EIGENVALUES OF HERMITIAN MATRICES AND CONES ARISING FROM QUIVERS

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Abstract. We study the set of the possible eigenvalues of Hermitian matrices with positive semi-definite sum of bounded rank. Our approach is based on quiver theory. We show that the spectral problem reduces to the study of cones of effective weights for quivers.

1. Introduction

Let $n \geq 1, m \geq 2$ and $0 \leq r \leq n$ be non-negative integers. We define $\mathcal{M}aj_r(n,m)$ to be the set of all m-tuples $(\lambda(1), \dots, \lambda(m))$ of weakly decreasing sequences of n real numbers for which there exit $n \times n$ complex Hermitian matrices $A(1), \dots, A(m)$ with eigenvalues $\lambda(1), \dots, \lambda(m)$ such that

$$\sum_{1 \leq i \leq m} A(i)$$

is positive semi-definite (i.e. has non-negative eigenvalues) and has rank at most r.

In [2], A. Buch, answering a question raised by A. Barvinok, has showed that $\mathcal{M}aj_r(n,m)$ is a rational convex polyhedral cone and found its facets. His proof is by induction on r, relaying on the case when $r = 0$. In this paper, we first show that the study of $\mathcal{M}aj_r(n,m)$ naturally fits into the framework of quiver theory and then using methods from quiver invariant theory we recover Buch's result. Moreover, we compute the dimension of the cone $\mathcal{M}aj_r(n,m)$ and find its lattice points. Our description of the lattice points of $\mathcal{M}aj_r(n,m)$ generalizes the saturation conjecture for Littlewood-Richardson coefficients proved by A. Knutson and T. Tao [14](see also H. Derksen and J. Weyman [6]).

When $r = 0$, we recover Horn's conjecture. If $r = n$, our main theorem gives a proof of the majorization problem [9] which does not rely on A. Klyachko's results [13].

1.1. Statements of the results.

The case $(r,m) = (0,2)$ is trivial and so we assume from now on that $(r,m) \neq (0,2)$.

Let $\mathcal{S}_{\mathcal{M}aj}(n,m)$ be the set of all m-tuples $I = (I_1, \dots, I_m)$ of subsets of the set $\{1, \dots, n\}$ of the same cardinality d with $1 \leq d \leq n$ and

$$c^{((n-d)^d)}_{\lambda(I_1), \dots, \lambda(I_m)} = 1.$$

The details of our notations can be found in the Notation paragraph at the end of this section.

Now, we can state our first result (see also [2, Theorem 1]):

Theorem 1.1. (1) *The set $\mathcal{M}aj_r(n,m)$ is a rational convex polyhedral cone and*

$$\dim \mathcal{M}aj_r(n,m) = nm - \delta_{0,r},$$

where $\delta_{x,y}$ denotes the Kronecker symbol.

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(2) *The cone $\mathcal{M}aj_r(n,m)$ consists of all m-tuples $(\lambda(1),\dots,\lambda(m))$ of weakly decreasing sequences of n real numbers satisfying*

$$\sum_{1\le i\le m}\sum_{j\in I_i}\lambda_j(i)\ge 0,$$

for all $(I_1,\dots,I_m)\in\mathcal{S}_{\mathcal{M}aj}(n,m)$ and

$$\sum_{1\le i\le m}\sum_{j\in J_i}\lambda_{n+1-j}(i)\le 0,$$

for all $(J_1,\dots,J_m)\in\mathcal{S}_{\mathcal{M}aj}(n-r,m)$.

The next theorem gives a description of the lattice points of the cone $\mathcal{M}aj_r(n,m)$:

Theorem 1.2. *Let $\lambda(1),\dots,\lambda(m)$ be m weakly decreasing sequences of n integers. Then $(\lambda(1),\dots,\lambda(m))\in\mathcal{M}aj_r(n,m)$ if and only if there exists a partition γ with at most r non-zero parts for which*

$$(\diamondsuit)\qquad c^{\gamma}_{\lambda(1),\dots,\lambda(m)}\neq 0.$$

Note that when $r=1$, $(\diamondsuit)$ just tells us that if we denote $\sum_{1\le i\le m}|\lambda(i)|$ by $|\underline{\lambda}|$ then $|\underline{\lambda}|\ge 0$ and the symmetric power $S^{|\underline{\lambda}|}(\mathbb{C}^n)$ occurs in the tensor product $S^{\lambda(1)}(\mathbb{C}^n)\otimes\cdots\otimes S^{\lambda(m)}(\mathbb{C}^n)$ with non-zero multiplicity.

If $r=0$, then $(\diamondsuit)$ tells us that

$$(S^{\lambda(1)}(\mathbb{C}^n)\otimes\cdots\otimes S^{\lambda(m)}(\mathbb{C}^n))^{\mathrm{GL}(n)}\neq 0.$$

The strategy in proving Theorem 1.1 and Theorem 1.2 is to bring the spectral problem into the framework of quiver theory. Let $Q=(Q_0,Q_1,t,h)$ be a quiver without oriented cycles where Q_0 is the set of vertices, Q_1 is the set of arrows and $t,h:Q_1\to Q_0$ assign to each arrow $a\in Q_1$ its tail ta and head ha, respectively. If $\beta\in\mathbb{Z}^{Q_0}_{\ge 0}$ is a dimension vector, we define the vector space $\mathrm{Rep}(Q,\beta)$ by

$$\mathrm{Rep}(Q,\beta)=\bigoplus_{a\in Q_1}\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\beta(ta)},\mathbb{C}^{\beta(ha)}).$$

For β a dimension vector and $\sigma\in\mathbb{R}^{Q_0}$ a real valued function on the set of vertices, we consider the following system of matrix equations

$$(\dagger)\quad \sum_{\{a\in Q_1|ta=x\}}W(a)^*W(a)-\sum_{\{a\in Q_1|ha=x\}}W(a)W(a)^*=\sigma(x)\mathrm{Id}_{\beta(x)},$$

for all $x\in Q_0$, where $W\in\mathrm{Rep}(Q,\beta)$ and $W(a)^*\in\mathrm{Hom}(\mathbb{C}^{\beta(ha)},\mathbb{C}^{\beta(ta)})$ denotes the adjoint of $W(a)$ with respect to the standard Hermitian inner product on $\mathbb{C}^n$.

As we will see in Section 3, for some star-like quivers the existence of a solution to $(\dagger)$ is equivalent to the existence of Hermitian matrices with prescribed (by σ) eigenvalues and such that the sum is positive semi-definite and has bounded rank.

We write $\beta_1\hookrightarrow\beta$ if every β- dimensional representation $W\in\mathrm{Rep}(Q,\beta)$ has a subrepresentation of dimension vector β_1. If $\sigma\in\mathbb{R}^{Q_0}$ and $\beta\in\mathbb{Z}^{Q_0}$ we define $\sigma(\beta)$ to be

$$\sigma(\beta)=\sum_{x\in Q_0}\sigma(x)\beta(x).$$

We define the *cone* $C(Q,\beta)$ *of effective weights* associated to (Q,β) as follows:

$$C(Q,\beta)=\{\sigma\in\mathbb{H}(\beta)\mid\sigma(\beta_1)\leq 0 \text{ for all } \beta_1\hookrightarrow\beta\},$$

where $\mathbb{H}(\beta)=\{\sigma\in\mathbb{R}^{Q_0}\mid\sigma(\beta)=0\}$. Now, let us state our next result:

Proposition 1.3. *Let Q be a quiver without oriented cycles, β be a dimension vector and $\sigma\in\mathbb{R}^{Q_0}$. Then the following statements are equivalent:*

(1) $\sigma\in C(Q,\beta)$;
(2) *there is a* $W\in\mathrm{Rep}(Q,\beta)$ *satisfying* (†).

The paper is organized as follows. In Section 2, we recall some facts about moment maps for quivers and prove Proposition 1.3. The star-like quiver setting $(Q_{\mathcal{M}aj},\beta)$ is introduced in Section 3. We also show how the description of the set $\mathcal{M}aj_r(n,m)$ can be reduced to the description of the cone $C(Q_{\mathcal{M}aj},\beta)$. In Section 4, we briefly recall Derksen and Weyman description of the facets of the cones of effective weights for quivers without oriented cycles. We prove Theorem 1.2 and Theorem 1.1 in Section 5. In Section 6, we show when the list of Horn type inequalities from Theorem 1.1(2) is minimal.

Notations. For a partition λ, we denote by λ' the partition conjugate to λ, i.e., the Young diagram of λ' is the Young diagram of λ reflected with respect to its main diagonal. We will often refer to partitions as Young diagrams. If $I=\{z_1<\cdots<z_l\}$ is an l-tuple of integers then $\lambda(I)$ is defined by $\lambda(I)=(z_l-l,\dots,z_1-1)$. If λ is a weakly decreasing sequence of n integers, we denote by $S^{\lambda}(V)$ the irreducible rational representation of $\mathrm{GL}(V)$ of highest weight λ, where $V=\mathbb{C}^n$. Let $\gamma,\lambda(1),\dots,\lambda(m)$ be weakly decreasing sequences of n integers. Then we define

$$c^{\gamma}_{\lambda(1),\dots,\lambda(m)}$$

to be the multiplicity of $S^{\gamma}(V)$ in $S^{\lambda(1)}(V)\otimes\cdots\otimes S^{\lambda(m)}(V)$. In other words,

$$c^{\gamma}_{\lambda(1),\dots,\lambda(m)}=\dim_{\mathbb{C}}(S^{\gamma}(V)^*\otimes S^{\lambda(1)}(V)\otimes\cdots\otimes S^{\lambda(m)}(V))^{\mathrm{GL}(V)},$$

where $S^{\gamma}(V)^*$ is the dual representation.

