ORBIT SEMIGROUPS AND THE REPRESENTATION TYPE OF QUIVERS

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Abstract. We show that a finite, connected quiver Q without oriented cycles is a Dynkin or Euclidean quiver if and only if all orbit semigroups of representations of Q are saturated.

1. Introduction

The representation type of a quiver reflects the complexity of its indecomposable representations. There are three distinct classes: *finite type*, *tame*, and *wild* quivers. A quiver is said to be of finite type if there are only finitely many indecomposable representations. We say that a quiver is tame if it is not of finite representation type, and in each dimension all but finitely many indecomposable representations come in a finite number of 1-parameter families. Finally, we call a quiver wild if its representation theory is at least as complicated as that of a free algebra in two (non-commuting) variables. For precise definitions, we refer to [3, Ch. 4].

Gabriel's classical result [16] identifies the connected quivers of finite type as being those whose underlying graphs are the Dynkin diagrams of types $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$. Later on, Nazarova [23] and Donovan-Freislich [15] found the tame, connected quivers. Their underlying graphs are the Euclidean diagrams of types $\widetilde{\mathbb{A}}$, $\widetilde{\mathbb{D}}$, or $\widetilde{\mathbb{E}}$. The remaining connected quivers are the wild ones.

It is an important and interesting task to find geometric characterizations of the representation type of a quiver (or more generally, of finite dimensional algebras). In [31, Theorem 1], Skowroński and Weyman showed that a finite, connected quiver is a Dynkin or Euclidean quiver if and only if the various algebras of semi-invariants are always complete intersections. In this paper, we provide a different characterization of the representation type in terms of saturated orbit semigroups.

Let Q be a quiver and β a dimension vector. Following [4, Definition 2.1], we define the orbit semigroup of a representation $W \in \mathrm{Rep}(Q,\beta)$ to be

$$S(W)_Q = \{\sigma \in \mathbb{Z}^{Q_0} \mid \exists f \in \mathrm{SI}(Q,\beta)_\sigma \text{ such that } f(W) \neq 0\}.$$

The cones generated by the semigroups $S(W)_Q$ play a fundamental role in the construction of the GIT-fans for quivers (see [8] and the reference therein). Furthermore, Derksen-Weyman saturation theorem [11] for semi-invariants tells us that $S(W)_Q$ are saturated for *generic* representations. However, there are quiver representations whose orbit semigroups are not saturated. We refer to Section 2 for background material on quiver invariant theory. Throughout this paper, we work over an algebraically closed field k of characteristic zero.

Now, we are ready to state our main result:

Theorem 1.1. *Let Q be a finite, connected quiver without oriented cycles. The following are equivalent:*

1. *Q is a Dynkin or Euclidean quiver;*
2. *for every dimension vector β, the semigroup $S(W)_Q$ is saturated for every $W \in \mathrm{Rep}(Q,\beta)$.*

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To show that orbit semigroups for Dynkin or Euclidean quivers are saturated, we use Derksen-Weyman spanning theorem and a theorem of Schofield on Kac's canonical decomposition for dimension vectors. This allows us to give a short, conceptual proof avoiding a case-by-case analysis. To deal with wild quivers, we use reflection functors, shrinking methods and exceptional sequences to reduce the list of wild quivers to the generalized Kronecker quiver with three arrows.

The layout of this paper is as follows. Background material on quiver invariant theory is reviewed in Section 2. In particular, we recall Derksen-Weyman spanning and saturation theorems. In Section 3, we first recall Schofield's theorem on canonical decompositions and then we prove the necessary part of our theorem. Reflection functors, the shrinking method, and exceptional sequences are reviewed in Section 4 where we show that these reduction methods behave nicely with respect to saturated orbit semigroups. We complete the proof of Theorem 1.1 in Section 5 by showing that for every wild quiver without oriented cycles there is a representation whose orbit semigroup is not saturated. The last section discusses the thin sincere case. In this case, we show that the orbit semigroups are saturated.

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2. Recollection from quiver invariant theory

Let $Q = (Q_0, Q_1, t, h)$ be a finite quiver, where Q_0 is the set of vertices, Q_1 is the set of arrows and $t, h : Q_1 \to Q_0$ assign to each arrow $a \in Q_1$ its tail ta and head ha, respectively.

