CLUSTER FANS, STABILITY CONDITIONS, AND DOMAINS OF SEMI-INVARIANTS

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ABSTRACT. We show that the cone of finite stability conditions of a quiver Q without oriented cycles has a fan covering given by (the dual of) the cluster fan of Q. Along the way, we give new proofs of Schofield's results [18] on perpendicular categories. From our results, we recover Igusa-Orr-Todorov-Weyman's theorem from [7] on cluster complexes and domains of semi-invariants for Dynkin quivers. For arbitrary quivers, we also give a description of the domains of semi-invariants labeled by real Schur roots in terms of quiver exceptional sets.

1. INTRODUCTION

Given a quiver Q without oriented cycles, the set of *almost positive real Schur roots* of Q is the set

$$\Psi(Q)_{\geq -1} = \{\beta \mid \beta \text{ is a real Schur root }\} \cup \{-\gamma_i \mid i \in Q_0\},$$

where γ_i is the dimension vector of the projective indecomposable representation at vertex i. For example, when Q is a Dynkin quiver, the real Schur roots are precisely the positive roots of the corresponding Dynkin diagram. However, in general the set of real Schur roots has a much more complicated structure.

The (possibly infinite) cluster fan $C(Q)$ on the ground set $\Psi(Q)_{\geq -1}$ consists of the rational convex polyhedral cones generated by the compatible subsets of $\Psi(Q)_{\geq -1}$. The details of our notations can be found in Section 2 and Section 4.

Our goal in this paper is to give an interpretation of $C(Q)$ in terms of the geometry of the representations of Q. Following Ingalls-Thomas [9], the cone $\mathcal{S}(Q)$ of *finite stability conditions* is, by definition, the set of all $\sigma \in \mathbb{Q}^{Q_0}$ for which there are finitely many σ-stable representations up to isomorphism.

Let $I : \mathbb{Q}^{Q_0} \to \mathbb{Q}^{Q_0}$ be the isomorphism defined by $I(\alpha) = \langle \alpha, \cdot \rangle_Q$ where $\langle \cdot, \cdot \rangle_Q$ is the Euler form of the quiver Q. Now, we can state our first result:

Theorem 1.1. *Let Q be a quiver without oriented cycles. Then $\mathcal{S}(Q)$ has a fan covering given by $\{I(\mathrm{Cone}(C)) \mid C$ is a compatible subset of $\Psi(Q)_{\geq -1}\}$.*

To prove the theorem above we use techniques from quiver invariant theory, developed mainly by Derksen and Weyman [4, 5], King [12] and Schofield [18]. Using the σ-stable decomposition for dimension vectors and the A_∞-formalism, we give a new proof of Schofield's Embedding Theorem [18] which plays a fundamental role in our study:

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Theorem 1.2. *(see also* [18, Theorem 2.5]*) Let α be a pre-homogeneous dimension vector and let σ be either $\langle \alpha, \cdot \rangle$ or $-\langle \cdot, \alpha \rangle$.*

1. *There are finitely many, up to isomorphism, σ-stable representations $E_1, \dots, E_l$ with $l \leq |Q_0| - 1$. Moreover, the E_i are exceptional representations.*
2. *If $\beta_i = \mathbf{dim} E_i$ then after rearranging $\mathcal{E} = (\beta_1, \dots, \beta_l)$ is a quiver exceptional sequence.*
3. *Let $Q(\mathcal{E})$ be the quiver with vertices $1, \dots, l$, and $-\langle \beta_i, \beta_j \rangle$ arrows from i to j for all $1 \leq i \neq j \leq l$. Then there exists an equivalence of categories from $\operatorname{rep}(Q(\mathcal{E}))$ to $\operatorname{rep}(Q)^{ss}_{\sigma}$ sending the simple representation S_i of $Q(\mathcal{E})$ at i to E_i. Consequently, if*

$$I : \mathbb{N}^{Q(\mathcal{E})_0} = \mathbb{N}^l \longrightarrow \mathbb{N}^{Q_0}$$

is defined by

$$I(\eta(1), \dots, \eta(l)) = \sum_{i=1}^{l} \eta(i)\beta_i,$$

then

$$\langle \eta, \gamma \rangle_{Q(\mathcal{E})} = \langle I(\eta), I(\gamma) \rangle_Q.$$

In [7], Igusa et al. initiated the study of cluster fans via domains of semi-invariants of quivers. In fact, their motivation was two-fold since domains of semi-invariants are also related to the Igusa-Orr [8] pictures from the homology of nilpotent groups. Let us briefly recall the definition of domains of semi-invariants (for further details, see Section 2). If β is a dimension vector of Q, the domain of semi-invariants $D(\beta)$ is defined by

$$D(\beta) = \{\alpha \in \mathbb{Q}^{Q_0} \mid \langle \alpha, \beta \rangle = 0 \text{ and } \langle \alpha, \beta' \rangle \leq 0, \forall \beta' \hookrightarrow \beta\}.$$

It was proved in [7, Theorem 8.1.7] that for a Dynkin quiver Q, the $(|Q_0|-1)$-skeleton of its cluster fan can be covered by the domains of semi-invariants labeled by the real Schur roots of Q. This result can also be obtained directly from our Theorem 1.1. In fact, we can show:

Theorem 1.3. *Let Q be a connected quiver without oriented cycles. Then Q is either a Dynkin quiver or a generalized Kronecker quiver if and only if*

$$\bigcup_{\beta} D(\beta) = \bigcup_{C} \operatorname{Cone}(C),$$

where the union on the left is over all real Schur roots β while the union on the right is over all compatible sets C with (at most) $|Q_0| - 1$ elements.

In order to describe the domains of semi-invariants for arbitrary quivers, we need to work with quiver exceptional sets instead of compatible ones.

Theorem 1.4. *Let Q be a quiver without oriented cycles and let β be a real Schur root. Then there are finitely many quiver exceptional sets $\mathcal{E}_1, \dots, \mathcal{E}_m$, each of size at most $|Q_0| - 1$, such that*

$$D(\beta) = \bigcup_{1 \leq i \leq m} \operatorname{Cone}(\mathcal{E}_i).$$

Consequently,

$$\bigcup_{\beta} D(\beta) = \bigcup_{\mathcal{E}} \operatorname{Cone}(\mathcal{E}),$$

where the union on the left is over all real Schur roots β *while the union on the right is over all quiver exceptional sets* $\mathcal{E}$ *of cardinality at most* $|Q_0| - 1$.

The layout of this paper is as follows. In Section 2, we recall the main tools from quiver invariant theory. This includes King's criterion for semi-stability of quiver representations, Derksen-Weyman's First Fundamental Theorem, the Saturation Theorem, and the Reciprocity Property for semi-invariants of quivers. Schofield's results on perpendicular categories are reviewed in Section 3 where we give new proofs of his results (see Theorem 1.2 and Theorem 3.5). Cluster fans and stability conditions for quivers are discussed in Section 4 where we also prove Theorem 1.1 and Theorem 1.3. In Section 5, we study domains of semi-invariants via exceptional sets and prove Theorem 1.4.

2. Recollection on quiver invariant theory

In this section, we review the main tools from quiver invariant theory that will be used in the latter sections. Let $Q = (Q_0, Q_1, t, h)$ be a finite quiver with vertex set Q_0 and arrow set Q_1. The two functions $t, h : Q_1 \to Q_0$ assign to each arrow $a \in Q_1$ its tail *ta* and head *ha*, respectively.

Throughout this paper, we work over an algebraically closed field K of characteristic zero. A representation V of Q over K is a collection $(V(i), V(a))_{i\in Q_0, a\in Q_1}$ of finite-dimensional K-vector spaces $V(i)$, $i \in Q_0$, and K-linear maps $V(a) \in \mathrm{Hom}_K(V(ta), V(ha))$, $a \in Q_1$. The dimension vector of a representation V of Q is the function $\mathbf{dim}V : Q_0 \to \mathbb{Z}$ defined by $(\mathbf{dim}V)(i) = \dim_K V(i)$ for $i \in Q_0$. A dimension vector $\alpha \in \mathbb{Z}^{Q_0}_{\geq 0}$ is said to be *sincere* if $\alpha(i) > 0$ for all $i \in Q_0$. Let S_i be the one-dimensional simple representation at vertex $i \in Q_0$ and let us denote its dimension vector by ε_i.

Given two representations V and W of Q, we define a morphism $\varphi : V \to W$ to be a collection of K-linear maps $(\varphi(i))_{i\in Q_0}$ with $\varphi(i) \in \mathrm{Hom}_K(V(i), W(i))$, $i \in Q_0$, and such that $\varphi(ha)V(a) = W(a)\varphi(ta)$ for all $a \in Q_1$. We denote by $\mathrm{Hom}_Q(V, W)$ the K-vector space of all morphisms from V to W. Let V and W be two representations of Q. We say that V is a subrepresentation of W if $V(i)$ is a subspace of $W(i)$ for all $i \in Q_0$ and $V(a)$ is the restriction of $W(a)$ to $V(ta)$ for all $a \in Q_1$. In this way, we obtain the abelian category $\mathrm{rep}(Q)$ of all quiver representations of Q.

From now on, we assume that our quivers are without oriented cycles. Let P_i be the projective indecomposable representation at vertex $i \in Q_0$ and let us denote its dimension vector by γ_i; we call γ_i a *projective root*.