For $l\geq 0$ an integer and a a real number, we denote the l-tuple $(a,\dots,a)$ by (a^l). If H is an $n\times n$ complex Hermitian matrix, we denote by $s(H)$ its spectrum consisting of the n eigenvalues (possibly with multiplicities) of H arranged in weakly decreasing order. If $\lambda=(\lambda_1,\dots,\lambda_n)$ is a sequence of numbers then we define $|\lambda|=\sum_{i=1}^{n}\lambda_i$.

2. Proof of Proposition 1.3

It has been proved by King [12] that if Q is a quiver without oriented cycles and $\sigma\in\mathbb{Z}^{Q_0}$ then the set of solutions to (†) (modulo a product of unitary groups) is the symplectic quotient description of a certain moduli space for quivers. Using King's [12] criterion for semi-stability and Schofield's [17] results on general representations, one can find necessary and sufficient linear homogeneous inequalities for the non-emptiness of the corresponding moduli space. More precisely, we have the following description (for a short proof see [4, Theorem 2.4]):

Theorem 2.1. *Let Q be a quiver without oriented cycles and let β be a dimension vector. If $\sigma\in\mathbb{Z}^{Q_0}$, then the following are equivalent:*

(1) $\sigma\in C(Q,\beta)$;
(2) *there exists* $W\in\mathrm{Rep}(Q,\beta)$ *satisfying* (†).

Remark 2.2. The paper [4] was the first one in which the authors used Theorem 2.1 to establish a direct link between quiver theory and inequalities for eigenvalues of Hermitian matrices. However, this was done for the less general case when $m = 3$, $r = 0$ (in our set up from the Introduction) and the eigenvalues of the Hermitian matrices involved were assumed to be integers.

Thus, it is desirable to know if Theorem 2.1 is in fact true for arbitrary points $\sigma \in C(Q, \beta)$. In the next subsection, we show that this is indeed the case (see Proposition 1.3).

2.1. **Moment maps for quivers.** We fix a quiver Q without oriented cycles and β a dimension vector. If $\mathrm{GL}(\beta) = \prod_{x \in Q_0} \mathrm{GL}(\beta(x))$ then $\mathrm{GL}(\beta)$ acts algebraically on $\mathrm{Rep}(Q, \beta)$ by simultaneous conjugation, i.e., for $g = (g(x))_{x \in Q_0} \in \mathrm{GL}(\beta)$ and $W = (W(a))_{a \in Q_1} \in \mathrm{Rep}(Q, \beta)$, we define $g \cdot W$ by

$$(g \cdot W)(a) = g(ha)W(a)g(ta)^{-1} \text{ for every } a \in Q_1.$$

If $\mathrm{U}(\beta(x))$ is the group of $\beta(x) \times \beta(x)$ unitary matrices for all $x \in Q_0$ then

$$\mathrm{U}(\beta) = \prod_{x \in Q_0} \mathrm{U}(\beta(x))$$

is a maximal compact subgroup of $\mathrm{GL}(\beta)$. The $\mathrm{U}(\beta)$-invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathrm{Rep}(Q, \beta)$ induced by the standard Hermitian inner product on $\mathbb{C}^{\beta(x)}$ is

$$\langle V, W \rangle = \sum_{a \in Q_1} \mathrm{Trace}(V(a)W(a)^*).$$

Viewing $\mathrm{Rep}(Q, \beta)$ as a symplectic manifold, the action of $\mathrm{U}(\beta)$ is Hamiltonian. To write down its moment map, we first note that the Lie algebra $\mathfrak{u}(\beta)$ of $U(\beta)$ is the Lie algebra of multiple skew-Hermitian matrices

$$\mathfrak{u}(\beta) = \{(A(x))_{x \in Q_0} \in \prod_{x \in Q_0} \mathrm{Mat}_{\beta(x) \times \beta(x)}(\mathbb{C}) \mid A(x) = -A(x)^*, \forall\, x \in Q_0\},$$

and the action of $A = (A(x))_{x \in Q_0} \in \mathfrak{u}(\beta)$ on $W = (W(a))_{a \in Q_1} \in \mathrm{Rep}(Q, \beta)$ is

$$(A \cdot W)(a) = A(ha)W(a) - W(a)A(ta), \forall\, a \in Q_1.$$

Consequently, the moment map $\Phi : \mathrm{Rep}(Q, \beta) \to \mathfrak{u}(\beta)^*$ is given by

$$\begin{aligned}\Phi(W)(A) &= \frac{i}{2} \langle A \cdot W, W \rangle \\ &= \sum_{x \in Q_0} -\frac{i}{2} \mathrm{Trace} \left(A(x) \left(\sum_{\substack{a \in Q_1 \\ ta = x}} W(a)^* W(a) - \sum_{\substack{a \in Q_1 \\ ha = x}} W(a) W(a)^* \right) \right).\end{aligned}$$

Furthermore, we can identify $\mathfrak{u}(\beta)^*$ with $i\mathfrak{u}(\beta) = \mathrm{Herm}(\beta)$, the space of multiple Hermitian matrices. The identification is done by taking each $H \in \mathrm{Herm}(\beta(x))$ to $(A \mapsto -\frac{i}{2} \mathrm{Trace}(AH))$. In this way, we rewrite the moment map $\Phi : \mathrm{Rep}(Q, \beta) \to \mathrm{Herm}(\beta)$ as

$$\Phi_x(W) = \sum_{\substack{a \in Q_1 \\ ta = x}} W(a)^* W(a) - \sum_{\substack{a \in Q_1 \\ ha = x}} W(a) W(a)^*, \forall\, x \in Q_0.$$

We define the *momentum image* $\Delta(Q,\beta)$ to be the set of all tuples $(\lambda(x))_{x\in Q_0}$ of weakly decreasing sequences such that there exists $W \in \mathrm{Rep}(Q,\beta)$ with $\lambda(x)$ equals to the spectrum of $\Phi_x(W)$ for all $x \in Q_0$. Inside $\Delta(Q,\beta)$, we consider the following set

$$C = \{\sigma \in \mathbb{R}^{Q_0} \mid \exists\ W \in \mathrm{Rep}(Q,\beta) \text{ with } s(\Phi_x(W)) = (\sigma(x)^{\beta(x)}), \forall\ x \in Q_0\}.$$

First, let us prove:

Lemma 2.3. *The set C is a rational convex polyhedral cone.*

Proof. Using a convexity result due to Sjamaar [19, Theorem 4.9], we know that $\Delta(Q,\beta)$ is a rational convex polyhedral cone. On the other hand, C is the intersection of $\Delta(Q,\beta)$ with a finite set of hyperplanes defined by linear equations with integer coefficients. Therefore, C must be a rational convex polyhedral cone. □

The proof of Proposition 1.3. From Theorem 2.1 and Lemma 2.3, we know that $C(Q,\beta)$ and C are rational convex polyhedral cones with the same lattice points. So, they have to be equal. □

3. The spectral problem and the star-like quiver setting

First, a simple linear algebra Lemma:

Lemma 3.1. *(a) Let $\sigma(1),\dots,\sigma(n-1)$ be non-positive real numbers. Then the following are equivalent:*
(a.1) there exist $W_i \in \mathrm{Mat}_{i\times(i+1)}(\mathbb{C}), 1 \le i \le n-1$ such that

$$W_i \cdot W_i^\star - W_{i-1}^\star \cdot W_{i-1} = -\sigma(i)\,\mathrm{Id}_{\mathbb{C}^i} \ \textit{for } 2 \le i \le n-1,$$
$$W_1 \cdot W_1^\star = -\sigma(1);$$

(a.2) there exists a $n \times n$ Hermitian matrix $H(= W_{n-1}^\star \cdot W_{n-1})$ with eigenvalues $\nu(i) = -\sum_{i\le j\le n-1} \sigma(j), \forall\ 1 \le i \le n-1$ and $\nu(n) = 0$.