In this paper, we work over an algebraically closed field k of characteristic zero. A representation V of Q over k is a collection $(V(x), V(a))_{x \in Q_0, a \in Q_1}$ of finite dimensional k-vector spaces $V(x)$, $x \in Q_0$, and k-linear maps $V(a) \in \mathrm{Hom}_k(V(ta), V(ha))$, $a \in Q_1$. If V is a representation of Q, we define its dimension vector $\underline{d}_V$ by $\underline{d}_V(x) = \dim_k V(x)$ for every $x \in Q_0$. Thus the dimension vectors of representations of Q lie in $\Gamma = \mathbb{Z}^{Q_0}$, the set of all integer-valued functions on Q_0.

Given two representations V and W of Q, we define a morphism $\varphi : V \to W$ to be a collection of k-linear maps $(\varphi(x))_{x \in Q_0}$ with $\varphi(x) \in \mathrm{Hom}_k(V(x), W(x))$, $x \in Q_0$, and such that $\varphi(ha)V(a) = W(a)\varphi(ta)$ for every arrow $a \in Q_1$. We denote by $\mathrm{Hom}_Q(V, W)$ the k-vector space of all morphisms from V to W. Let W and V be two representations of Q. We say that V is a subrepresentation of W if $V(x)$ is a subspace of $W(x)$ for all vertices $x \in Q_0$ and $V(a)$ is the restriction of $W(a)$ to $V(ta)$ for all arrows $a \in Q_1$. In this way, we obtain the abelian category $\mathrm{Rep}(Q)$ of all quiver representations of Q.

A representation W is said to be a *Schur representation* if $\mathrm{End}_Q(W) \cong k$. The dimension vector of a Schur representation is called a *Schur root*.

From now on, we assume that our quivers are without oriented cycles. For two quiver representations V and W, consider Ringel's canonical exact sequence [25] :

$$0 \to \mathrm{Hom}_Q(V, W) \to \bigoplus_{x \in Q_0} \mathrm{Hom}_k(V(x), W(x)) \xrightarrow{d^V_W} \bigoplus_{a \in Q_1} \mathrm{Hom}_k(V(ta), W(ha)), \tag{1}$$

where $d^V_W((\varphi(x))_{x \in Q_0}) = (\varphi(ha)V(a) - W(a)\varphi(ta))_{a \in Q_1}$ and $\mathrm{Ext}^1_Q(V, W) = \mathrm{coker}(d^V_W)$.

If α, β are two elements of Γ, we define the Euler inner product by

$$\langle \alpha, \beta \rangle_Q = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha). \tag{2}$$

(When no confusion arises, we drop the subscript Q.) It follows from (1) and (2) that

$$\langle \underline{d}_V, \underline{d}_W \rangle = \dim_k \operatorname{Hom}_Q(V,W) - \dim_k \operatorname{Ext}^1_Q(V,W).$$

2.1. **Semi-invariants of quivers.** For every vertex x, we denote by e_x the simple dimension vector corresponding to x, i.e., $e_x(y) = \delta_{x,y}, \forall y \in Q_0$, where $\delta_{x,y}$ is the Kronecker symbol.

Let β be a dimension vector of Q. The representation space of $\beta-$dimensional representations of Q is

$$\operatorname{Rep}(Q,\beta) = \bigoplus_{a \in Q_1} \operatorname{Hom}_k(k^{\beta(ta)}, k^{\beta(ha)}).$$

If $\mathrm{GL}(\beta) = \prod_{x\in Q_0} \mathrm{GL}(\beta(x))$ then $\mathrm{GL}(\beta)$ acts algebraically on $\operatorname{Rep}(Q,\beta)$ by simultaneous conjugation, i.e., for $g = (g(x))_{x\in Q_0} \in \mathrm{GL}(\beta)$ and $W = (W(a))_{a\in Q_1} \in \operatorname{Rep}(Q,\beta)$, we define $g \cdot W$ by

$$(g\cdot W)(a) = g(ha)W(a)g(ta)^{-1} \text{ for each } a \in Q_1.$$

Hence, $\operatorname{Rep}(Q,\beta)$ is a rational representation of the linearly reductive group $\mathrm{GL}(\beta)$ and the $\mathrm{GL}(\beta)-$orbits in $\operatorname{Rep}(Q,\beta)$ are in one-to-one correspondence with the isomorphism classes of $\beta-$dimensional representations of Q. As Q is a quiver without oriented cycles, one can show that there is only one closed $\mathrm{GL}(\beta)-$orbit in $\operatorname{Rep}(Q,\beta)$ and hence the invariant ring $\mathrm{I}(Q,\beta) = k[\operatorname{Rep}(Q,\beta)]^{\mathrm{GL}(\beta)}$ is exactly the base field k.