A representation V is said to be a *Schur* representation if $\mathrm{End}_Q(V) \cong K$. We say that V is a *rigid* representation if $\mathrm{Ext}^1_Q(V, V) = 0$. Finally, we say that V is an *exceptional* representation if V is a rigid Schur representation. The dimension vector of a Schur representation is called a *Schur root* while the dimension vector an exceptional representation is called a *real Schur root*. For example, the projective roots are real Schur roots.

Given two representations V and W of Q, we have the Ringel's [16] canonical exact sequence:

$$(1) \qquad 0 \to \mathrm{Hom}_Q(V, W) \to \bigoplus_{i\in Q_0} \mathrm{Hom}_K(V(i), W(i)) \xrightarrow{d^V_W} \bigoplus_{a\in Q_1} \mathrm{Hom}_K(V(ta), W(ha)),$$

where $d^V_W((\varphi(i))_{i\in Q_0}) = (\varphi(ha)V(a) - W(a)\varphi(ta))_{a\in Q_1}$ and $\mathrm{Ext}^1_Q(V, W) = \mathrm{coker}(d^V_W)$.

The Euler form of Q is the $\mathbb{Z}$-bilinear form on $\mathbb{Z}^{Q_0}$ defined by

$$\langle \alpha, \beta \rangle_Q = \sum_{i \in Q_0} \alpha(i)\beta(i) - \sum_{a \in Q_1} \alpha(ta)\beta(ha). \tag{2}$$

Of course, we can extend this bilinear form to $\mathbb{R}^{Q_0}$. (When no confusion arises, we drop the subscript Q.)

It follows from (1) and (2) that

$$\langle \mathbf{dim} V, \mathbf{dim} W \rangle = \dim_K \operatorname{Hom}_Q(V, W) - \dim_K \operatorname{Ext}^1_Q(V, W). \tag{3}$$

2.1. **Semi-invariants and semi-stable representations.** For a given dimension vector β of Q, the representation space of $\beta-$dimensional representations of Q is defined by

$$\operatorname{rep}(Q, \beta) = \bigoplus_{a \in Q_1} \operatorname{Hom}_K(K^{\beta(ta)}, K^{\beta(ha)}).$$

If $\mathrm{GL}(\beta) = \prod_{i \in Q_0} \mathrm{GL}_{\beta(i)}(K)$ then $\mathrm{GL}(\beta)$ acts algebraically on $\operatorname{rep}(Q, \beta)$ by simultaneous conjugation, i.e., for $g = (g(i))_{i \in Q_0} \in \mathrm{GL}(\beta)$ and $W = (W(a))_{a \in Q_1} \in \operatorname{rep}(Q, \beta)$, we define $g \cdot W$ by

$$(g \cdot W)(a) = g(ha)W(a)g(ta)^{-1}, \forall a \in Q_1.$$

Hence, $\operatorname{rep}(Q, \beta)$ is a rational representation of the linearly reductive group $\mathrm{GL}(\beta)$ and the $\mathrm{GL}(\beta)-$orbits in $\operatorname{rep}(Q, \beta)$ are in one-to-one correspondence with the isomorphism classes of $\beta-$dimensional representations of Q. As Q is a quiver without oriented cycles, one can show that there is only one closed $\mathrm{GL}(\beta)-$orbit in $\operatorname{rep}(Q, \beta)$ and hence the invariant ring $\mathrm{I}(Q, \beta) = K[\operatorname{rep}(Q, \beta)]^{\mathrm{GL}(\beta)}$ is exactly the base field K.

Now, consider the subgroup $\mathrm{SL}(\beta) \subseteq \mathrm{GL}(\beta)$ defined by

$$\mathrm{SL}(\beta) = \prod_{i \in Q_0} \mathrm{SL}_{\beta(i)}(K).$$

Although there are only constant $\mathrm{GL}(\beta)-$invariant polynomial functions on $\operatorname{rep}(Q, \beta)$, the action of $\mathrm{SL}(\beta)$ on $\operatorname{rep}(Q, \beta)$ provides us with a highly non-trivial ring of semi-invariants. Note that any $\sigma \in \mathbb{Z}^{Q_0}$ defines a rational character of $\mathrm{GL}(\beta)$ by

$$(g(i))_{i \in Q_0} \in \mathrm{GL}(\beta) \mapsto \prod_{i \in Q_0} (\det g(i))^{\sigma(i)}.$$

In this way, we can identify $\Gamma := \mathbb{Z}^{Q_0}$ with the group $X^\star(\mathrm{GL}(\beta))$ of rational characters of $\mathrm{GL}(\beta)$, assuming that β is a sincere dimension vector. In general, we have only the natural epimorphism $\Gamma \to X^*(\mathrm{GL}(\beta))$. We also refer to the rational characters of $\mathrm{GL}(\beta)$ as (integral) weights.

Let $\mathrm{SI}(Q, \beta) = K[\operatorname{rep}(Q, \beta)]^{\mathrm{SL}(\beta)}$ be the ring of semi-invariants. As $\mathrm{SL}(\beta)$ is the commutator subgroup of $\mathrm{GL}(\beta)$ and $\mathrm{GL}(\beta)$ is linearly reductive, we have

$$\mathrm{SI}(Q, \beta) = \bigoplus_{\sigma \in X^\star(\mathrm{GL}(\beta))} \mathrm{SI}(Q, \beta)_\sigma,$$

where

$$\mathrm{SI}(Q, \beta)_\sigma = \{f \in K[\operatorname{rep}(Q, \beta)] \mid g \cdot f = \sigma(g) f \text{ for all } g \in \mathrm{GL}(\beta)\}$$

is the space of semi-invariants of weight σ.

In a seminal paper [12], King constructed, via GIT, moduli spaces for finite-dimensional algebras. In what follows, we recall King's main results. Note that the one-dimensional torus

$$T = \{(t \operatorname{Id}_{\beta(i)})_{i \in Q_0} \mid t \in K^*\} \subseteq \mathrm{GL}(\beta)$$

acts trivially on $\mathrm{rep}(Q, \beta)$ and so there is a well-defined action of $\mathrm{PGL}(\beta) = \mathrm{GL}(\beta)/T$ on $\mathrm{rep}(Q, \beta)$.

Definition 2.1. [12, Definition 2.1] Let β be a dimension vector of Q and $\sigma \in \mathbb{Z}^{Q_0}$ an integral weight. A representation $W \in \mathrm{rep}(Q, \beta)$ is said to be:

1. σ-*semi-stable* if there exists a semi-invariant $f \in \mathrm{SI}(Q, \beta)_{m\sigma}$ with $m \geq 1$, such that $f(W) \neq 0$;
2. σ-*stable* if there exists a semi-invariant $f \in \mathrm{SI}(Q, \beta)_{m\sigma}$ with $m \geq 1$, such that $f(W) \neq 0$ and, furthermore, the $\mathrm{GL}(\beta)$-action on the principal open subset defined by f is closed and $\dim \mathrm{GL}(\beta) W = \dim \mathrm{PGL}(\beta)$.

Note that any σ-stable representation is, in particular, a Schur representation. Consider the (possibly empty) open subsets

$$\mathrm{rep}(Q, \beta)^{ss}_{\sigma} = \{W \in \mathrm{rep}(Q, \beta) \mid W \text{ is } \sigma\text{-semi-stable}\}$$

and

$$\mathrm{rep}(Q, \beta)^{s}_{\sigma} = \{W \in \mathrm{rep}(Q, \beta) \mid W \text{ is } \sigma\text{-stable}\}$$

of β-dimensional σ(-semi)-stable representations.

The GIT-quotient of $\mathrm{rep}(Q, \beta)^{ss}_{\sigma}$ by $\mathrm{PGL}(\beta)$ is

$$\mathcal{M}(Q, \beta)^{ss}_{\sigma} = \mathrm{Proj}(\bigoplus_{m \geq 0} \mathrm{SI}(Q, \beta)_{m\sigma}).$$

This is an irreducible projective variety whose closed points parameterize the closed $\mathrm{GL}(\beta)$-orbits in $\mathrm{rep}(Q, \beta)^{ss}_{\sigma}$. For given $\beta, \sigma \in \mathbb{R}^{Q_0}$, we define

$$\sigma(\beta) = \sum_{i \in Q_0} \sigma(i)\beta(i).$$

In [12], King found a representation-theoretic description of the (semi-)stable representations and of the closed orbits in $\mathrm{rep}(Q, \beta)^{ss}_{\sigma}$:

Proposition 2.2. [12, Proposition 3.1, 3.2] *Let β be a (non-zero) dimension vector and σ an integral weight of Q. For a given representation $W \in \mathrm{rep}(Q, \beta)$, the following are true:*

1. *W is σ-semi-stable if and only if $\sigma(\mathbf{dim} W) = 0$ and $\sigma(\mathbf{dim} W') \leq 0$ for every subrepresentation W' of W;*
2. *W is σ-stable if and only if $\sigma(\mathbf{dim} W) = 0$ and $\sigma(\mathbf{dim} W') < 0$ for every proper subrepresentation $0 \neq W' \subsetneq W$;*
3. *$\mathrm{GL}(\beta) W$ is closed in $\mathrm{rep}(Q, \beta)^{ss}_{\sigma}$ if and only if W is a direct sum of σ-stable representations. We call such a representation* σ-poly-stable*.*

Note that we can use this result to define σ-(semi-)stable representations with respect to any real-valued function $\sigma \in \mathbb{R}^{Q_0}$. We say that a dimension vector β is σ*(-semi)-stable* if there exists σ(-semi)-stable representation $W \in \mathrm{rep}(Q, \beta)$.