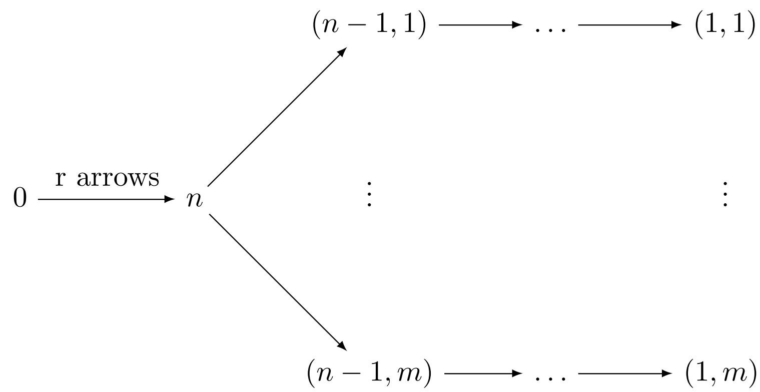
(b) A Hermitian matrix $X \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is of rank at most r and has non-negative eigenvalues if and only if there exist matrices $X(i) \in \mathrm{Mat}_{n\times 1}(\mathbb{C})$, $1 \le i \le r$ such that

$$X = \sum_{1\le i\le r} X(i) \cdot X(i)^\star.$$

Proof. See [4]. □

We define the star like quiver setting $(Q_{\mathcal{M}aj},\beta)$ as follows.

(1) The connected quiver $Q_{\mathcal{M}aj}$ has m flags $\mathcal{A}(i),\ 1 \le i \le m$ going out from the central vertex n. There is also a vertex 0 and r arrows from vertex 0 to vertex n :



If $r = 0$ then we delete the vertex 0 so that our quiver stays connected.

(2) The dimension vector β is given by $\beta(j, i) = j$ for all $1 \leq j \leq n-1$, $1 \leq i \leq m$, and $\beta(n) = n$; if $r \geq 1$ then $\beta(0) = 1$, i.e., the dimension vector β is equal to

$$\begin{array}{cccc} & & n-1 \;\; \cdots \;\; 1 \\ 1 & n & \\ & & n-1 \;\; \cdots \;\; 1. \end{array}$$

Remark 3.2. Before we move on, let us see which $\sigma \in \mathbb{R}^{Q_0}$ are allowed in the cone of effective weights $C(Q_{\mathcal{M}aj}, \beta)$. By definition, if $\sigma \in C(Q_{\mathcal{M}aj}, \beta)$ then $\sigma(\beta_1) \leq 0$ for all $\beta_1 \hookrightarrow \beta$. Now, let $\varepsilon_{(j,i)}$ be the dimension vector which is one at vertex (j, i) and zero elsewhere. Then it is easy to see that $\varepsilon_{(j,i)} \hookrightarrow \beta$ and $\sigma(\varepsilon_{(j,i)}) = \sigma(j, i)$ for all $1 \leq j \leq n-1,\ 1 \leq i \leq m$. Therefore, an effective weight σ must satisfy

$$\sigma(j, i) \leq 0,$$

for all $1 \leq j \leq n-1$ and $1 \leq i \leq m$. We call these the *chamber inequalities*.

Lemma 3.3. *Let $\sigma \in \mathbb{H}(\beta)$ be such that it satisfies the chamber inequalities. We have that $\sigma \in C(Q_{\mathcal{M}aj}, \beta)$ if and only if there exist $n \times n$ Hermitian matrices $A(i)$ with eigenvalues $a(i), \forall\ 1 \leq i \leq m$ and*

$$\sum_{1 \leq i \leq m} A(i) - \sigma(n) \cdot \mathrm{Id}_n$$

is positive semi-definite of rank at most r. Here, the n-tuple $a(i)$ is defined by

$$a(i) = (-\sum_{1 \leq j \leq n-1} \sigma(j, i), -\sum_{2 \leq j \leq n-1} \sigma(j, i), \ldots, -\sigma(n-1, i), 0)$$

for all $1 \leq i \leq m$.

Proof. Let us denote by $y_{n-1}(i)$ the arrow going from vertex n to vertex $(n-1, i), \forall\ 1 \leq i \leq m$. From Proposition 1.3, it follows that $\sigma \in C(Q_{\mathcal{M}aj}, \beta)$ if and only if there exists $W \in \mathrm{Rep}(Q_{\mathcal{M}aj}, \beta)$ satisfying the matrix equations (†). Note that the matrix equations coming from the first $n-1$ vertices of the flag $\mathcal{A}(i)$ are those from Lemma 3.1(a). So, they are equivalent to the existence of a Hermitian matrix $A(i) = W(y_{n-1}(i))^* W(y_{n-1}(i))$ with eigenvalues given by $a(i)$. There are two more equations that we need to take into account; the one coming from the central vertex n and the one coming from the vertex 0. Let us denote by $X(1), \ldots, X(r)$ the $n \times 1$ complex matrices given by W along the r arrows going from vertex 0 to vertex n.

It is now clear that $\sigma \in C(Q_{\mathcal{M}aj}, \beta)$ if and only if there exist $n \times n$ Hermitian matrices $A(i)$ with eigenvalues $a(i), \forall\ 1 \leq i \leq m$ together with $n \times 1$ matrices $X(l), \forall\ 1 \leq l \leq r$ such that

$$\sum_{1\leq i\leq m} A(i) - \sum_{1\leq i\leq r} X(i)\cdot X(i)^* = \sigma(n)\,\mathrm{Id}_n \tag{1}$$

and

$$\sum_{1\leq i\leq r} X(i)^*\cdot X(i) = \sigma(0). \tag{2}$$

Note that condition $\sigma \in \mathbb{H}(\beta)$ is equivalent to

$$\sigma(0) = \sum_{1\leq i\leq m} |a(i)| - n\sigma(n),$$

and hence taking traces in (1) we obtain equation (2). The proof follows now from Lemma 3.1(b). □

Let $\lambda(1), \dots, \lambda(m)$ be weakly decreasing sequences of n real numbers. Then we define the weight σ_λ by

$$\sigma_\lambda(j,i) = -\lambda_j(i) + \lambda_{j+1}(i), \forall\ 1 \leq j \leq n-1, \forall\ 1 \leq i \leq m \tag{3}$$

and

$$\sigma_\lambda(n) = -\sum_{1\leq i\leq m} \lambda_n(i). \tag{4}$$

If $r \geq 1$ then $\sigma_\lambda(0)$ is defined to be $\sum_{1\leq i\leq m} |\lambda(i)|$.

With this notation we have:

Proposition 3.4. *Let $\lambda(1), \dots, \lambda(m)$ be weakly decreasing sequences of n real numbers. Then the following are equivalent:*

(1) *there exist $n \times n$ complex Hermitian matrices $A(1), \dots, A(m)$ with eigenvalues $\lambda(1)$, $\dots$, $\lambda(m)$ such that*

$$\sum_{1\leq \imath\leq m} A(i)$$

is positive semi-definite of rank at most r;

(2) $\sigma_\lambda \in C(Q_{\mathcal{M}aj}, \beta)$.

Proof. This is a direct consequence of the previous lemma. Indeed, let $A(1), \dots, A(m)$ be $n \times n$ Hermitian matrices with eigenvalues $\lambda(1), \dots, \lambda(m)$. Then $A'(i) = A(i) - \lambda_n(i)\cdot \mathrm{Id}_n$ are Hermitian matrices with the smallest eigenvalue equals to zero. Furthermore,

$$\sum_{1\leq i\leq m} A(i)$$

is positive semi-definite of rank at most r if and only if

$$\sum_{1\leq i\leq m} A'(i) - \sigma_\lambda(n)\cdot \mathrm{Id}_n$$

is positive semi-definite of rank at most r. But this is now equivalent to $\sigma_\lambda \in C(Q_{\mathcal{M}aj}, \beta)$ by Lemma 3.3. □

Proposition 3.5. *We have that $\mathcal{M}aj_r(n,m)$ is a rational convex polyhedral cone and*

$$\mathcal{M}aj_r(n,m) \cong C(Q_{\mathcal{M}aj}, \beta) \times \mathbb{R}^{m-1}.$$

Proof. The chamber inequalities of Remark 3.2 and Proposition 3.4 show that

$$\begin{aligned} \mathcal{M}aj_r(n,m) &\longrightarrow C(Q_{\mathcal{M}aj}, \beta) \times \mathbb{R}^{m-1} \\ \lambda = (\lambda(1), \dots, \lambda(m)) &\longrightarrow (\sigma_\lambda, \lambda_n(1), \dots, \lambda_n(m-1)) \end{aligned}$$

is an isomorphism of cones. □

Remark 3.6. At this point it is clear that an explicit description of the cone $\mathcal{M}aj_r(n,m)$ follows from that of $C(Q_{\mathcal{M}aj}, \beta)$. Let us point out that in the particular case when $r = 0$ and $m = 3$, W. Crawley-Boevey and C. Geiss [4] found inequalities describing the lattice points of the cone of effective weights. Also, this list of inequalities obtained in [4] is essentially the same as the one conjectured by A. Horn [10], which is known to contain redundant inequalities.

Our general setup requires more powerful methods than those from [4]. We recall the methods that we need in the next section.

4. Semi-invariants for quivers

In this section we recall some important facts about semi-invariants for quivers. In particular, we recall Derksen and Weyman's [5] description of the facets of the cone of effective weights for quivers without oriented cycles.