Now, consider the subgroup $\mathrm{SL}(\beta) \subseteq \mathrm{GL}(\beta)$ defined by

$$\mathrm{SL}(\beta) = \prod_{x\in Q_0} \mathrm{SL}(\beta(x)).$$

Although there are only constant $\mathrm{GL}(\beta)-$invariant polynomial functions on $\operatorname{Rep}(Q,\beta)$, the action of $\mathrm{SL}(\beta)$ on $\operatorname{Rep}(Q,\beta)$ provides us with a highly non-trivial ring of semi-invariants. Note that any $\sigma \in \mathbb{Z}^{Q_0}$ defines a rational character of $\mathrm{GL}(\beta)$ by

$$(g(x))_{x\in Q_0} \in \mathrm{GL}(\beta) \mapsto \prod_{x\in Q_0} (\det g(x))^{\sigma(x)}.$$

In this way, we can identify $\Gamma = \mathbb{Z}^{Q_0}$ with the group $X^\star(\mathrm{GL}(\beta))$ of rational characters of $\mathrm{GL}(\beta)$, assuming that β is a sincere dimension vector (i.e., $\beta(x) > 0$ for all vertices $x \in Q_0$). In general, we have only the epimorphism $\Gamma \to X^*(\mathrm{GL}(\beta))$, but we usually do not distinguish between σ and its image in $X^*(\mathrm{GL}(\beta))$. We also refer to the rational characters of $\mathrm{GL}(\beta)$ as (integral) weights.

Let $\mathrm{SI}(Q,\beta) = k[\operatorname{Rep}(Q,\beta)]^{\mathrm{SL}(\beta)}$ be the ring of semi-invariants. As $\mathrm{SL}(\beta)$ is the commutator subgroup of $\mathrm{GL}(\beta)$ and $\mathrm{GL}(\beta)$ is linearly reductive, we have

$$\mathrm{SI}(Q,\beta) = \bigoplus_{\sigma \in X^\star(\mathrm{GL}(\beta))} \mathrm{SI}(Q,\beta)_\sigma,$$

where

$$\mathrm{SI}(Q,\beta)_\sigma = \{f \in k[\operatorname{Rep}(Q,\beta)] \mid g\cdot f = \sigma(g)f \text{ for all } g \in \mathrm{GL}(\beta)\}$$

is the space of semi-invariants of weight σ.

If $\alpha \in \mathbb{Z}^{Q_0}$, we define the weight $\sigma = \langle \alpha, \cdot \rangle$ by

$$\sigma(x) = \langle \alpha, e_x \rangle, \forall x \in Q_0.$$

Conversely, it is easy to see that for any weight $\sigma \in \mathbb{Z}^{Q_0}$ there is a unique $\alpha \in \mathbb{Z}^{Q_0}$ (not necessarily a dimension vector) such that $\sigma = \langle \alpha, \cdot \rangle$. Similarly, one can define $\mu = \langle \cdot, \alpha \rangle$.

In [27], Schofield constructed semi-invariants of quivers with remarkable properties. Let α, β be two dimension vectors such that $\langle \alpha, \beta \rangle = 0$. Following [27], we define

$$c : \mathrm{Rep}(Q, \alpha) \times \mathrm{Rep}(Q, \beta) \to k$$

by $c(V, W) = \det(d^V_W)$. Note that d^V_W from (1) is indeed a square matrix since $\langle \alpha, \beta \rangle = 0$. Fix $(V, W) \in \mathrm{Rep}(Q, \alpha) \times \mathrm{Rep}(Q, \beta)$. Then it is easy to see that $c^V = c(V, \cdot) : \mathrm{Rep}(Q, \beta) \to k$ is a semi-invariant of weight $\langle \alpha, \cdot \rangle$ and $c_W = c(\cdot, W) : \mathrm{Rep}(Q, \alpha) \to k$ is a semi-invariant of weight $-\langle \cdot, \beta \rangle$.

Remark 2.1. We should point out that if V is an α-dimensional representation not necessarily in $\mathrm{Rep}(Q, \alpha)$, the semi-invariant c^V is well-defined on $\mathrm{Rep}(Q, \beta)$ up to a non-zero scalar.

Given two representations V and W, we say that V is orthogonal to W, and write $V \perp W$, if $\mathrm{Ext}^1_Q(V, W) = \mathrm{Hom}_Q(V, W) = 0$.

Remark 2.2. Using the exact sequence (1) for two representations V and W, we deduce that $c^V(W) \neq 0$ if and only if $V \perp W$.

2.2. **The Spanning and Saturation Theorems.** A very important property of the Schofield semi-invariants is that each weight space of semi-invariants is spanned by such semi-invariants. This is a fundamental result proved by Derksen and Weyman [11] (see also [29]).