2.2. **The σ-stable decomposition.** In this section, we recall Derksen and Weyman's [5] notion of σ-stable decomposition of dimension vectors which proves to be a very powerful tool for studying semi-invariants of quivers.

Given a rational-valued function $\sigma \in \mathbb{Q}^{Q_0}$, we define $\operatorname{rep}(Q)^{ss}_{\sigma}$ to be the full subcategory of $\operatorname{rep}(Q)$ consisting of all σ-semi-stable representations, i.e., those representations that satisfy Proposition 2.2(1). Similarly, we define $\operatorname{rep}(Q)^{s}_{\sigma}$ to be the full subcategory of $\operatorname{rep}(Q)$ consisting of all σ-stable representations. (Of course, the zero representation is always semi-stable but not stable.)

It is easy to see that $\operatorname{rep}(Q)^{ss}_{\sigma}$ is a full exact subcategory, closed under extensions, and whose simple objects are precisely the σ-stable representations. Moreover, $\operatorname{rep}(Q)^{ss}_{\sigma}$ is Artinian and Noetherian, and hence, every σ-semi-stable representation has a Jordan-Hölder filtration in $\operatorname{rep}(Q)^{ss}_{\sigma}$.

Let α, β be two dimension vectors. We define

$$\operatorname{ext}_Q(\alpha, \beta) = \min\{\dim_K \operatorname{Ext}^1_Q(V, W) \mid (V, W) \in \operatorname{rep}(Q, \alpha) \times \operatorname{rep}(Q, \beta)\}$$

and

$$\hom_Q(\alpha, \beta) = \min\{\dim_K \operatorname{Hom}_Q(V, W) \mid (V, W) \in \operatorname{rep}(Q, \alpha) \times \operatorname{rep}(Q, \beta)\}.$$

It is not difficult to show that the dimensions of Ext^1_Q and Hom_Q spaces are upper-semi-continuous as functions on $\operatorname{rep}(Q, \alpha) \times \operatorname{rep}(Q, \beta)$. Hence, the above minimal values are attained on open subsets of $\operatorname{rep}(Q, \alpha) \times \operatorname{rep}(Q, \beta)$.

Let β be a (non-zero) σ-semi-stable dimension vector where $\sigma \in \mathbb{Z}^{Q_0}$. We say that

$$\beta = \beta_1 \dotplus \beta_2 \dotplus \ldots \dotplus \beta_s$$

is the *σ-stable decomposition* of β if a general representation in $\operatorname{rep}(Q, \beta)$ has a Jordan-Hölder filtration in $\operatorname{rep}(Q)^{ss}_{\sigma}$ with factors of dimensions $\beta_1, \ldots, \beta_s$ (in some order). We write $c \cdot \beta$ instead of $\beta \dotplus \beta \dotplus \ldots \dotplus \beta$ (c times).

The next proposition gives some basic properties of the dimension vectors occurring in the σ-stable decomposition of a dimension vector. It is essential for proving Proposition 2.8, Schofield's Embedding Theorem 1.2, and Theorem 1.4.

Proposition 2.3. [5, Proposition 3.18] *Let β be a σ-semi-stable dimension vector and let*

$$\beta = c_1 \cdot \beta_1 \dotplus c_2 \cdot \beta_2 \dotplus \ldots \dotplus c_l \cdot \beta_l$$

be the σ-stable decomposition of β with the β_i distinct. Then:

1. *the β_i are Schur roots;*
2. $\hom_Q(\beta_i, \beta_j) = 0$ *for all* $i \neq j$*;*
3. *after rearranging, we can assume that* $\operatorname{ext}_Q(\beta_i, \beta_j) = 0$ *for all* $1 \leq i < j \leq l$.

2.3. **Domains of semi-invariants.** Let α and β be two dimension vectors. We write $\alpha \hookrightarrow \beta$ if every representation of dimension vector β has a subrepresentation of dimension vector α.

Recall that if β is a dimension vector of Q, the domain of semi-invariants associated to (Q, β) is

$$D(\beta) = \{\alpha \in \mathbb{Q}^{Q_0} \mid \langle \alpha, \beta \rangle = 0 \text{ and } \langle \alpha, \beta' \rangle \leq 0 \text{ for all } \beta' \hookrightarrow \beta\}.$$

Remark 2.4. Let β be a dimension vector and $i \in Q_0$. Then, it is easy to see that $\beta_i = 0$ if and only if $\gamma_i \in D(\beta)$ if and only if $-\gamma_i \in D(\beta)$ (see for example [7, Lemma 6.5.6]).

Lemma 2.5. [7, Lemma 6.5.7] *Let $\alpha, \beta \in \mathbb{Z}^{Q_0}$ be two integer valued functions.*

(1) *Assume that β is a sincere dimension vector and $\alpha \in D(\beta)$. Then α is also a dimension vector.*
(2) *Dually, if α is a sincere dimension vector, $\langle \alpha, \beta \rangle = 0$, and $\langle \alpha', \beta \rangle \geq 0$ for all $\alpha' \hookrightarrow \alpha$ then β is also a dimension vector.*

Remark 2.6. Note that the lemma above can also be deduced from [4, Theorem 1].

When β is a sincere dimension vector, a description of the lattice points of $D(\beta)$ in terms of perpendicular categories was obtained independently in [3], [4], and [20]. An extension of this result to the case of arbitrary dimension vectors was obtained by Igusa-Orr-Todorov-Weyman in [7]. For a dimension vector δ, we define

$$P_\delta = \bigoplus_{i \in Q_0} P_i^{\delta(i)}.$$

Now, we can state:

Theorem 2.7. [7] *Let β be a dimension vector of Q and $\alpha \in \mathbb{Z}^{Q_0}$ an integral weight.*

(1) *There are unique dimension vectors α' and δ such that*

$$\alpha = \alpha' - \mathbf{dim} P_\delta \text{ and } \operatorname{supp}(\alpha') \cap \operatorname{supp}(\delta) = \emptyset.$$

Furthermore, in the special case when $\alpha \in \mathbb{Z}_{\geq 0}^{Q_0}$, one has $\alpha = \alpha'$ and $\delta = 0$.
(2) *The following statements are equivalent:*
  (a) $\alpha \in D(\beta)$;
  (b) *there is an α'-dimensional representation V such that*
    (i) $\operatorname{Hom}_Q(V, W) = \operatorname{Ext}^1_Q(V, W) = 0$ *for some (equivalently, a generic) $W \in \operatorname{rep}(Q, \beta)$;*
    (ii) $\operatorname{supp}(\beta) \cap \operatorname{supp}(\delta) = \emptyset$.

Proof. The first part of the theorem is proved in [7, Lemma 5.3.2]. The second part follows from Proposition 5.1.4, Corollary 6.2.2 and Theorem 6.5.11 in [7]. □

If $\alpha \in \mathbb{Z}^{Q_0}$, we define the weight $\sigma = \langle \alpha, \cdot \rangle$ by

$$\sigma(i) = \langle \alpha, \varepsilon_i \rangle, \forall i \in Q_0.$$

Conversely, it is easy to see that for any weight $\sigma \in \mathbb{Z}^{Q_0}$ there is a unique $\alpha \in \mathbb{Z}^{Q_0}$ (not necessarily a dimension vector) such that $\sigma = \langle \alpha, \cdot \rangle$. Similarly, one can define $\mu = \langle \cdot, \alpha \rangle$.

Proposition 2.8. *Let β be a dimension vector and $\sigma \in \mathbb{Z}^{Q_0}$ an integral weight.*

(1) *β is σ-semi-stable if and only if $\sigma(\beta) = 0$ and $\sigma(\beta') \leq 0$ for all $\beta' \hookrightarrow \beta$.*
(2) *β is σ-stable if and only if β is non-zero, $\sigma(\beta) = 0$, and $\sigma(\beta') < 0$ for all $\beta' \hookrightarrow \beta$ and $\beta' \neq 0$ or β.*

Remark 2.9. This result is undoubtedly well-known. The implication " $\Longrightarrow$ " of both (1) and (2) is proved in Proposition 2.2. The implication " $\Longleftarrow$ " of (1) was proved independently in [4] and [20] (for a proof, see [3, Theorem 2.4]) for the case where σ is of the form $\sigma = \langle \alpha, \cdot \rangle$ with α a dimension vector. For the lack of a reference for arbitrary α or for the implication " $\Longleftarrow$ " of (2), we include a proof below.

Proof. Working with the full subquiver of Q whose set of vertices is $\mathrm{supp}(\beta)$ and using Lemma 2.5, we can assume that $\sigma = \langle \alpha, \cdot \rangle$ with α a dimension vector. The proof of (1) now follows from the remark above.