Let Q be a quiver without oriented cycles. The dimension vectors of representations of Q lie in $\Gamma = \mathbb{Z}^{Q_0}$, the set of all integer-valued functions on Q_0. For each vertex x, we denote by ε_x the simple dimension vector corresponding to x, i.e. $\varepsilon_x(y) = \delta_{x,y}, \forall\ y \in Q_0$, where $\delta_{x,y}$ is the Kronecker symbol. If α, β are two elements of Γ, we define the Euler form by

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

Let β be a dimension vector of Q. Recall that the representation space of $\beta-$dimensional representations of Q is defined by

$$\operatorname{Rep}(Q, \beta) = \bigoplus_{a \in Q_1} \operatorname{Hom}(\mathbb{C}^{\beta(ta)}, \mathbb{C}^{\beta(ha)}).$$

If $\operatorname{GL}(\beta) = \prod_{x \in Q_0} \operatorname{GL}(\beta(x))$ then $\operatorname{GL}(\beta)$ acts algebraically on $\operatorname{Rep}(Q, \beta)$ by simultaneous conjugation, i.e., for $g = (g(x))_{x \in Q_0} \in \operatorname{GL}(\beta)$ and $W = (W(a))_{a \in Q_1} \in \operatorname{Rep}(Q, \beta)$, we define $g \cdot W$ by

$$(g \cdot W)(a) = g(ha)W(a)g(ta)^{-1} \text{ for every } a \in Q_1.$$

In this way, $\operatorname{Rep}(Q, \beta)$ becomes a rational representation of the linearly reductive group $\operatorname{GL}(\beta)$ and the $\operatorname{GL}(\beta)-$orbits in $\operatorname{Rep}(Q, \beta)$ are in one-to-one correspondence with the isomorphism classes of $\beta-$dimensional representations of Q. As Q is a quiver without oriented cycles, one can show that there is only one closed $\operatorname{GL}(\beta)-$orbit in $\operatorname{Rep}(Q, \beta)$ and hence the invariant ring $\operatorname{I}(Q, \beta) = \mathbb{C}[\operatorname{Rep}(Q, \beta)]^{\operatorname{GL}(\beta)}$ is exactly the base field $\mathbb{C}$. Although there are only constant $\operatorname{GL}(\beta)-$invariant polynomial functions on $\operatorname{Rep}(Q, \beta)$, the action of $\operatorname{SL}(\beta)$ on $\operatorname{Rep}(Q, \beta)$ provides us with a highly non-trivial ring of semi-invariants.

Let $\mathrm{SI}(Q,\beta)=\mathbb{C}[\mathrm{Rep}(Q,\beta)]^{\mathrm{SL}(\beta)}$ be the ring of semi-invariants. As $\mathrm{SL}(\beta)$ is the commutator subgroup of $\mathrm{GL}(\beta)$ and $\mathrm{GL}(\beta)$ is linearly reductive, we have that

$$\mathrm{SI}(Q,\beta)=\bigoplus_{\sigma\in X^\star(\mathrm{GL}(\beta))}\mathrm{SI}(Q,\beta)_\sigma,$$

where $X^\star(\mathrm{GL}(\beta))$ is the group of rational characters of $\mathrm{GL}(\beta)$ and

$$\mathrm{SI}(Q,\beta)_\sigma=\{f\in\mathbb{C}[\mathrm{Rep}(Q,\beta)]\mid gf=\sigma(g)f\text{ for all }g\in\mathrm{GL}(\beta)\}$$

is the space of semi-invariants of weight σ. Note that a character or weight of $\mathrm{GL}(\beta)$ is of the form

$$\{g(x)\mid x\in Q_0\}\in\mathrm{GL}(\beta)\mapsto\prod_{x\in Q_0}(\det g(x))^{\sigma(x)}$$

with $\sigma(x)\in\mathbb{Z}$ for all $x\in Q_0$. Therefore, we can identify $X^\star(\mathrm{GL}(\beta))$ with $\mathbb{Z}^{Q_0}$. If $\alpha\in\Gamma$, we define $\sigma=\langle\alpha,\cdot\rangle$ by

$$\sigma(x)=\langle\alpha,\varepsilon_x\rangle,\forall\ x\in Q_0.$$

Similarly, one can define $\sigma=\langle\cdot,\alpha\rangle$.

Given a quiver Q and a dimension vector β, we define the set $\Sigma(Q,\beta)$ of *(integral) effective weights* by

$$\Sigma(Q,\beta)=\{\sigma\in\mathbb{Z}^{Q_0}\mid\mathrm{SI}(Q,\beta)_\sigma\neq0\}.$$

In [16], Schofield constructed some very useful semi-invariants for quivers. A fundamental result due to Derksen and Weyman [6] (see also [18]) states that these semi-invariants span all spaces of semi-invariants. An important consequence of this spanning theorem is the following description of $\Sigma(Q,\beta)$ (for a proof see [3, Theorem 2.21]).

Theorem 4.1. *Let Q be a quiver and β be a dimension vector. If $\sigma=\langle\alpha,\cdot\rangle\in\mathbb{Z}^{Q_0}$ is a weight then the following are equivalent:*

(1) $\dim\mathrm{SI}(Q,\beta)_\sigma\neq0$, *i.e.,* $\sigma\in\Sigma(Q,\beta)$;
(2) $\sigma(\beta)=0$ *and* $\sigma(\beta')\leq0$ *for all* $\beta'\hookrightarrow\beta$, *i.e.,* $\sigma\in C(Q,\beta)\bigcap\mathbb{Z}^{Q_0}$;
(3) α *must be a dimension vector,* $\sigma(\beta)=0$ *and* $\alpha\hookrightarrow\alpha+\beta$.

If one of these equivalent conditions is satisfied we say that β is σ-semi-stable.

The following reciprocity property will turn out to be quite useful.

Lemma 4.2. [6, Corollary 1] *Let α and β be two dimension vectors. Then*

$$\dim\mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle}=\dim\mathrm{SI}(Q,\alpha)_{-\langle\cdot,\beta\rangle}.$$

In this case, we define $\alpha\circ\beta$ to be

$$\alpha\circ\beta=\dim\mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle}=\dim\mathrm{SI}(Q,\alpha)_{-\langle\cdot,\beta\rangle}.$$

With this notation, Theorem 4.1 says that if α and β are two dimension vectors than

$$\alpha\circ\beta\neq0\Longleftrightarrow\beta\text{ is }\langle\alpha,\cdot\rangle-\text{semi-stable}\Longleftrightarrow\alpha\text{ is }-\langle\cdot,\beta\rangle-\text{semi-stable}.$$

An important consequence of Theorem 4.1 and Lemma 4.2 is that the dimension $\beta_1\circ\beta_2$ is *saturated*, i.e., if $s_1,s_2\geq1$ are integers then

$$s_1\beta_1\circ s_2\beta_2\neq0\Longleftrightarrow\beta_1\circ\beta_2\neq0.$$

Some of the inequalities defining the cone $C(Q,\beta)$ are redundant. As it turns out, one can find a minimal list defining this cone. First, let us define Schur roots:

Theorem 4.3. [17, Theorem 6.1] *Let $\beta \in \mathbb{Z}_{\geq 0}^{Q_0}$ be a dimension vector. The following are equivalent:*

(1) *there exists W a β-dimensional representation such that*

$$\mathrm{End}_Q(W, W) \cong \mathbb{C};$$

(2) $\sigma_\beta(\beta') < 0, \forall\ \beta' \hookrightarrow \beta,\ \beta' \neq 0, \beta,$ *where* $\sigma_\beta = \langle \beta, \cdot \rangle - \langle \cdot, \beta \rangle$.

If one of the two equivalent conditions is satisfied, we say that β is a Schur root.

In what follows, we explain Derksen and Weyman's [5] description of the facets of $C(Q, \beta)$. The following Proposition will be very useful for us (see [3, Proposition 2.26] or [5]):

Proposition 4.4. *Let Q be a quiver with N vertices and let us assume that β is a Schur root. Then:*

(1) $\dim C(Q, \beta) = N - 1$;

(2) *if $\mathcal{F}$ is a facet of the cone $C(Q, \beta)$ then $\mathcal{F}$ has to be of the form*

$$\mathbb{H}(\beta_1) \bigcap C(Q, \beta),$$

where $\beta = c_1\beta_1 + c_2\beta_2$ with β_1, β_2 Schur roots, $\beta_1 \circ \beta_2 = 1$, and c_i positive integers with $c_i = 1$ whenever $\langle \beta_i, \beta_i \rangle < 0$.

Remark 4.5. It is worth pointing out that if β_1 and β_2 satisfy the conditions of Proposition 4.4(2) then β_1 gives a valid inequality for $C(Q, \beta)$. In fact, to show that $\sigma(\beta_1) \leq 0$ on $C(Q, \beta)$, we only need to know that $\beta = c_1\beta_1 + c_2\beta_2$ and $\beta_1 \circ \beta_2 \neq 0$. First, note that

$$\beta_1 \circ \beta_2 \neq 0 \Rightarrow c_1\beta_1 \circ c_2\beta_2 \neq 0$$

and the latter implies $c_1\beta_1 \hookrightarrow \beta$ by Theorem 4.1. So, we must have $\sigma(\beta_1) \leq 0$ on $C(Q, \beta)$.

