}^\omega \) and \( \mathcal{X}^{\omega^{-1}} \), where \( \omega = e^{2\pi i/3} \). Both twisted sectors are isomorphic to a \( \mathcal{B}_{\mathbb{A}^1} \)-gerbe over \( \mathbb{P}^1 \). The K-theory of each sector is a quotient of the representation ring \( \mathbb{R}(C^+) \).

Precisely, we have

\[
K(\mathcal{X}^1) = \mathbb{Z}[\chi]/(\chi - 1)(\chi^3 - 1)^2, \text{ and } K(\mathcal{X}^\omega) = K(\mathcal{X}^{\omega^{-1}}) = \mathbb{Z}[\chi]/(\chi + 1)(\chi^3 - 1)^2.
\]

I
\( \mathbb{Z}[\chi]/\langle (\chi^3 - 1)^2 \rangle \), where \( \chi \) is the defining character of \( \mathbb{C}^* \). The projection formula in equivariant K-theory implies that any inertial product is determined by the products \( 1_g * 1_g \in K(\mathcal{M}^{g*}) \), where \( 1_g \) is the K-theoretic fundamental class on the sector \( \mathcal{M}^g \).

The usual orbifold product is represented by the following symmetric matrix:

\[
\begin{pmatrix}
\mathcal{T}^1 & \mathcal{T}^\omega & \mathcal{T}^{\omega^{-1}} \\
\mathcal{T}^\omega & 1 & 1 & \text{eu}(\chi) \\
\mathcal{T}^{\omega^{-1}} & 1 & \text{eu}(\chi) & 1 \\
\end{pmatrix}
\]

The virtual and orbifold cotangent products are represented by the following matrices:

\[
\begin{pmatrix}
\mathcal{V}^1 & \mathcal{V}^\omega & \mathcal{V}^{\omega^{-1}} \\
\mathcal{V}^\omega & \text{eu}(\chi) & \text{eu}(\chi)^2 & 1 \\
\mathcal{V}^{\omega^{-1}} & \text{eu}(\chi) & 1 & \text{eu}(\chi) \\
\end{pmatrix}
\]

If we denote by \( t = c_1(\mathcal{T}) \in A^*(\mathbb{B}^\mathcal{T} \mathbb{C}^*) \), then the inertial products on Chow and cohomology groups can also be represented by matrices as above. After tensoring with \( \mathbb{C} \), the Chow groups of the sectors are \( A^*(\mathcal{T})_{\mathbb{C}} = \mathbb{C}[t]/\langle t^3 \rangle \) and \( A^*(\mathcal{T}^\omega)_{\mathbb{C}} = A^*(\mathcal{T}^{\omega^{-1}})_{\mathbb{C}} = \mathbb{C}[t]/\langle t^2 \rangle \). The corresponding matrices for the virtual and cotangent orbifold products are the following:

\[
\begin{pmatrix}
\mathcal{V}^1 & \mathcal{V}^\omega & \mathcal{V}^{\omega^{-1}} \\
\mathcal{V}^\omega & \text{eu}(\chi) & \text{eu}(\chi)^2 & 1 \\
\mathcal{V}^{\omega^{-1}} & \text{eu}(\chi) & 1 & \text{eu}(\chi) \\
\end{pmatrix}
\]

The automorphism of \( A^*(\mathcal{T})_{\mathbb{C}} \) which is the identity on \( A(\mathcal{T}^1)_{\mathbb{C}} \) and which acts by multiplication by \( e^{2\pi i/3} \) on \( A^*(\mathcal{T}^\omega)_{\mathbb{C}} \) and \( e^{\pi i/3} \) on \( A^*(\mathcal{T}^{\omega^{-1}})_{\mathbb{C}} \) defines a ring isomorphism between these products.

5. The localized orbifold product on \( K(\mathcal{M}) \otimes \mathbb{C} \)

If an algebraic group \( G \) acts with finite stabilizer on smooth variety \( Y \), then there is a decomposition of \( K_G(Y) \otimes \mathbb{C} \) as a sum of localizations \( \bigoplus_\Psi K_G(Y)_{m_\Psi} \). Here the sum is over conjugacy classes \( \Psi \subset G \) such that \( I(\Psi) \neq \emptyset \), and \( m_\Psi \in \text{Spec } R(G) \) is the maximal ideal of class functions vanishing on the conjugacy class \( \Psi \).