Now, let us prove " $\Longleftarrow$ " of (2). From (1) we know that β is σ-semi-stable and let us consider the σ-stable decomposition of β:

$$\beta = c_1 \cdot \beta_1 \dotplus c_2 \cdot \beta_2 \dotplus \ldots \dotplus c_l \cdot \beta_l,$$

where the β_i satisfies the conditions $(1) - (3)$ of Proposition 2.3. It is clear that

$$\mathrm{ext}_Q(c_1\beta_1, \sum_{2 \leq i \leq l} c_i\beta_i) = 0$$

and hence $c_1\beta_1 \hookrightarrow \beta$ by [19, Theorem 3.2]. If β is not σ-stable then either $\beta' = \beta_1$ (when $l = 1$) or $\beta' = c_1\beta_1$ (when $l \geq 2$) is a proper dimension sub-vector of β with $\beta' \hookrightarrow \beta$ and $\sigma(\beta') = 0$. But this is a contradiction. □

2.4. **Derksen-Weyman Saturation and Reciprocity Properties.** Let α, β be two dimension vectors such that $\langle \alpha, \beta \rangle = 0$. In [18], Schofield discovered some very important semi-invariants of quivers. Consider the following polynomial function

$$c : \mathrm{rep}(Q, \alpha) \times \mathrm{rep}(Q, \beta) \to K$$
$$c(V, W) = \det(d^V_W).$$

Note that d^V_W from Ringel's canonical exact sequence (1) is indeed a square matrix since $\langle \alpha, \beta \rangle = 0$. Fix $(V, W) \in \mathrm{rep}(Q, \alpha) \times \mathrm{rep}(Q, \beta)$. Then it is easy to see that $c^V = c(V, \cdot) : \mathrm{rep}(Q, \beta) \to K$ is a semi-invariant of weight $\langle \alpha, \cdot \rangle$ and $c_W = c(\cdot, W) : \mathrm{rep}(Q, \alpha) \to K$ is a semi-invariant of weight $-\langle \cdot, \beta \rangle$.

Remark 2.10. We should point out that if V is an α-dimensional representation in $\mathrm{rep}(Q, \alpha)$, the semi-invariant c^V is well-defined on $\mathrm{rep}(Q, \beta)$ up to a non-zero scalar.

Remark 2.11. Given $(V, W) \in \mathrm{rep}(Q, \alpha) \times \mathrm{rep}(Q, \beta)$, we have

$$\mathrm{Hom}_Q(V, W) = \mathrm{Ext}^1_Q(V, W) = 0 \Longleftrightarrow d^V_W \text{ is invertible}$$

and this implies that V is $-\langle \cdot, \beta \rangle$-semi-stable and W is $\langle \alpha, \cdot \rangle$-semi-stable.

It is rather easy to see that the Schofield semi-invariants behave nicely with respect to exact sequences. In fact, we have:

Lemma 2.12. [4, Lemma 1] *Let α and β be two dimension vectors such that $\langle \alpha, \beta \rangle = 0$. Let W be a β-dimensional representation which has a filtration*

$$F_\bullet(W) : 0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{l-1} \subsetneq W_l = W,$$

with $\langle \alpha, \mathbf{dim} W_i/W_{i-1} \rangle = 0, \forall 1 \leq i \leq l$. Then $c_W = \prod_{i=1}^{l} c_{W_i/W_{i-1}}$ on $\mathrm{rep}(Q, \alpha)$.

In [4, Theorem 1] (see also [20] and [6]), Derksen and Weyman proved a fundamental result showing that each weight space of semi-invariants is spanned by Schofield semi-invariants. This is known as the *First Fundamental Theorem* for semi-invariants of quivers. Using the FFT, Derksen and Weyman derived some remarkable consequences.

Theorem 2.13 (Saturation Theorem for Quivers)**.** [4, Theorem 2] *Let β be a dimension vector and $\sigma \in \mathbb{Z}^{Q_0}$ a weight. Then the following are equivalent:*

(i) $\dim_K \mathrm{SI}(Q,\beta)_\sigma > 0$;
(ii) $\sigma(\beta) = 0$ *and* $\sigma(\beta') \leq 0$ *for all* $\beta' \hookrightarrow \beta$.

Remark 2.14. It is worth pointing out that when Q is the triple star quiver, the theorem above immediately implies the Saturation Conjecture for Littlewood-Richardson coefficients. See [13] and [4, Corollary 2].

We also have the so-called reciprocity property:

Proposition 2.15 (Reciprocity Property)**.** [4, Corollary 1] *Let α and β be two dimension vectors. Then*

$$\dim_K \mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle} = \dim_K \mathrm{SI}(Q,\alpha)_{-\langle\cdot,\beta\rangle}.$$

For two dimension vectors α and β, we define

$$\alpha \circ \beta = \dim_K \mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle} = \dim_K \mathrm{SI}(Q,\alpha)_{-\langle\cdot,\beta\rangle}.$$

The next lemma is especially useful for proving Theorem 1.2 and Theorem 1.4:

Lemma 2.16. *Assume that α is a dimension vector such that $\mathrm{GL}(\alpha)$ acts with a dense orbit on $\mathrm{rep}(Q,\alpha)$ and let σ be either $\langle\alpha,\cdot\rangle$ or $-\langle\cdot,\alpha\rangle$. If δ is a σ-stable dimension vector then δ is a real Schur root.*

Proof. We know that the dimension vector of any stable representation is a Schur root. We only need to show that $\mathrm{GL}(\delta)$ acts with a dense orbit on $\mathrm{rep}(Q,\delta)$ which is equivalent to showing

$$\dim_K \mathrm{SI}(Q,\delta)_\mu \leq 1$$

for all weights μ of $\mathrm{GL}(\delta)$.

It is well-known (and easy to see) that $\mathrm{GL}(\alpha)$ acts with a dense orbit on $\mathrm{rep}(Q,\alpha)$ if and only if there exists $V \in \mathrm{rep}(Q,\alpha)$ with $\mathrm{Ext}^1_Q(V,V) = 0$. Consequently, $\mathrm{GL}(n\alpha)$ acts with a dense orbit on $\mathrm{rep}(Q,n\alpha)$ for any integer $n > 0$. From this and the Reciprocity Property 2.15, we deduce that

$$\dim_K \mathrm{SI}(Q,\delta)_{n\sigma} = 1,$$

for any integer $n > 0$.

Now, let μ be a weight such that $\mathrm{SI}(Q,\delta)_\mu \neq 0$; in particular, δ is μ-semi-stable. As δ is σ-stable and using Proposition 2.8, we can always find a sufficiently large integer $n > 0$ such that $n\sigma(\delta') - \mu(\delta') \leq 0$ for all $\delta' \hookrightarrow \delta$. But this is equivalent to $\mathrm{SI}(Q,\delta)_{n\sigma-\mu} \neq \{0\}$ by the Saturation Theorem 2.13. Multiplying the semi-invariants in $\mathrm{SI}(Q,\delta)_\mu$ by a fixed non-zero semi-invariant in $\mathrm{SI}(Q,\delta)_{n\sigma-\mu}$, we get an injective linear map from $\mathrm{SI}(Q,\delta)_\mu$ into $\mathrm{SI}(Q,\delta)_{n\sigma}$. Consequently, $\dim_K \mathrm{SI}(Q,\delta)_\mu \leq \dim_K \mathrm{SI}(Q,\delta)_{n\sigma} = 1$, and so, δ is a real Schur root. □

3. Schofield's Embedding Theorem

In this section, we review Schofield's results on perpendicular categories from [18]. We give new proofs of his results by using some of the tools we have already discussed and the A_∞-formalism.

For a given representation V, the right perpendicular category of V, denoted by $V^\perp$, is the full subcategory of representations W such that $\mathrm{Hom}_Q(V,W) = \mathrm{Ext}^1_Q(V,W) = 0$ (we also write $V \perp W$ in this case). Similarly, one defines the left perpendicular category ${}^\perp V$.

Now, let α be a dimension vector. We define $\alpha^{\perp}$ to be the full subcategory consisting of all those representations W with $V \perp W$ for some (or equivalently generic) $V \in \mathrm{rep}(Q, \alpha)$. Similarly, we define ${}^{\perp}\alpha$.

A dimension vector α is said to be *pre-homogeneous* if $\mathrm{GL}(\alpha)$ acts with a dense orbit on the representation space $\mathrm{rep}(Q, \alpha)$. Our goal in this section is to understand the categories $\alpha^{\perp}$ (and ${}^{\perp}\alpha$) when α is a pre-homogeneous dimension vector. For such a dimension vector α, we claim that

$$\alpha^{\perp} = \mathrm{rep}(Q)^{ss}_{\sigma} \text{ and } {}^{\perp}\alpha = \mathrm{rep}(Q)^{ss}_{\mu}$$

where $\sigma = \langle \alpha, \cdot \rangle$ and $\mu = -\langle \cdot, \alpha \rangle$. Indeed, if $V \in \mathrm{rep}(Q, \alpha)$ is a rigid representation then each non-zero weight space of semi-invariants of the form $\mathrm{SI}(Q, \beta)_{m\sigma}$ is one-dimensional being spanned by $(c^V)^m$. Using this observation it is now easy to see that $\alpha^{\perp} = \mathrm{rep}(Q)^{ss}_{\sigma}$. Similarly, one can prove the other claim.

A sequence $\mathcal{E} = (\beta_1, \dots, \beta_l)$ of dimension vectors is said to be a *quiver exceptional sequence* if

(i) each β_i is a real Schur root;
(ii) $\beta_i \perp \beta_j$ for all $1 \leq i < j \leq l$;
(iii) $\langle \beta_j, \beta_i \rangle \leq 0$ for all $1 \leq i < j \leq l$.

If we drop condition (iii), we call $\mathcal{E}$ just an *exceptional sequence*. A sequence $(E_1, \dots, E_l)$ of exceptional representations is said to be a (quiver) exceptional sequence (of representations) if $(\mathbf{dim} E_1, \dots, \mathbf{dim} E_l)$ is a (quiver) exceptional sequence. We say that $\mathcal{E}$ is *complete* if $l = |Q_0|$.