To give a precise description of the facets of the cone $C(Q, \beta)$, we need to introduce:

Definition 4.6. Let β be a dimension vector. We define $W_2(Q, \beta)$ to be the set of all pairs (β_1, β_2) for which:

(1) $\beta = c_1\beta_1 + c_2\beta_2$ for some integers $c_1,\ c_2 \geq 1$;
(2) β_1 and β_2 are Schur roots;
(3) $s_1\beta_1 \circ s_2\beta_2 = 1$ for all $s_1, s_2 \geq 1$;
(4) if $\langle \beta_i, \beta_i \rangle < 0$ then $c_i = 1$.

It has been essentially proved in [5, Proposition 28] (see also [3, Proposition 2.34]) that the facets of $C(Q, \beta)$ are in one to one correspondence with $W_2(Q, \beta)$. If $\mathcal{F}(Q, \beta)$ denotes the set of all facets of $C(Q, \beta)$ then:

Proposition 4.7. *Let Q be a quiver and let β be a Schur root. Then the map*

$$W_2(Q, \beta) \to \mathcal{F}(Q, \beta)$$

that sends a pair

$$(\beta_1, \beta_2) \in W_2(Q, \beta) \mapsto C(Q, \beta_1) \bigcap C(Q, \beta_2) = \mathbb{H}(\beta_1) \bigcap C(Q, \beta)$$

is a bijection. Therefore, we obtain that $\sigma(\beta) = 0$ together with

$$\sigma(\beta_1) \leq 0, \forall\ (\beta_1, \beta_2) \in W_2(Q, \beta)$$

form a minimal list of linear homogeneous inequalities defining the cone $C(Q, \beta)$.

The following conjecture explains the difference between Proposition 4.4 and Proposition 4.7.

Conjecture 4.8 (The rigidity conjecture)**.** *In* [5, Conjecture 30] *it is conjectured that*

$$s_1\beta_1 \circ s_2\beta_2 = 1, \forall\ s_1, s_2 \geq 1 \Longleftrightarrow \beta_1 \circ \beta_2 = 1.$$

Remark 4.9. A more general version of the rigidity conjecture would be

$$s_1\beta_1 \circ s_2\beta_2 = 1 \text{ for all } s_1, s_2 \geq 1 \Longleftrightarrow s_1\beta_1 \circ s_2\beta_2 = 1 \text{ for some } s_1, s_2 \geq 1.$$

As it turns out, this version is in fact equivalent to Conjecture 4.8. Let β_1 and β_2 be two dimension vectors. Note that $s_1\beta_1 \circ s_2\beta_2 \geq \beta_1 \circ \beta_2$, for all $s_1, s_2 \geq 1$. Using this observation together with the saturation property for $\beta_1 \circ \beta_2$ (see discussion after Lemma 4.2), we obtain that if $s_1\beta_1 \circ s_2\beta_2 = 1$ for some $s_1, s_2 \geq 1$ then $\beta_1 \circ \beta_2 = 1$. This clearly shows that the two conjectures are equivalent.

What is known about Conjecture 4.8 is that it requires only a finite number of steps:

Proposition 4.10. [3, Lemma 2.31] *Let α and β be two dimension vectors. Then there exists an integer $d \geq 1$ such that*

$$s_1\alpha \circ s_2\beta = 1, \forall\ s_1, s_2 \geq 1 \Longleftrightarrow l\alpha \circ \beta = 1, \forall\ 1 \leq l \leq d.$$

We will come back to Conjecture 4.8 in Corollary 5.6 and Section 6.

5. The Horn type inequalities

For the remaining of the paper, we work with the star-like quiver setting defined in Section 3. First let us prove the following:

Lemma 5.1. *The dimension vector β is a Schur root for $Q_{\mathcal{M}aj}$.*

Proof. If $n = 1$ or $(r, n, m) = (0, 2, 3)$ then our quiver $Q_{\mathcal{M}aj}$ is a single vertex or it is of type $\mathbb{A}_2$ or $\mathbb{D}_4$. In these cases, β is a (real) Schur root. Next, if $(r, n, m) = (1, n, 2)$, then it is known that β is a Schur root (see for example [3, Lemma 3.10]).

In the remaining cases, β lies in the fundamental region, i.e., the support of β is connected and $\langle \varepsilon_x, \beta \rangle + \langle \beta, \varepsilon_x \rangle \leq 0$ for all $x \in Q_0$. Also, the greatest common divisor of the coordinates of β is one. It follows from [11, Theorem B(d)] that β is a Schur root. □

Definition 5.2. A face $\mathcal{F}$ of $C(Q_{\mathcal{M}aj}, \beta)$ is called *regular* if $\mathcal{F}$ is not of the form $\mathbb{H}(\beta_1) \bigcap C(Q_{\mathcal{M}aj}, \beta)$ with $\beta_1 = \varepsilon_{(j,i)}$ for some $1 \leq j \leq n-1,\ 1 \leq i \leq m$.

Remark 5.3. The inequalities coming from those faces which are not regular just tell us that a weight σ_λ must satisfy the chamber inequalities which is equivalent to $\lambda(1), \dots, \lambda(m)$ being weakly decreasing sequences of real numbers. This is something that we will always assume.

Lemma 5.4. *The regular facets of $C(Q_{\mathcal{M}aj}, \beta)$ are of the form*

$$\mathbb{H}(\beta_1) \bigcap C(Q_{\mathcal{M}aj}, \beta),$$

where β_1 is weakly increasing with jumps of at most one along the m flags, $\beta_1 \neq \beta$, and $\beta_1 \circ (\beta - \beta_1) = 1$.

Proof. Let $\mathcal{F}$ be a facet of $C(Q_{\mathcal{M}aj}, \beta)$. From Proposition 4.4 we know that $\mathcal{F}$ is of the form

$$\mathbb{H}(\beta_1) \bigcap C(Q_{\mathcal{M}aj}, \beta),$$

where $\beta = c_1\beta_1 + c_2\beta_2$ with β_1, β_2 Schur roots and $\beta_1 \circ \beta_2 = 1$.

Let us denote $c_1\beta_1 = \beta', c_2\beta_2 = \beta''$. We have $\beta' \neq \varepsilon_{(l,i)}, \forall\ 1 \leq l \leq n-1,\ 1 \leq i \leq m$ as $\mathcal{F}$ is a regular facet. Since $\beta' \circ \beta'' \neq 0$ it follows from Theorem 4.1 that any representation of dimension vector β has a subrepresentation of dimension vector β'. If we choose a β-dimensional representation which is surjective along the m flags $\mathcal{A}(i)$ then β' can increase by at most one along each flag (from right to left).

Next, we will show that β' is weakly increasing along each flag. For convenience, let us write

$$\mathcal{A}(i) : n \longrightarrow n-1 \quad \cdots \quad 2 \longrightarrow 1.$$

Assume to the contrary that $\beta'(l+1) - \beta'(l) < 0$ for some $l \in \{1, \dots n-1\}$ which implies that $\beta' - \varepsilon_l \hookrightarrow \beta'$. On the other hand, we know that $\beta' \circ \beta'' \neq 0$ is equivalent to β' being $-\langle \cdot, \beta'' \rangle$-semi-stable by Theorem 4.1. Thus, we have $\langle \beta' - \varepsilon_l, \beta'' \rangle \geq 0$. So, $\beta''(l) \leq \beta''(l-1)$ which implies $\beta'(l) \geq 1 + \beta'(l-1)$. As β' has jumps of at most one along the flag, we get $\beta'(l) = 1 + \beta'(l-1)$. This says that $c_1 = 1$ and $\varepsilon_l \hookrightarrow \beta'$. Note that $\beta' = \beta_1$ is a Schur root and $\varepsilon_l, \beta' - \varepsilon_l \hookrightarrow \beta'$, with $\beta' \neq \varepsilon_l$. Therefore, $\sigma_{\beta'}(\varepsilon_l) < 0$ and $\sigma_{\beta'}(\beta' - \varepsilon_l) < 0$ by Theorem 4.3. But this is a contradiction. We have shown that β' is weakly increasing with jumps of at most one along the m flags $\mathcal{A}(i)$.

Now, let us show that $c_1 = c_2 = 1$. Since $\beta' = c_1\beta_1$ has jumps of at most one along each flag, we obtain $0 \leq c_1(\beta_1(l+1,i) - \beta_1(l,i)) \leq 1$ for all $l \in \{1, \dots, n-1\}$ and $i \in \{1, \dots, m\}$. If there exist l, i such that $\beta_1(l+1,i) - \beta_1(l,i) \neq 0$ then $c_1 = 1$. Otherwise, there exists an i such that $\beta'(1,i) = 1$ and so $c_1 = 1$. Similarly, one can show that $c_2 = 1$. In conclusion, $\beta = \beta_1 + \beta_2$ with β_1 weakly increasing with jumps of at most one along the m flags, $\beta_1 \neq \beta$ and $\beta_1 \circ (\beta - \beta_1) = 1$. □

Our next goal is to give a closed form to those inequalities coming from the regular facets obtained in Lemma 5.4.