Given a conjugacy class \( \Psi \subset G \) and a choice of \( h \in \Psi \), denote the centralizer of \( h \) in \( G \) by \( Z = Z_G(h) \). The conjugacy class of \( h \) in \( Z \) is just \( h \) alone, and there is a corresponding maximal ideal \( m_\Psi \in \text{Spec } R(Z) \). As described in [EG05 §4.3], the localization \( K_G(I(\Psi))_{m_\Psi} \) is a summand of the localization \( K_G(I(\Psi))_{m_\Psi} \), and this summand is independent of the choice of \( h \). This is called the central summand of \( \Psi \) and is denoted by \( K_G(I(\Psi))_{(\Psi)} \).

Since \( G \) acts with finite stabilizer, the projection \( f_\Psi : I(\Psi) \to Y \) is a finite l.c.i. morphism. The non-Abelian localization theorem of [EG05] states that the pullback \( f_\Psi^* : K_G(Y) \otimes \mathbb{C} \to K_G(I(\Psi)) \otimes \mathbb{C} \) induces an isomorphism between the localization of \( K_G(Y) \) at \( m_\Psi \) and the central summand \( K_G(I(\Psi))_{(\Psi)} \subset K_G(I(\Psi))_{m_\Psi} \). The inverse to \( f_\Psi^* \) is the map \( \alpha \mapsto f_\Psi^*(\alpha \cdot \text{eu}(N_{\Psi}))^{-1} \). If we let \( f \) be the global stabilizer map \( I_GY \to Y \), then, after summing over all conjugacy
classes $\Psi$ in the support of $K_G(Y) \otimes \mathbb{C}$, we obtain an isomorphism
\[
f^*: K_G(Y) \otimes \mathbb{C} \longrightarrow K_G(I_G Y)_c,
\]
where $K_G(I_G Y)_c = \bigoplus K_G(I(\Psi))_c(\Psi)$. The inverse is $f_{eu}$. Applying this construction with $Y = I_G X$ allows us to define a product we call the localized orbifold product

**Definition 5.0.1.** The localized orbifold product on $K_G(I_G X) \otimes \mathbb{C}$ is defined by the formula
\[
\alpha *_{LO} \beta = I f_*(\mathcal{I} f^* \alpha \cdot \mathcal{I} f^* \beta) \otimes eu(N_f)^{-1},
\]
where $*$ is the usual orbifold product on $K_G(I_G X)_c$, and $I f: I_G(I_G X) \longrightarrow I_G X$ is the projection.

**Remark 5.0.2.** It should be noted that $I_G(I_G X)$ is not the same as $\mathbb{P}_G^2 X$. The inertia $I_G(I_G X) = \{(h, g, x) | hx = gx = x, hg = gh\}$ is a closed subspace of $\mathbb{P}_G^2 X$.

The localized product can be interpreted in the context of the $*_V$ product, where the vector bundle $V$ is replaced by the virtual bundle $-N_f$. Observe that the pullback of $\mathbb{T}$ to $I_G X$ splits as $\mathbb{T} = \mathbb{T}_{I_G X} + N_f$, where $N_f$ is the normal bundle to the finite l.c.i. map $I_G X \longrightarrow X$. Although $N_f$ is not a bundle on $X$, we can still compute $N_f^+(g_1, g_2)$ on $\mathbb{P}_G^2 X$.

The same formal argument used in the proof of Theorem 4.1.2 yields the following result.

**Proposition 5.0.3.** The class $eu(LR(\mathbb{T}) + R^+(-N_f))$ is well defined in localized $K$-theory and
\[
\alpha *_{LO} \beta = \alpha *_{(-N_f)} \beta.
\]

**Remark 5.0.4.** The inertial pair corresponding the localized product is the formal pair $(LR(\mathbb{T}) + R^+(-N_f), \mathcal{X} \mathcal{T} + S^+(-N_f))$. However, the Chern character corresponding to this inertial pair is the usual orbifold Chern character and the corresponding product on $A^*(I \mathcal{X})$ is the usual orbifold product. The reason is that the orbifold Chern character isomorphism factors through $K_G(I_G X)_c$, the localization of $K_G(I_G X)$ at the augmentation ideal of $R(G)$. This localization corresponds to the untwisted sector of $I_G X$ where $f$ restricts to the identity map.