Remark 3.1. Let σ be an integral weight and β a non-zero σ-semi-stable dimension vector. It follows from Proposition 2.3 that the Schur roots occurring in the σ-stable decomposition of β form, possible after reordering, a quiver exceptional sequence.

Remark 3.2. Let $(\beta_1, \dots, \beta_l)$ be a quiver exceptional sequence. We claim that $\hom_Q(\beta_i, \beta_j)$ is zero for all $i \neq j$. This is clearly true for $i < j$. Since β_i and β_j are Schur roots and $\mathrm{ext}_Q(\beta_i, \beta_j) = 0$ for $i < j$, we know that either $\hom_Q(\beta_j, \beta_i) = 0$ or $\mathrm{ext}_Q(\beta_j, \beta_i) = 0$ by [19, Theorem 4.1]. From this and the fact that $\langle \beta_j, \beta_i \rangle \leq 0$, we finally deduce that $\hom_Q(\beta_j, \beta_i) = 0$ for $i < j$. In particular, $\mathrm{ext}_Q(\beta_i, \beta_j) = -\langle \beta_i, \beta_j \rangle$ for all $1 \leq i \neq j \leq l$. Moreover, the matrix $(\langle \beta_i, \beta_j \rangle)_{i,j}$ is lower triangular with ones on the diagonal, and hence, the β_i are linearly independent over $\mathbb{R}$.

Now, we are ready to give a new proof of Schofield's Embedding Theorem from [18, Theorem 2.5].

Proof of Theorem 1.2. We prove the theorem for $\sigma = \langle \alpha, \cdot \rangle$. The case where $\sigma = -\langle \cdot, \alpha \rangle$ is completely analogous.

(1) We know from Lemma 2.16 that every σ-stable representation is exceptional and that in each dimension vector there is at most one σ-stable representation up to isomorphism. Next, we claim that if $E_1, \dots, E_m$ are pairwise non-isomorphic σ-stable representations, their dimension vectors $\mathbf{dim} E_i$ must be linearly independent over $\mathbb{Z}$. Assume to the contrary that there are integers $k_l \in \mathbb{Z}_{>0}$ such that

$$\beta = \sum_{i \in I} k_i \mathbf{dim} E_i = \sum_{j \in J} k_j \mathbf{dim} E_j,$$

with $I \bigcap J = \emptyset$. Set $E_I = \oplus_{i \in I} E_i$ and $E_J = \oplus_{j \in J} E_j$ and note that E_I and E_J are σ-poly-stable representations of the same dimension β. Since α is pre-homogeneous, we know that any of its positive integer multiples is pre-homogeneous, and hence, $\dim_K \mathrm{SI}(Q, l\alpha)_{-\langle \cdot, \beta \rangle} = 1$ for all integers $l \geq 0$. By the Reciprocity Property 2.15, this is equivalent to $\dim_K \mathrm{SI}(Q, \beta)_{l\sigma} = 1$, and so, the moduli space $\mathcal{M}(Q, \beta)^{ss}_{\sigma}$ is just a point. From Proposition 2.2(3) it follows that $E_I \cong E_J$ which is a contradiction. The first part of the theorem now follows.

(2) Since the β_i are linearly independent, we deduce that

$$\beta_0 = \sum_{i=1}^{l} \beta_i$$

is the σ-stable decomposition of β_0. It follows from Proposition 2.3 that after rearranging $\mathcal{E} = (\beta_1, \dots, \beta_l)$ is a quiver exceptional sequence.

(3) Let $\mathrm{filt}(\mathcal{E})$ be the full subcategory of $\mathrm{rep}(Q)$ whose objects have a finite filtration with factors among the E_i. We clearly have that $\mathrm{filt}(\mathcal{E}) = \mathrm{rep}(Q)^{ss}_{\sigma}$. Using the A_∞-formalism, Keller [10, Section 2.3] (see also [11, Section 7.7]) proved that $\mathrm{filt}(\mathcal{E})$ is determined by the Yoneda algebra $\mathrm{Ext}^*_Q(\bigoplus_{i=1}^{l} E_i, \bigoplus_{i=1}^{l} E_i)$ equipped with its A_∞-algebra structure. More precisely, let $\mathcal{A}$ be the A_∞-category with objects $X_1, \dots, X_l$ and morphism spaces $\mathrm{Hom}^*_{\mathcal{A}}(X_i, X_j) = \mathrm{Ext}^*_Q(E_i, E_j)$ and let $\mathrm{twist}(\mathcal{A})$ be the category of twisted stalks over $\mathcal{A}$. Since the are no higher Ext^i_Q spaces (with $i \geq 2$) over path algebras, the objects of $\mathrm{twist}(\mathcal{A})$ can be described as pairs (X, δ), where $X = (\{X_i\}_{1 \leq i \leq l}, \{V_i\}_{1 \leq i \leq l})$, formally written as $X = \bigoplus_{i=1}^{l} V_i \otimes X_i$ with the V_i finite dimensional vector spaces, called multiplicity spaces, and $\delta = (\delta_{ji})_{1 \leq j, i \leq l}$ is a matrix of morphisms $\delta_{ji} \in \mathrm{Hom}_K(V_i, V_j) \otimes \mathrm{Hom}^1_{\mathcal{A}}(X_i, X_j)$. (Note that δ is an upper triangular matrix and it satisfies the Maurer-Cartan equation.) Using the fact that $\mathrm{Hom}^0_{\mathcal{A}}(X_i, X_j) = \{0\}$ for $i \neq j$ and that it is just the base field K when $i = j$, it is easy to see that $\mathrm{twist}(\mathcal{A}) = \mathrm{rep}(Q(\mathcal{E}))$. From [11, Proposition 2.3] we get the desired equivalence of categories. The fact that the map I preserves the Euler forms of Q and $Q(\mathcal{E})$ follows immediately from formula (3). $\square$

Remark 3.3. It follows from the proof above that l is at most $|Q_0|$ minus the number r of non-isomorphic indecomposable direct summands of the generic α-dimensional representation. In fact, Schofield showed in [18] that $l = |Q_0| - r$.

Remark 3.4. Let $\mathcal{E} = (E_1, \dots, E_l)$ be a quiver exceptional sequence and let $\mathrm{filt}(\mathcal{E})$ be the full subcategory of $\mathrm{rep}(Q)$ whose objects have a filtration with factors among the E_i. It is now clear that Theorem 1.2(3) remains true for $\mathcal{E}$ (see also [5, Theorem 2.39]).

The next theorem, which is the main result of [18, Theorem 4.3], provides us with algebraically independent generators of the algebra of semi-invariants $\mathrm{SI}(Q, \alpha)$ for the case where α is pre-homogeneous. Although it is not needed for our direct purposes, we include a new proof for completeness:

Theorem 3.5. *Let α be a sincere pre-homogeneous dimension vector. If $E_1, \dots, E_l$ are the pairwise non-isomorphic $\langle \alpha, \cdot \rangle$-stable representations then the Schofield semi-invariants c_{E_i} are algebraically independent and*

$$\mathrm{SI}(Q, \alpha) = K[c_{E_1}, \dots, c_{E_l}].$$

Remark 3.6. Schofield's original proof of the theorem above uses Luna's [14] étale slice machinery and Sato-Kimura [17] classification of pre-homogeneous vector spaces. Our proof is much simpler in nature and it follows immediately from Theorem 1.2 and some of the basic properties of the Schofield semi-invariants.

Proof. Note that the weights $-\langle \cdot, \mathbf{dim} E_i\rangle$ of the semi-invariants c_{E_i} are linearly independent over $\mathbb{Z}$ by Theorem 1.2(2) and Remark 3.2. Therefore, these semi-invariants are algebraically independent. (To conclude this, we need the assumption that α is sincere.)

It remains to show that each non-zero weight-space $\mathrm{SI}(Q,\alpha)_\mu$ is spanned by a monomial in the c_{E_i}. Since α is sincere, we know that $\mu = -\langle \cdot, \beta\rangle$ with β a dimension vector by Lemma 2.5. Now, it easy to see that $\mathrm{SI}(Q,\alpha)_\mu$ is spanned by a semi-invariant of the form c_W with W a σ-semi-stable β-dimensional representation where $\sigma = \langle \alpha, \cdot\rangle$. Consider a Jordan-Hölder filtration of W

$$F_\bullet(W) : 0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{l-1} \subsetneq W_l = W,$$

with W_i/W_{i-1} one of the σ-stable representations E_j. Using Lemma 2.12, we can write

$$c_W = \prod_{i=1}^{l} c_{W_i/W_{i-1}},$$

and this finishes the proof. □

4. Cluster fans and cones of finite-stability conditions

In this section, we first recall the construction of the cluster fan of a quiver Q without oriented cycles. Recall that the set of almost positive real Schur roots is

$$\Psi(Q)_{\geq -1} = \{\beta \mid \beta \text{ is a real Schur root of } Q\} \cup \{-\gamma_i \mid i \in Q_0\}.$$

We should point out that in general the set of all real Schur roots depends on the orientation of Q.