Let $\beta = \beta_1 + \beta_2$ with β_1, β_2 weakly increasing along the m flags. Define the following sets

$$I_i = \{j \mid \beta_1(j,i) > \beta_1(j-1,i), 1 \leq j \leq n\},$$

with the convention that $\beta_1(0,i) = 0$ for all $i \in \{1, \dots, r\}$. Then $|I_i| = \beta_1(n)$ for all $i \in \{1, \dots, m\}$.

Conversely, let $I = (I_1, \dots, I_m)$ be an m-tuple of subsets of $\{1, \dots, n\}$ of the same cardinality. Along the m flags we can define the dimension vector whose jump sets are given by the m-tuple I. If $r \geq 1$, we extend this dimension vector to β_I and β'_I by letting $\beta_I(0) = 0$ and $\beta'_I(0) = 1$.

We are interested in computing $s_1\beta_I \circ s_2(\beta - \beta_I)$ and $s_1\beta'_I \circ s_2(\beta - \beta'_I)$ for s_1, s_2 two positive integers. As it turns out, these dimensions can be viewed as Littlewood-Richardson coefficients. To write down a general formula, we need to introduce some notation. If $\lambda = (\lambda_1, \dots, \lambda_l)$ is a partition and s is a positive integer, we define λ^s to be the partition obtained from λ by repeating each part s times, i.e., $\lambda^s = (\underbrace{\lambda_1, \dots, \lambda_1}_{s}, \dots, \underbrace{\lambda_l, \dots, \lambda_l}_{s})$.

Also, if t is a positive integer then $t\lambda$ denotes the stretched partition, i.e., $t\lambda = (t\lambda_1, \dots, t\lambda_l)$. For example, if s_1 and s_2 are two positive integers and $I = \{1 \leq i_1 < \dots < i_l \leq n\}$ is a subset of $\{1, \dots, n\}$ then $s_2\lambda^{s_1}(I) = (\underbrace{s_2(i_l - l), \dots, s_2(i_l - l)}_{s_1}, \dots, \underbrace{s_2(i_1 - 1), \dots, s_2(i_1 - 1)}_{s_1})$.

Lemma 5.5. *Let s_1 and s_2 be two positive integers.*

(1) *We have that*

$$s_1\beta_I \circ s_2(\beta - \beta_I) = c^{((s_2n-s_2d)^{s_1d})}_{s_2\lambda^{s_1}(I_1),\dots,s_2\lambda^{s_1}(I_m)},$$

where $d = |I_i|$. In particular, we have

$$\beta_I \circ (\beta - \beta_I) = c^{((n-d)^d)}_{\lambda(I_1),\dots,\lambda(I_m)}.$$

(2) *Similarly, if $r \geq 1$ and $\beta'_I \circ (\beta - \beta'_I) \neq 0$ then $d \geq r$ and*

$$s_1\beta'_I \circ s_2(\beta - \beta'_I) = c^{((s_2n-s_2d)^{s_1(d-r)})}_{s_2\lambda^{s_1}(I_1),\dots,s_2\lambda^{s_1}(I_m)}.$$

In particular, we have

$$\beta'_I \circ (\beta - \beta'_I) = c^{((n-d)^{(d-r)})}_{\lambda(I_1),\dots,\lambda(I_m)}.$$

Proof. (1) We denote $s_1\beta_I$ by β_1 and $s_2(\beta - \beta_I)$ by β_2. Since

$$\beta_1 \circ \beta_2 = \dim \mathrm{SI}(Q_{\mathcal{M}aj}, \beta_1)_{-\langle \cdot, \beta_2 \rangle}$$

and $\beta_1(0) = 0$ we can reduce our quiver to the star quiver obtained by removing the vertex 0 and the r arrows going out from the removed vertex. We denote the star quiver by Q and we keep the same notation for the restriction of β_1 and β_2 to the star quiver. If we denote $\langle \beta_1, \cdot \rangle$ by σ_1 then

$$\beta_1 \circ \beta_2 = \dim \mathrm{SI}(Q, \beta_2)_{\sigma_1}$$

by the reciprocity property (see Property 4.2). Now the fact that β_2 is weakly increasing along the flags (from right to left) and the weight σ_1 is non-positive along the flags (except for the central vertex n) will be quite useful for us.

To compute $\dim \mathrm{SI}(Q, \beta_2)_{\sigma_1}$, we first decompose the affine coordinate ring of $\mathrm{Rep}(Q, \beta_2)$ as a (rather complicated) direct sum in which the summands are tensor products of irreducible representations of $GL(n)$'s. Then, we will sort out those summands that give us semi-invariants of weight σ_1. For simplicity, let us define $V_j(i) = \mathbb{C}^{\beta_2(j,i)}, \forall\ 1 \leq i \leq m,\ 1 \leq j \leq n$, and $V = \mathbb{C}^{\beta_2(n)}$.

It will be convenient to introduce the following space:

$$V_{\mathcal{A}(l)} = \oplus_{j=1}^{n-1} \mathrm{Hom}(V_{j+1}(l), V_j(l)).$$

Since

$$\mathrm{Rep}(Q, \beta_2) = \oplus_{l=1}^{m} V_{\mathcal{A}(l)},$$

we have

$$\mathbb{C}[\mathrm{Rep}(Q, \beta_2)] = \otimes_{l=1}^{m} \mathbb{C}[V_{\mathcal{A}(l)}].$$

In what follows, we determine the contribution of each flag $\mathcal{A}(l)$ to the space of semi-invariants $\mathrm{SI}(Q, \beta_2)_{\sigma_1}$. First, let us write

$$\mathbb{C}[V_{\mathcal{A}(l)}] = \otimes_{j=1}^{n-1} \mathbb{C}[\mathrm{Hom}(V_{j+1}(l), V_j(l))].$$

Using Cauchy's formula [7, page 121], we have

$$\begin{aligned}\mathbb{C}[\mathrm{Hom}(V_{j+1}(l), V_j(l))] &= S(V_j^*(l) \otimes V_{j+1}(l)),\\ &= \oplus S^{\gamma^j(l)} V_j^*(l) \otimes S^{\gamma^j(l)} V_{j+1}(l),\end{aligned}$$

where the sum is over partitions $\gamma^j(l)$, for all $1 \leq j \leq n-1,\ 1 \leq l \leq m$. Therefore, we can write

$$\mathbb{C}[V_{\mathcal{A}(l)}] = \oplus S^{\gamma^1(l)} V_1^*(l) \otimes \otimes_{j=2}^{n-1} \left(S^{\gamma^{j-1}(l)} V_j(l) \otimes S^{\gamma^j(l)} V_j^*(l) \right) \otimes S^{\gamma^{n-1}(l)} V_n(l),$$

where the sum is taken over partitions $\gamma^1(l), \ldots, \gamma^{n-1}(l)$.

When computing semi-invariants, we see that $\left(S^{\gamma^1(l)} V_1^*(l) \right)^{\mathrm{SL}(V_1(l))}$ is non-zero if and only if it is one dimensional. In this case, $\gamma^1(l)$ is a $\beta_2(1,l) \times w$ rectangle and the space is spanned by a semi-invariant of weight $-w$. So, $\left(S^{\gamma^1(l)} V_1^*(l) \right)^{\mathrm{SL}(V_1(l))}$ contains non-zero semi-invariants of weight $\sigma_1(1,l)$ if and only if $\sigma_1(1,l) \leq 0$ and $\gamma^1(l) = (-\sigma_1(1,l)^{\beta_2(1,l)})$, i.e.,

$$\gamma^1(l) = (\beta_2(1,l)^{-\sigma_1(1,l)})'.$$

Next, we look at the space

$$\left(S^{\gamma^1(l)} V_2(l) \otimes S^{\gamma^2(l)} V_2^*(l) \right)^{\mathrm{SL}(V_2(l))}$$

which is canonically isomorphic to $\mathrm{Hom}_{\mathrm{SL}(V_2(l))}(S^{\gamma^2(l)} V_2(l), S^{\gamma^1(l)} V_2(l))$. Now, this space is non-zero if and only if it is one dimensional in which case $\gamma^2(l)$ is $\gamma^1(l)$ plus some extra columns of length $\beta_2(2,l)$. The weight of a semi-invariant spanning this space is $-t$, where t is the number of the extra columns. Consequently,

$$\left(S^{\gamma^1(l)} V_2(l) \otimes S^{\gamma^2(l)} V_2^*(l) \right)^{\mathrm{SL}(V_2(l))}$$

contains non-zero semi-invariants of weight $\sigma_1(2,l)$ if and only if $\sigma_1(2,l) \leq 0$ and $\gamma^2(l)$ is $\gamma^1(l)$ plus $(-\sigma_1(2,l))$ columns of length $\beta_2(2,l)$, i.e.,

$$\gamma^2(l) = (\beta_2(2,l)^{-\sigma_1(2,l)}, \beta_2(1,l)^{-\sigma_1(1,l)})'.$$