**Remark 5.0.5.** Identifying $K_G(I_G X)_c$ with the localization of $K_G(I_G(I_G X))_c$ allows us to invert the class $eu(N_f)$. In [Kau10 Section 3.4] the author gives a framework for defining similar products after formally inverting the Euler classes of normal bundles.

### 5.1. An example with $\mathbb{P}(1,2)$

We consider the weighted projective line $\mathcal{X} = \mathbb{P}(1, 2) = [(\mathbb{A}^2 \setminus \{0\})/\mathbb{C}^*]$, where $\mathbb{C}^*$ acts with weights $(1, 2)$. The inertia stack $I \mathcal{X}$ has two sectors $\mathcal{X}^1$ and $\mathcal{X}^{-1} = \mathcal{X}_{g2}$. We have $K(I \mathcal{X}^1) \otimes \mathbb{C} = \mathbb{C}[\chi] / (\chi^2 - 1)$ and $K(I \mathcal{X}^{-1}) \otimes \mathbb{C} = \mathbb{C}[\chi] / (\chi^2 - 1)$. In particular $K(I \mathcal{X}) \otimes \mathbb{C}$ is supported at $\pm 1 \in \mathbb{C}^*$. As was the case in Section 4.3, inertial ring structures are determined by the products $1_{g_1} * 1_{g_2} \in K(I \mathcal{X}^{g_1, g_2})$. In terms of the localization decomposition, $K(I \mathcal{X}) \otimes \mathbb{C} = K(I \mathcal{X})_{(1)} \oplus K(I \mathcal{X})_{(-1)}$. The localized product is determined by computing the corresponding orbifold product on each localized piece using the decomposition of the element $1_\mathcal{X}$ into its localized pieces and the product $1_{g_1} *_{LO} 1_{g_2}$ decomposes as
\[
(1_{g_1})_{(1)} *_{LO} (1_{g_2})_{(1)} + (1_{g_1})_{(-1)} *_{LO} (1_{g_2})_{(-1)}.
\]
The multiplication matrix for $K(I\mathcal{X})_{(1)}$ is the usual orbifold matrix, which in this case is the following.

$$
\begin{array}{cc}
\mathcal{X}^{-1} & \mathcal{X}^{-1} \\
1 & 1 \\
1 & \text{eu}(\chi)
\end{array}
$$

The multiplication matrix for the localized product on $K(I\mathcal{X})_{(-1)}$ is the same as the multiplication matrix for the orbifold product on $\mathcal{B}\mu_2$ which is the following.

$$
\begin{array}{cc}
\mathcal{X}^{-1} & \mathcal{X}^{-1} \\
1 & 1 \\
1 & 1
\end{array}
$$

Thus we see that the only nontrivial product is $1_{(-1)} \ast_{LO} 1_{(-1)}$. To obtain a single multiplication matrix we use the decomposition

$$
1_{(-1)} = \frac{(1 + \chi)}{2} + \frac{(1 - \chi)}{2} \in K(\mathcal{X}^{-1}) \otimes \mathbb{C},
$$

where $(1 + \chi)/2$ is supported at 1 and $(1 - \chi)/2$ is supported at $-1$. The final result is the following.

$$
\begin{array}{cc}
\mathcal{X}^{-1} & \mathcal{X}^{-1} \\
1 & 1 \\
\frac{(1 + \chi)^2 \text{eu}(\chi) + (1 - \chi)^2}{4}
\end{array}
$$

Because the twisted sector $\mathcal{X}^{-1}$ has dimension 0, both the orbifold and usual Chern characters on this sector compute the virtual rank. The untwisted sector $\mathbb{P}(1,2)$ has Chow ring $\mathbb{C}[t]/(t^2)$ where $t = c_1(\chi)$. Thus $\text{ch}(\text{eu}(\chi)) = t \in A^*(\mathbb{P}(1,2)) \otimes \mathbb{C}$. Observe that

$$
\text{ch}\left(\frac{(1 + \chi)^2 \text{eu}(\chi) + (1 - \chi)^2}{4}\right) = \frac{(2 + t)^2(t) + (-t)^2}{4} = t
$$

in $\mathbb{C}[t]/(t^2)$ as well.

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