To construct the (possibly infinite) cluster fan $C(Q)$ on the ground set $\Psi(Q)_{\geq -1}$, we need some definitions first. If $\beta_1, \beta_2 \in \Psi(Q)_{\geq -1}$, their compatibility degree is defined by

$$(\beta_1 \| \beta_2)_Q = \begin{cases} \mathrm{ext}_Q(\beta_1,\beta_2) + \mathrm{ext}_Q(\beta_2,\beta_1) & \text{if the } \beta_i \text{ are real Schur roots} \\ \beta(i) & \text{if } \{\beta_1,\beta_2\} = \{\beta, -\gamma_i\} \text{ with } \beta \text{ a real Schur root} \\ 0 & \text{otherwise} \end{cases}$$

A subset $C \subseteq \Psi(Q)_{\geq -1}$ is said to be compatible if $(\beta_1\|\beta_2)_Q = 0$ for all $\beta_1, \beta_2 \in C$. A maximal (with respect to inclusion) compatible set is called a *cluster*.

Remark 4.1. Note that the compatibility degrees are dimensions of Ext spaces between indecomposable objects in the cluster category $\mathcal{C}_Q$ associated to Q (see [1]). Now, let ϕ be the bijection that sends $-\gamma_i$ to $-\varepsilon_i$ and is the identity map on the set of real Schur roots. Then $C \subseteq \Psi(Q)_{\geq -1}$ is compatible if and only if $\phi(C)$ is compatible in the sense of [15], assuming Q is a Dynkin quiver.

Remark 4.2. It is not difficult to see that any compatible subset C of $\Psi(Q)_{\geq -1}$ is linearly independent over $\mathbb{R}$. Indeed, write $C = \{-\gamma_{i_1}, \ldots, \gamma_{i_l}, \beta_{l+1}, \ldots, \beta_n\}$ and assume, without loss of generality, that $\gamma_{i_j} \perp \gamma_{i_k}$ for all $1 \leq j < k \leq l$. (This is always possible since Q has no oriented cycles.) Next, using [19, Theorem 2.4], we can rearrange the β_j so that $\hom_Q(\beta_m, \beta_p) = 0$ for all $l+1 \leq m < p \leq n$. Now, let $\alpha_j = \gamma_{i_j}$ for $1 \leq j \leq l$, and $\alpha_j = \beta_j$ for $l+1 \leq j \leq n$. Note that the matrix $(\langle \alpha_i, \alpha_j\rangle)_{i,j}$ is lower triangular with ones on the diagonal, and hence, is invertible. Consequently, the elements of C are linearly independent over $\mathbb{R}$; in particular, C must have at most $|Q_0|$ linearly independent elements.

For $C \subseteq \mathbb{Q}^N$ a finite set of points, let $\mathrm{Cone}(C)$ be the rational convex polyhedral cone (in $\mathbb{Q}^N$) generated by C, i.e., $\mathrm{Cone}(C) = \{x \in \mathbb{Q}^N \mid x = \sum_{c \in C} \lambda_c c \text{ with } \lambda_c \in \mathbb{Q}_{\geq 0}\}$. Next, let us record the following well-known result which can be easily proved using basic results from tilting theory:

Theorem 4.3. *Let Q be a quiver with N vertices. Then the collection of cones*

$$C(Q) = \{\mathrm{Cone}(C) \mid C \text{ is a compatible subset of } \Psi(Q)_{\geq -1}\}$$

is a smooth fan of pure dimension N. We call $C(Q)$ the cluster fan *of Q.*

Our goal is to give a more geometric interpretation of the cluster fan of Q. Recall that the cone (not necessarily convex) of finite stability conditions is the set of all $\sigma \in \mathbb{Q}^{Q_0}$ for which there are finitely many (possibly none) σ-stable representations up to isomorphism.

Proposition 4.4. *Let $\sigma = \langle \alpha, \cdot \rangle \in \mathbb{Z}^{Q_0}$ be a weight with $\alpha \in \mathbb{Z}^{Q_0}$. The following are equivalent:*

(1) $\sigma \in \mathcal{S}(Q)$;
(2) $\alpha \in \mathrm{Cone}(C)$ *for some compatible subset C of $\Psi(Q)_{\geq -1}$.*

Proof. First, let us prove the implication " $\Longrightarrow$ ". Let $\beta_1, \dots, \beta_l$ be the σ-stable dimension vectors. We clearly have that β_0 is σ-semi-stable where $\beta_0 = \sum_{i=1}^{l} \beta_i$. (In case there are no σ-stable dimension vectors, we set $\beta_0 = 0$). From Theorem 2.7, we know

$$\alpha = \alpha' - \mathbf{dim} P_\delta,$$

where $\mathrm{supp}(\alpha') \cap \mathrm{supp}(\delta) = \mathrm{supp}(\beta_0) \cap \mathrm{supp}(\delta) = \emptyset$, $\alpha' \in D(\beta_0)$. Let Q' be the full subquiver of Q with $Q'_0 = \mathrm{supp}(\alpha')$ and denote by the same letter the restriction of α' to Q'.

Since there are only finitely many σ-stable representations, we know that there are only finitely many σ-polystable representations in each dimension vector. In other words, each moduli space $\mathcal{M}(Q, \beta)^{ss}_{\sigma}$ is either empty or a point and so

$$\dim_K \mathrm{SI}(Q, \beta)_\sigma \leq 1, \tag{4}$$

for each dimension vector β.

Let β' be a dimension vector of Q' and extend it trivially to a dimension vector β of Q. Denote the dimension vector of P_δ by α''. From Remark 2.4 and the fact that β and δ have disjoint supports, we deduce that $-\alpha'', \alpha'' \in D(\beta)$ which is equivalent to β being semi-stable with respect to both $\langle \alpha'', \cdot \rangle$ and $-\langle \alpha'', \cdot \rangle$ by Proposition 2.8. From this observation and (4), one can easily see that $(\alpha' \circ \beta')_{Q'} = (\alpha' \circ \beta)_Q \leq 1$. As α' is a sincere dimension vector of Q', an effective weight of $\mathrm{GL}(\alpha')$ is of the form $-\langle \cdot, \beta' \rangle$ with β' some dimension vector of Q' by Lemma 2.5. This shows that

$$\dim_K \mathrm{SI}(Q', \alpha')_{\sigma'} \leq 1,$$

for all weights σ' of Q', and consequently, $\mathrm{GL}(\alpha')$ acts with a dense orbit on $\mathrm{rep}(Q', \alpha')$. So, the dimension vectors of the indecomposable direct summands of the generic α'-dimensional representation of Q' form a compatible subset of $\Psi(Q')_{\geq -1}$. The (trivial) extension of this compatible subset to a compatible subset of $\Psi(Q)_{\geq -1}$ together with $\{-\gamma_i \mid i \in \mathrm{supp}(\delta)\}$ form a compatible subset $C \subseteq \Psi(Q)_{\geq -1}$ with $\alpha \in \mathrm{Cone}\, C$.

To prove the other implication " $\Longleftarrow$ ", let us assume that

$$n\alpha = \sum_{j=1}^{l} \eta(j)\beta_j - \mathbf{dim} P_\delta,$$

where n and the $\eta(j) > 0$ are positive integers and $\{\beta_1, \ldots, \beta_l\} \bigcup \{-\gamma_i \mid i \in \mathrm{supp}(\delta)\}$ is a compatible subset of $\Psi(Q)_{\geq -1}$. Denote $\sum_{j=1}^{l} \eta(j)\beta_j$ by α' and $\mathbf{dim} P_\delta$ by α''. Note that α' and δ have disjoint supports, and $\mathrm{GL}(\alpha')$ acts with a dense orbit on $\mathrm{rep}(Q, \alpha')$.

Now, let β be a σ-stable dimension vector. Since this is equivalent to β being $n\sigma$-stable, we can assume without loss of generality that $n = 1$. From Theorem 2.7, we know that $\alpha' \in D(\beta)$ and $\mathrm{supp}(\beta) \cap \mathrm{supp}(\delta) = \emptyset$. In particular, we have $\langle \alpha', \beta' \rangle = \langle \alpha, \beta' \rangle$ for all $\beta' \leq \beta$ (coordinatewise).

Using Proposition 2.8, we deduce that β is $\langle \alpha', \cdot \rangle$-stable. As $\mathrm{GL}(\alpha')$ acts with a dense orbit on $\mathrm{rep}(Q, \alpha')$ we know that there are only finitely many $\langle \alpha', \cdot \rangle$-stable dimension vectors by Theorem 1.2. This finishes the proof. $\square$

Remark 4.5. The last part of the proof above together with Lemma 2.16 shows that if β is $\langle \alpha, \cdot \rangle$-stable with $\alpha \in \mathcal{S}(Q)$ then β is a real Schur root.

Proof of Theorem 1.1. It now follows from Theorem 4.3 and Proposition 4.4. $\square$

The cone $\widetilde{\mathcal{S}}(Q)$ of *effective finite stability conditions* of Q is, by definition, the set of all $\sigma \in \mathbb{Q}^{Q_0}$ for which there exists at least one, but finitely many σ-stable representations up to isomorphism.

Theorem 4.6. *Let Q be a quiver with N vertices. Then the cone $\widetilde{\mathcal{S}}(Q)$ has a fan covering given by the dual of the $(N-1)$-skeleton of the cluster fan of Q.*

Proof. It follows from Remark 3.3 that $\alpha \in \mathrm{relint}(\mathrm{Cone}(C))$ for some cluster C if and only if there are no $\langle \alpha, \cdot \rangle$-stable representations. From this observation and Theorem 1.1 we obtain the desired result. $\square$

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. From Theorem 4.6 and Remark 4.5, we deduce that

$$\bigcup_C \mathrm{Cone}(C) = \{\alpha \in \mathbb{Q}^{Q_0} \mid \langle \alpha, \cdot \rangle \in \widetilde{\mathcal{S}}(Q)\} \subseteq \bigcup_\beta D(\beta), \tag{5}$$

where the union on the left is over all compatible sets C with (at most) $|Q_0| - 1$ elements while the union of the right is over all real Schur roots β.