Reasoning in this way, we see that the vertices of this flag $\mathcal{A}(l)$, except the central vertex $(n,l) = n$, give nonzero spaces of semi-invariants (in which case they must be one dimensional) of weight $\sigma_1(1,l), \ldots, \sigma_1(n-1,l)$ if and only if $\gamma^1(l)$ is a $\beta_2(1,l) \times (-\sigma_1(1,l))$ rectangle and $\gamma^j(l)$ is $\gamma^{j-1}(l)$ plus $(-\sigma_1(j,l))$ columns of length $\beta_2(j,l)$ for all $j \in \{2, \ldots, n-1\}$, i.e.,

$$\gamma^{n-1}(l) = (\beta_2(n-1,l)^{-\sigma_1(n-1,l)}, \ldots, \beta_2(1,l)^{-\sigma_1(1,l)})'.$$

We have proved that each flag $\mathcal{A}(l)$ contributes to the space of semi-invariants $\mathrm{SI}(Q, \beta_2)_{\sigma_1}$ with

$$S^{\gamma^{n-1}(l)} V_n(l).$$

Putting all together, we obtain that

$$\beta_1 \circ \beta_2 = \dim_{\mathbb{C}}(S^{\mu(1)} V \otimes \cdots \otimes S^{\mu(m)} V)^{\mathrm{SL}(V)},$$

where

$$\mu(l) = (\beta_2(n-1,l)^{-\sigma_1(n-1,l)}, \ldots, \beta_2(1,l)^{-\sigma_1(1,l)})', \forall\ 1 \leq l \leq m.$$

Since $\sigma_1 = \langle \beta_1, \cdot \rangle$ it easy to see that

$$\sigma_1(j,l) = \begin{cases} -s_1 & \text{if } j+1 \in I_l \\ 0 & \text{otherwise} \end{cases},$$

for all $1 \leq j \leq n-1,\ 1 \leq l \leq m$.

Furthermore, if $I_l = \{l_1 < \cdots < l_d\}$ then $\beta_I(l_k, l) = k$ and so

$$\beta_2(l_k, l) = s_2(l_k - k) = \beta_2(l_k - 1, l)$$

for all $k \in \{1, \ldots, d\}$. Therefore,

$$\mu(l) = (s_2 \lambda^{s_1}(I_l))'$$

for all $1 \leq l \leq m$. Taking into account the weight $\sigma_1(n)$ coming from the central vertex, we can write

$$\beta_1 \circ \beta_2 = \dim_{\mathbb{C}}(S^{(s_2\lambda^{s_1}(I_1))'}(V) \otimes \cdots \otimes S^{(s_2\lambda^{s_1}(I_m))'}(V) \otimes \det_{\beta_2(n)}^{-\sigma_1(n)})^{\mathrm{GL}(V)}.$$

On the other hand, $\sigma_1(n) = \beta_1(n) = s_1 d$ and $\beta_2(n) = s_2(n-d)$. Hence,

$$\beta_1 \circ \beta_2 = \dim_{\mathbb{C}}(S^{(s_2\lambda^{s_1}(I_1))'}(V) \otimes \cdots \otimes S^{(s_2\lambda^{s_1}(I_m))'}(V) \otimes \det_{s_2(n-d)}^{-s_1 d})^{\mathrm{GL}(V)}$$

has the desired form.

(2) From $\beta'_I \circ (\beta - \beta'_I) \neq 0$ and Theorem 4.1(3) it follows that any $W \in \mathrm{Rep}(Q_{\mathcal{M}aj}, \beta)$ has a subrepresentation of dimension vector β'_I. Take a representation $W \in \mathrm{Rep}(Q_{\mathcal{M}aj}, \beta)$ such that the one dimensional images of W along the r arrows from vertex 0 to vertex n are linearly independent. This clearly implies that $r \leq \beta'_I(n) = d$. To show that $s_1\beta'_I \circ s_2(\beta - \beta'_I)$ has the desired form, one follows the same arguments as in part (1). □

The next corollary will be particularly useful in Lemma 6.1.

Corollary 5.6. *Keep the same notation as in Proposition 5.5. We have that*

$$s_1\beta_I \circ s_2(\beta - \beta_I) = 1, \forall\ s_1, s_2 \geq 1 \Longleftrightarrow \beta_I \circ (\beta - \beta_I) = 1,$$

and

$$s_1\beta'_I \circ s_2(\beta - \beta'_I) = 1, \forall\ s_1, s_2 \geq 1 \Longleftrightarrow \beta'_I \circ (\beta - \beta'_I) = 1.$$

Proof. First, let us recall some facts about multiple Littlewood-Richardson coefficients. Let γ and $\lambda(1), \ldots, \lambda(m)$ be partitions. Form [20, Proposition 9] follows that the multiple Littlewood-Richardson coefficient $c^{\gamma}_{\lambda(1),\ldots,\lambda(m)}$ is in fact a Littlewood-Richardson coefficient. Moreover, we can apply Fulton's conjecture [15] (see also [1]) on stretched Littlewood-Richardson coefficients to conclude that:

$$c^{s\gamma}_{s\lambda(1),\ldots,s\lambda(m)} = 1, \forall\ s \geq 1 \Longleftrightarrow c^{\gamma}_{\lambda(1),\ldots,\lambda(m)} = 1. \tag{5}$$

We also obtain directly from [20, Proposition 9] that

$$c^{\gamma'}_{\lambda'(1),\ldots,\lambda'(m)} = c^{\gamma}_{\lambda(1),\ldots,\lambda(m)}. \tag{6}$$

From (5) and Lemma 5.5, we have

$$s_1\beta_I \circ s_2(\beta - \beta_I) = 1, \forall\ s_1, s_2 \geq 1 \Longleftrightarrow s_1\beta_I \circ (\beta - \beta_I) = 1, \forall\ s_1 \geq 1.$$

Next, using (6) and Lemma 5.5 again, we can write

$$s_1\beta_I \circ (\beta - \beta_I) = c^{((s_1 d)^{n-d})}_{s_1\lambda'(I_1),\ldots,s_1\lambda'(I_m)}.$$

Finally, applying (5) and then (6) one more time we get:

$$s_1\beta_I \circ (\beta - \beta_I) = 1,\ \forall s_1 \geq 1 \Longleftrightarrow \beta_I \circ (\beta - \beta_I) = 1$$

and this proves the first part of our corollary. The second part can be proved in a similar way. □

Proposition 5.7. *Let $\lambda(1), \dots, \lambda(m)$ be weakly decreasing sequences of n real numbers. The following are equivalent:*

(1)
$$\sigma_\lambda \in C(Q_{\mathcal{M}aj}, \beta);$$

(2)
$$\sum_{1 \le i \le m} \sum_{j \in I_i} \lambda_j(i) \ge 0$$

for all $(I_1, \dots, I_m) \in \mathcal{S}_{\mathcal{M}aj}(n, m)$ and
$$\sum_{1 \le i \le m} \sum_{j \in J_i} \lambda_{n+1-j}(i) \le 0,$$

for all $(J_1, \dots, J_m) \in \mathcal{S}_{\mathcal{M}aj}(n-r, m)$.

Proof. Let $I = (I_1, \dots, I_m)$ be an m-tuple of subsets of $\{1, \dots, n\}$ of the same cardinality, say d. Then
$$\sigma_\lambda(\beta_I) = -\sum_{1 \le i \le m} \sum_{j \in I_i} \lambda_j(i).$$
and
$$\sigma_\lambda(\beta'_I) = \sum_{1 \le i \le m} \sum_{j \notin I_i} \lambda_j(i).$$

If $r \ge 1$ and $\beta'_I \circ (\beta - \beta'_I) = 1 \ne 0$ then the Young diagram of $\lambda(I_i)$ has to be included in the rectangle $(d-r) \times (n-d)$ by Proposition 5.5(2). This clearly implies $\{1, \dots, r\} \subseteq I_i$. Let us define $J_i = \{i \mid n+1-i \notin I_i\}$. Then $\lambda(I_i)' = \lambda(J_i)$ and so $(J_1, \dots, J_m) \in \mathcal{S}_{\mathcal{M}aj}(n-r, m)$. Also, we have
$$\sigma_\lambda(\beta'_I) = \sum_{1 \le i \le m} \sum_{j \in J_i} \lambda_{n+1-j}(i).$$

Next, if $r = 0$ then the list of Horn type inequalities from (2) can be rewritten as
$$\sum_{1 \le i \le m} |\lambda(i)| = 0,$$
and
$$\sum_{1 \le i \le m} \sum_{j \in I_i} \lambda_j(i) \ge 0,$$
for all $(I_1, \dots, I_m) \in \mathcal{S}_{\mathcal{M}aj}(n, m)$ with $|I_i| < n$.

Now, using Lemma 5.5 we can easily see that the list of Horn type inequalities from (2) can be rewritten as $\sigma_\lambda(\beta) = 0$ together with
$$\sigma_\lambda(\beta_1) \le 0,$$
for every β_1 weakly increasing with jumps of at most one along the m flags, $\beta_1 \ne \beta$ and $\beta_1 \circ (\beta - \beta_1) = 1$. The proof follows now from Lemma 5.4. □

Proof of Theorem 1.1. (1) The fact that $\mathcal{M}aj_r(n, m)$ is a rational convex polyhedral cone follows from Proposition 3.4. As β is a Schur root and Proposition 4.4 we know that
$$\dim C(Q_{\mathcal{M}aj}, \beta) = (n-1)m + 1 - \delta_{0,r}.$$
The proof of (1) follows now from Proposition 3.5.