Now, let us prove the implication " $\Longrightarrow$". First, let us look into the case where Q is a Dynkin quiver. If $\alpha \in D(\beta)$ then there is at least one $\langle \alpha, \cdot \rangle$-stable representation, and furthermore, there can be only finitely many, up to isomorphism, stable representations as Q is a Dynkin quiver. So, the inclusion in (5) is an equality for Dynkin quivers.

Next, let us assume that Q is a generalized Kronecker quiver. Pick an $\alpha \in D(\beta) \cap \mathbb{Z}^{Q_0}$ where β is a real Schur root. From Theorem 2.7, we know that

$$\alpha = \alpha' - \mathbf{dim} P_\delta,$$

where α' and δ are dimension vectors such that $\mathrm{supp}(\alpha') \cap \mathrm{supp}(\delta) = \mathrm{supp}(\beta) \cap \mathrm{supp}(\delta) = \emptyset$ and $\alpha' \in D(\beta)$.

If δ is the zero dimension vector then $\alpha = \alpha'$ is a $-\langle \cdot, \beta \rangle$-semi-stable dimension vector. Looking at the $-\langle \cdot, \beta \rangle$-stable decomposition of α and using Lemma 2.16, we see that the Schur roots that occur in this decomposition of α are real Schur roots. Since the space of all vectors $\alpha'' \in \mathbb{Q}^{Q_0}$ with $\langle \alpha'', \beta \rangle = 0$ is one dimensional, we deduce that α is just a positive integer multiple of a real Schur root, i.e., $\alpha \in \mathrm{Cone}(C)$ with C a compatible subset with one element.

If δ is not the zero dimension vector then β is just one of the two simple roots while α' must be the zero dimension vector. So, the inclusion in (5) is an equality for generalized Kronecker quivers, as well.

For the other implication " $\Longleftarrow$", let W be an exceptional representation and V a representation such that $V \perp W$; in particular, $\mathbf{dim} V \in D(\mathbf{dim} W)$. It follows from Theorem 2.7(1) that

$$n \mathbf{dim} V = \sum_{j=1}^{l} k_j \beta_j,$$

where $n \geq 1$ is an integer, the k_j are non-negative integers, and $\{\beta_1, \ldots, \beta_l\}$ is a compatible subset of $\Psi(Q)_{\geq -1}$. This implies that $\langle \mathbf{dim} V, \mathbf{dim} V \rangle > 0$ for all non-zero representations V with $V \perp W$. Consequently, if W is the indecomposable injective representation at some vertex i, we get that the quiver $Q \setminus \{i\}$ is a (union of) Dynkin quivers. Hence, Q is either a Dynkin, or a generalized Kronecker quiver, or a Euclidean quiver with at least three vertices. In what follows, we show that the last case cannot occur.

Assume to the contrary that Q is a Euclidean quiver with at least three vertices. Denote by δ_Q the isotropic Schur root of Q and choose a vertex i such that $Q \setminus \{i\}$ is a Dynkin quiver. Without loss of generality, let us assume that i is a source. For $\beta_1 = \delta_Q - \varepsilon_i$ and $\beta_2 = \varepsilon_i$, we can see that $\mathcal{E} = (\beta_1, \beta_2)$ is a quiver exceptional sequence with $\langle \beta_2, \beta_1 \rangle = -2$. Hence, $Q(\mathcal{E})$ is the Kronecker quiver:

$$\cdot \Longleftarrow \cdot$$

Since $\mathcal{E}$ is not a complete exceptional sequence, we can always find a real Schur root β such that $\beta_1, \beta_2 \in D(\beta)$. Indeed, this follows from the extension theorem for exceptional sequences due to Crawley-Boevey [2]. Next, using Remark 3.4, we deduce that the Tits quadratic form of $Q(\mathcal{E})$ is weakly positive definite which is a contradiction. $\square$

We end this section with some observations about the cluster fan and the GIT-classes of a quiver Q. Given two weights $\sigma_1, \sigma_2 \in \mathbb{Q}^{Q_0}$, we say that they are *GIT-equivalent* if

$$\mathrm{rep}(Q)^{ss}_{\sigma_1} = \mathrm{rep}(Q)^{ss}_{\sigma_2}.$$

The *GIT-class of a weight* $\sigma \in \mathbb{Q}^{Q_0}$, denoted by $\langle \sigma \rangle$, is

$$\langle \sigma \rangle = \{\sigma' \in \mathbb{Q}^{Q_0} \mid \mathrm{rep}(Q)^{ss}_{\sigma} = \mathrm{rep}(Q)^{ss}_{\sigma'}\}.$$

Now, let $C = \{\beta_j \mid 1 \leq j \leq l\} \cup \{-\gamma_{i_k} \mid l+1 \leq k \leq m\}$ be a compatible subset of $\Psi(Q)_{\geq -1}$ and pick $\alpha = \sum_{j=1}^{l} \eta(j)\beta_j - \sum_{k=l+1}^{m} c_k \gamma_{i_k} \in \mathrm{Cone}(C)$ with $\eta(j)$ and c_k positive integers. Denote $\langle \alpha, \cdot \rangle$ by σ. It follows from the proof of Theorem 1.1 that

$$\mathrm{rep}(Q)^{ss}_{\sigma} = \{W \in \mathrm{rep}(Q) \mid \beta_j \perp W, 1 \leq j \leq l, \text{ and } W(i_k) = 0, l+1 \leq k \leq m\}.$$

(Here, by $\beta \perp W$, we simply mean that $U \perp W$ for some β-dimensional representation U.) Let $\alpha_C = \sum_{j=1}^{l} \beta_j - \sum_{k=l+1}^{m} \gamma_{i_k}$ and $\sigma_C = \langle \alpha_C, \cdot \rangle$. It is easy so see that

$$\mathrm{relint}(\mathrm{I}(\mathrm{Cone}(C))) \subseteq \langle \sigma_C \rangle.$$

It is clear that the inclusion above is strict whenever C is a cluster. In fact, if C is a cluster then $\mathrm{rep}(Q)^{ss}_{\sigma_C}$ consists of only the zero representation, and so, we have

$$\langle \sigma_C \rangle = \bigcup_{C'} \mathrm{relint}(\mathrm{I}(\mathrm{Cone}(C'))),$$

where the union on the right is over all clusters C'. That is to say, the clusters form one single GIT-class.

Now, let assume that Q has at least three vertices and let β be a real Schur root of Q. Using Theorem 1.2, we can always find compatible sets C_1 and C_2, each consisting of $|Q_0| - 1$ real Schur roots of Q, such that the $\alpha_i := \sum_{\alpha \in C_i} \alpha$, $1 \leq i \leq 2$, are two distinct prehomogeneous dimension vectors for ${}^{\perp}\beta$. Denote $\langle \alpha_i, \cdot \rangle$ by σ_i, $1 \leq i \leq 2$. Using Theorem 1.2 again, we see that σ_1 and σ_2 are GIT-equivalent since the σ_i-stable representations are precisely the β-dimensional exceptional representations for each $i \in \{1, 2\}$. We conclude that

$$\mathrm{relint}(\mathrm{I}(\mathrm{Cone}(C_1))) \cup \mathrm{relint}(\mathrm{I}(\mathrm{Cone}(C_2))) \subseteq \langle \sigma_1 \rangle = \langle \sigma_2 \rangle.$$

So, the GIT-equivalence relation does not distinguish among the relative interiors of the cones generated by the compatible subsets of $\Psi(Q)_{\geq -1}$. Nonetheless, it would be interesting to find a (geometric) equivalence relation on $\mathcal{S}(Q)$ such that equivalence classes are precisely the relative interiors of the cones $\mathrm{I}(\mathrm{Cone}(C))$ with C compatible subsets of $\Psi(Q)_{\geq -1}$.

5. Domains of semi-invariants and quiver exceptional sets

Our goal in this section is to find an extension of [7, Theorem 8.1.7] to arbitrary quivers by keeping the domains of semi-invariants in our attention. For this, we need to work with (quiver) exceptional sets instead of compatible sets.

Let

$$\mathcal{E} = \{\beta_1, \ldots, \beta_l, -\gamma_{i_{l+1}}, \ldots, -\gamma_{i_m}\}$$

be a subset of $\Psi(Q)_{\geq -1}$. We say that $\mathcal{E}$ is a *(quiver) exceptional set* if

(1) $\beta_j(i_k) = 0$, $1 \leq j \leq l$, $l+1 \leq k \leq m$,
(2) the β_j can be rearranged so that $(\beta_1, \ldots, \beta_l)$ is a (quiver) exceptional sequence.

Proof of Theorem 1.4. Let us denote $-\langle \cdot, \beta \rangle$ by σ. We know that there are only finitely many σ-stable dimension vectors and they are real Schur roots by Theorem 1.2. Let $\mathcal{E}_1, \ldots, \mathcal{E}_m$ be all quiver exceptional sets such that each of them consists of σ-stable dimension vectors and integral vectors of the form $-\gamma_i$ with $i \notin \mathrm{supp}(\beta)$. By construction, we have

$$\bigcup_{1 \le i \le m} \mathrm{Cone}(\mathcal{E}_i) \subseteq D(\beta).$$

Note that the size of each of the $\mathcal{E}_i$ is at most $|Q_0| - 1$.

Now, let us prove the other inclusion. Pick $\alpha \in D(\beta) \bigcap \mathbb{Z}^{Q_0}$. From Theorem 2.7, we know that there are dimension vectors α', δ such that α' is σ-semi-stable, $\alpha = \alpha' - \mathbf{dim} P_\delta$, and $\mathrm{supp}(\beta) \cap \mathrm{supp}(\delta) = \mathrm{supp}(\alpha') \cap \mathrm{supp}(\delta) = \emptyset$. If $\mathrm{supp}(\delta) = \{i_{l+1} \dots, i_m\}$ then each γ_{i_k} is in $D(\beta)$ as $\beta(i_k) = 0$, and of course, $\mathbf{dim} P_\delta$ is a nonnegative linear combination of the γ_{i_k}.

Now, consider the σ-stable decomposition of α':

$$\alpha' = c_1 \cdot \beta_1 \dotplus \cdots \dotplus c_l \cdot \beta_l,$$

where the β_i are distinct σ-stable dimension vectors and the c_i are positive integers. From Lemma 2.16, it follows that the β_i are real Schur roots. Moreover, we know that after rearranging $(\beta_1, \dots, \beta_l)$ is a quiver exceptional sequence by Proposition 2.3. Consequently, the set $\mathcal{E} = \{\beta_1, \dots, \beta_l, -\gamma_{i_{l+1}}, \dots, -\gamma_{i_m}\}$ is one of the $\mathcal{E}_i$, and furthermore, $\alpha \in \mathrm{Cone}(\mathcal{E}) \subseteq D(\beta)$. This finishes the first part of our theorem.

To prove the last part, let $\mathcal{E} = \{\alpha_1, \dots, \alpha_l\}$ be a quiver exceptional set with $l \le N - 1$. If $\alpha_k = -\gamma_{i_k}$ for all $1 \le k \le l$ then one can choose β to be the simple root corresponding to some vertex $i \in Q_0 \setminus \{i_1, \dots, i_l\}$. For such β, we clearly have $\mathcal{E} \subseteq D(\beta)$.

Now, let assume that

$$\mathcal{E} = \{\alpha_1 = \beta_1, \dots, \alpha_l = \beta_l, \alpha_{l+1} = -\gamma_{i_{l+1}}, \dots, \alpha_m = -\gamma_{i_m}, \},$$

with $1 \le l \le m \le n-1$. Then we can rearrange the β_i so that $(\beta_1, \dots, \beta_l)$ is an exceptional sequence for $\widetilde{Q} = Q \setminus \{i_{l+1}, \dots, i_m\}$. From the extension theorem for exceptional sequences due to Crawley-Boevey [2], we know that there exists a real Schur root β of $\widetilde{Q}$ such that $\beta_i \in D_{\widetilde{Q}}(\beta)$. Extend β (trivially) to a real Schur root of Q. Then, $\gamma_{i_k} \in D(\beta)$ as $i_k \notin \mathrm{supp}(\beta)$, and so, $\mathcal{E} \subseteq D(\beta)$. □

Remark 5.1. We should point out that in case β is a sincere dimension vector, it follows from Theorem 1.2 and Lemma 2.5 that $D(\beta) = \mathrm{Cone}(\mathcal{E})$ where $\mathcal{E}$ is the quiver exceptional set consisting of all $-\langle \cdot, \beta \rangle$-stable dimension vectors. However, this fails when β is not sincere. Indeed, if $\beta(i) = 0$ for some $i \in Q_0$, the cone $D(\beta)$ is not strongly convex as it contains both γ_i and $-\gamma_i$. So, $D(\beta)$ cannot even be simplicial in the non-sincere case.

Remark 5.2. We would like to point out that Theorem 1.4 remains true if instead of quiver exceptional sets we work with just exceptional sets.

Let $\mathcal{E} = \{\alpha_1 = \beta_1, \dots, \alpha_l = \beta_l, \alpha_{l+1} = -\gamma_{i_{l+1}}, \dots, \alpha_m = -\gamma_{i_m}\}$ be a quiver exceptional set. Define $Q(\mathcal{E})$ to be the quiver with vertices $1, \dots, l$, and with $-\langle \beta_i, \beta_j \rangle$ arrows from vertex i to vertex j for all $1 \le i \ne j \le l$. Recall that for a quiver exceptional set $\mathcal{E}$, $\mathrm{ext}(\beta_i, \beta_j) = -\langle \beta_i, \beta_j \rangle$ for all $1 \le i \ne j \le l$. For this reason, we also call $Q(\mathcal{E})$ the Ext-quiver of $\mathcal{E}$.

We call a quiver exceptional set *representation-finite* if $Q(\mathcal{E})$ is a (union of) Dynkin quivers.

Example 5.3. Let us give some examples of quiver exceptional sets whose Ext-quivers are easy to describe.

- (Dynkin case) Let Q be a Dynkin quiver and $\mathcal{E}$ a quiver exceptional set. From Theorem 1.2(3) and formula (3), we deduce that the Tits quadratic form of $Q(\mathcal{E})$ is weakly positive and hence $Q(\mathcal{E})$ is also a Dynkin quiver.
- (Euclidean case) Let Q be a Euclidean quiver and denote by δ_Q the isotropic Schur root of Q. Choose i to be a vertex such that $Q \setminus \{i\}$ is a Dynkin quiver. Without loss of generality, let us assume that i is a source. In this case, we take $\beta_1 = \delta_Q - \varepsilon_i$ and $\beta_2 = \varepsilon_i$. Then, the set $\mathcal{E} = \{\beta_1, \beta_2\}$ is a quiver exceptional set with $\langle \beta_2, \beta_1 \rangle = -2$. Hence, $Q(\mathcal{E})$ is the Kronecker quiver:

$$\begin{tikzcd} \bullet & \bullet \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

- (Wild case) Let $T_{4,3,4}$ be the wild star quiver with the following orientation:

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] & \bullet \arrow[l] & \\
 & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[ul] \arrow[dl] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l] &
\end{tikzcd}$$

Let us consider the exceptional set $\mathcal{E} = \{\beta_1, \beta_2\}$ of $T_{4,3,4}$ given by:

$$\beta_1 = \begin{array}{cccc} 1 & 2 & 3 & \\ & 0 & 3 & 4, \\ 1 & 2 & 3 & \end{array}$$

and

$$\beta_2 = \begin{array}{cccc} 0 & 0 & 0 & \\ & 1 & 0 & 0. \\ 0 & 0 & 0 & \end{array}$$

Since $\langle \beta_2, \beta_1 \rangle = -3$, $T_{4,3,4}(\mathcal{E})$ is the generalized Kronecker quiver:

$$\begin{tikzcd} \bullet & \bullet \arrow[l, shift left] \arrow[l] \arrow[l, shift right] \end{tikzcd}$$

Next, we compare quiver exceptional sets with clusters:

Proposition 5.4. *If $\mathcal{E}$ is a representation-finite quiver exceptional set then there are finitely many compatible sets $C_1, \ldots, C_r$ such that*

$$\mathrm{Cone}(\mathcal{E}) = \bigcup_{i=1}^{r} \mathrm{Cone}(C_i).$$

Proof. Write $\mathcal{E} = \{\alpha_1 = \beta_1, \ldots, \alpha_l = \beta_l, \alpha_{l+1} = -\gamma_{i_{l+1}}, \ldots, \alpha_m = -\gamma_{i_m}\}$ and let

$$\alpha = \sum_{j=1}^{l} \eta(j)\beta_j - \sum_{k=l+1}^{m} c_k \gamma_{i_k} \in \mathrm{Cone}(\mathcal{E})$$

with $\eta(j)$ and c_k non-negative integers. We can assume that $(\beta_1, \ldots, \beta_l)$ is a quiver exceptional sequence which, by some abuse, we denote by the same letter $\mathcal{E}$. From Theorem 1.2(3), we know that there exists a full exact embedding of $\mathrm{rep}(Q(\mathcal{E}))$ into $\mathrm{rep}(Q)$ and let $I : \mathbb{N}^{Q(\mathcal{E})_0} = \mathbb{N}^l \longrightarrow \mathbb{N}^{Q_0}$ be the isometry induced by $\mathcal{E}$.

Since $Q(\mathcal{E})$ is a Dynkin quiver, we know that $\eta = (\eta(1), \ldots, \eta(l))$ is a pre-homogeneous dimension vector. If $\eta_1, \ldots, \eta_r$ are the dimension vectors of the indecomposable direct summands of a η-dimensional rigid representation of $Q(\mathcal{E})$ then it is easy to see that

$$C = \{I(\eta_1), \ldots, I(\eta_r)\} \bigcup \{-\gamma_{i_{l+1}}, \ldots, -\gamma_{i_m}\}$$

is a compatible subset of $\Psi(Q)_{\geq -1}$ and $\alpha \in \mathrm{Cone}(C)$. Furthermore, it is clear that there are only finitely many such compatible sets C as $Q(\mathcal{E})$ has finitely many positive roots. $\square$

Remark 5.5. Note that [7, Theorem 8.1.7] can also be deduced from the proposition above, Example 5.3, and Theorem 1.4.

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