(2) It follows from Proposition 3.4 and Proposition 5.7. □

Proof of Theorem 1.2. Let us assume that $\lambda(i)$ are weakly decreasing sequences of n integers. Following the same arguments as in Lemma 5.5 with β_2 replaced by β and σ_1 replaced by σ_λ, we obtain that $\dim \mathrm{SI}(Q_{\mathcal{M}aj}, \beta)_{\sigma_\lambda}$ is equal to

$$\sum \dim_{\mathbb{C}}(S^{a_1}(\mathbb{C}^n)^* \otimes \cdots \otimes S^{a_r}(\mathbb{C}^n)^* \otimes S^{\lambda(1)}(\mathbb{C}^n) \otimes \cdots \otimes S^{\lambda(m)}(\mathbb{C}^n))^{\mathrm{GL}(n)},$$

where the sum is taken over all non-negative integers $a_1, \ldots, a_r$ for which $\sum_{i=1}^r a_i = \sum_{i=1}^m |\lambda(i)|$. Thus, we have

$$\dim \mathrm{SI}(Q_{\mathcal{M}aj}, \beta)_{\sigma_\lambda} \neq 0 \Leftrightarrow c^{\mu}_{\lambda(1),\ldots,\lambda(m)} \neq 0,$$

where $\mu = (a_1, \ldots, a_r)$ is some partition with at most r non-zero parts. Now, using Proposition 3.5 and Theorem 4.1 we obtain

$$(\lambda(1), \ldots, \lambda(m)) \in \mathcal{M}aj_r(n,m) \bigcap \mathbb{Z}^{nm} \Leftrightarrow \dim \mathrm{SI}(Q_{\mathcal{M}aj})_{\sigma_\lambda} \neq 0,$$

and so the proof follows.

□

Corollary 5.8 (**Horn's conjecture**). *Let $\lambda(1)$, $\lambda(2)$, and $\lambda(3)$ be three weakly decreasing sequences of n real numbers. Then the following are equivalent:*

(1) *there exist $n \times n$ Hermitian matrices $A(1), A(2), A(3)$ with eigenvalues $\lambda(1)$, $\lambda(2)$, $\lambda(3)$ such that*
$$A(2) = A(1) + A(3);$$
(2)
$$|\lambda(2)| = |\lambda(1)| + |\lambda(3)|$$
and
$$\sum_{j \in I_2} \lambda_j(2) \leq \sum_{j \in I_1} \lambda_j(1) + \sum_{j \in I_3} \lambda_j(3)$$
for all triples (I_1, I_2, I_3) of subsets of $\{1, \ldots, n\}$ of the same cardinality $r < n$ such that $c^{\lambda(I_2)}_{\lambda(I_1),\lambda(I_3)} = 1$.
Assume that $\lambda(1)$, $\lambda(2)$, and $\lambda(3)$ are sequences of n integers. Then (1) and (2) are equivalent to:

(3)
$$c^{\lambda(2)}_{\lambda(1),\lambda(3)} \neq 0.$$

Corollary 5.9 (**Majorization problem**). *Let $\lambda(1), \lambda(2), \lambda(3)$ be three weakly decreasing sequences of n real numbers. Then the following are equivalent:*

(1) *there exist $n \times n$ Hermitian matrices $A(1), A(2), A(3)$ with eigenvalues $\lambda(1)$, $\lambda(2)$, $\lambda(3)$ such that*
$$A(2) \leq A(1) + A(3);$$
(2)
$$|\lambda(2)| \leq |\lambda(1)| + |\lambda(3)|$$
and
$$\sum_{j \in I_2} \lambda_j(2) \leq \sum_{j \in I_1} \lambda_j(1) + \sum_{j \in I_3} \lambda_j(3)$$
for all triples (I_1, I_2, I_3) of subsets of $\{1, \ldots, n\}$ of the same cardinality $r < n$ such that $c^{\lambda(I_2)}_{\lambda(I_1),\lambda(I_3)} = 1$.

Proof. Apply Theorem 1.1 with $r = n$ and $m = 3$. Since $r = n$ we have that $\mathcal{S}_{\mathcal{M}aj}(n-n, m) = \emptyset$ and so the proof follows. □

6. Minimality of the list of Horn type inequalities

In this last section, we will find a minimal list of Horn type inequalities describing the cone $\mathcal{M}aj_r(n, m)$.

Lemma 6.1. *Let us assume that $m \geq 3$. Then the regular facets of $C(Q_{\mathcal{M}aj}, \beta)$ are precisely those of the form*

$$\mathbb{H}(\beta_1) \bigcap C(Q_{\mathcal{M}aj}, \beta),$$

where β_1 is weakly increasing with jumps of at most one along the m flags, $\beta_1 \neq \beta$, and $\beta_1 \circ (\beta - \beta_1) = 1$.

Proof. We have proved in Lemma 5.4 that any regular facet has the desired form. Conversely, let us consider the face

$$\mathbb{H}(\beta_1) \bigcap C(Q_{\mathcal{M}aj}, \beta),$$

where β_1 is weakly increasing with jumps of at most one along the m flags, $\beta_1 \neq \beta$, and $\beta_1 \circ (\beta - \beta_1) = 1$.

We are going to show that if $m \geq 3$ then $(\beta_1, \beta_2 = \beta - \beta_1) \in W_2(Q_{\mathcal{M}aj}, \beta)$. First, let us show that β_1 and β_2 are both Schur roots. For this, we shrink the quiver $Q_{\mathcal{M}aj}$ such that β_1 restricted to the shrunk quiver has jumps all equal to one along the flags. Furthermore, it is easy to see that β_1 is a Schur root if the restriction of β_1 to the shrunk quiver is a Schur root. But now for the shrunk quiver one can use the same arguments as in Lemma 5.1 to show that the restriction of β_1, and hence β_1, is a Schur root. Similarly, one can show that β_2 is a Schur root. Now, using Corollary 5.6 we obtain that $s_1\beta_1 \circ s_2\beta_2 = 1, \forall\ s_1, s_2 \geq 1$ and so $(\beta_1, \beta_2) \in W_2(Q_{\mathcal{M}aj}, \beta)$. The proof follows now from Proposition 4.7. □

Proposition 6.2. *Let us assume that $m \geq 3$.*

1. *If $r \geq 1$ then the list of Horn type inequalities of Theorem 1.1(2) is minimal.*
2. *If $m \geq 3$ and $r = 0$ then $\mathcal{M}aj_0(n, m)$ consists of all m-tuples $(\lambda(1), \dots, \lambda(m))$ of weakly decreasing sequences of n real numbers satisfying*

$$\sum_{1 \leq i \leq m} |\lambda(i)| = 0$$

and

$$\sum_{1 \leq i \leq m} \sum_{j \in I_i} \lambda_j(i) \geq 0,$$

for all $(I_1, \dots, I_m) \in \mathcal{S}_{\mathcal{M}aj}(n, m)$ with $|I_i| < n$. Furthermore, this is now a minimal list of Horn type inequalities.

Proof. The proof follows from Proposition 5.7 and Lemma 6.1. □

Remark 6.3. To find the minimal list of Horn type inequalities of Proposition 6.2, we used Lemma 6.1 which relies on the following conjecture of W. Fulton [8, page 239]

$$(*) \qquad c^{\lambda(2)}_{\lambda(1),\lambda(3)} = 1 \Leftrightarrow c^{n\lambda(2)}_{n\lambda(1),n\lambda(3)} = 1, \forall\ n \geq 1.$$

This conjecture was first proved by Knutson, Tao and Woodward [15] using some combinatorial gadgets called puzzles. Recently, P. Belkale [1] gave a geometric proof for $(*)$. He also suggested that it should be possible to apply his methods to quiver theory and prove the more general Conjecture 4.8.

Example 6.4. Let $m = 2$ and $r \geq 1$. Then arguing as in Lemma 5.4 it easy to see that $\beta_I, \beta - \beta_I$ are Schur roots if and only if $|I_1| = |I_2| = 1$. If this the case then

$$I = (I_1, I_2) \in \mathcal{S}_{\mathcal{M}aj}(n, 2) \Longleftrightarrow n_1 + n_2 = n + 1$$

where $I_i = \{n_i\}$.

Similarly, one has that $\beta'_I, \beta - \beta'_I$ are both Schur roots if and only if $|I_1| = |I_2| = n - 1$. In this case,

$$I = (I_1, I_2) \in \mathcal{S}_{\mathcal{M}aj}(n - r, 2) \Longleftrightarrow n_1 + n_2 = n + 1 + r$$

where $\{1, \ldots, n\} \setminus I_i = \{n_i\}$.

So, we obtain that $\mathcal{M}aj_r(n, 2)$ consists of all pairs $(\lambda(1), \lambda(2))$ of weakly decreasing sequences of n real numbers for which

$$\lambda_i(1) + \lambda_{n+1-i}(2) \geq 0$$

for all $1 \leq i \leq n$, and

$$\lambda_i(1) + \lambda_{n+1+r-i}(2) \leq 0$$

for all $r + 1 \leq i \leq n$. This is now a minimal list of Horn type inequalities.

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