Theorem 2.3 (The Spanning Theorem)**.** [11] *Let β be a sincere dimension vector of Q and let $\sigma = \langle \alpha, \cdot \rangle$ be a weight with $\alpha \in \mathbb{Z}^{Q_0}$. If the weight space of semi-invariants $\mathrm{SI}(Q, \beta)_\sigma$ is non-zero then α is a dimension vector, and moreover, $\mathrm{SI}(Q, \beta)_\sigma$ is spanned by the semi-invariants c^V with $V \in \mathrm{Rep}(Q, \alpha)$.*

Recall that for a given representation $W \in \mathrm{Rep}(Q, \beta)$, its orbit semigroup is

$$S(W)_Q = \{\sigma \in \mathbb{Z}^{Q_0} \mid \exists f \in \mathrm{SI}(Q, \beta)_\sigma \text{ such that } f(W) \neq 0\}.$$

(When no confusion arises, we drop the subscript Q.)

Now, let Q' be the full subquiver of Q whose vertex set is $\mathrm{supp}(W) = \{x \in Q_0 \mid W(x) \neq \{0\}\}$ and denote by W' the restriction of W to Q'. If $\sigma \in \mathbb{Z}^{Q_0}$ is a weight, we denote by σ' its restriction to Q'. Also, any representation V' of Q' is viewed as a representation of Q in a natural way. Note that $S(W)_Q$ is just the inverse image in $\mathbb{Z}^{Q_0}$ of $S(W')_{Q'}$.

From Remark 2.2 and Theorem 2.3, we deduce:

Proposition 2.4. *Let $W \in \mathrm{Rep}(Q, \beta)$ be a representation and $\sigma = \langle \alpha, \cdot \rangle$ a weight with $\alpha \in \mathbb{Z}^{Q_0}_{\geq 0}$. Then,*

$$\sigma \in S(W) \Longleftrightarrow \exists V \in \mathrm{Rep}(Q, \alpha) \textit{ such that } V \perp W.$$

Consequently,

$$S(W) = \{\sigma \in \mathbb{Z}^{Q_0} \mid \sigma' = \langle \alpha', \cdot \rangle_{Q'} \textit{ with } \alpha' \in \mathbb{Z}^{Q'_0}_{\geq 0} \textit{ and } \exists V' \in \mathrm{Rep}(Q', \alpha') \textit{ such that } V' \perp W\}. \tag{3}$$

At this point, we can use (3) to define orbit semigroups for arbitrary representations. In this way, the orbit semigroup $S(W)$ is independent of the choice of the isomorphism class of W. We should mention that Proposition 2.4 together with Theorem 3.1 below plays a crucial role in our study.

Using the Spanning Theorem and Schofield's theory of general representations [28], Derksen and Weyman proved the following remarkable result:

Theorem 2.5 (The Saturation Theorem)**.** [11] *Let Q be a quiver and β a dimension vector. Consider the semigroup of integral effective weights:*

$$\Sigma(Q,\beta) = \{\sigma \in \mathbb{Z}^{Q_0} \mid \mathrm{SI}(Q,\beta)_\sigma \neq 0\}.$$

Then, $\Sigma(Q,\beta)$ is saturated, i.e., for any positive integer n and $\sigma \in \mathbb{Z}^{Q_0}$,

$$n\sigma \in \Sigma(Q,\beta) \Longleftrightarrow \sigma \in \Sigma(Q,\beta).$$

Remark 2.6. Let $\{f_1,\dots,f_m\}$ be a generating system of semi-invariants for $\mathrm{SI}(Q,\beta)$. Then, every representation $W \in \mathrm{Rep}(Q,\beta)$ with $f_i(W) \neq 0, \forall 1 \leq i \leq m$ has the property that $S(W) = \Sigma(Q,\beta)$. This shows that orbit semigroups are saturated for generic representations.

Remark 2.7. It is worth pointing out that for star quivers, Theorem 2.5 implies the Saturation Conjecture for Littlewood-Richardson coefficients (for more details, see [11] and [12]).

3. Kac's canonical decomposition

One of the main tools that we use in this paper is Kac's canonical decomposition of dimension vectors. Let Q be a quiver and α a dimension vector. Following [19], we say that

$$\alpha = \alpha_1 \oplus \dots \oplus \alpha_l$$

is the canonical decomposition of α if there is a non-empty open subset $\mathcal{U} \subseteq \mathrm{Rep}(Q,\alpha)$ such that every $V \in \mathcal{U}$ decomposes as a direct sum of indecomposables of dimension vectors $\alpha_1,\dots,\alpha_l$. It was proved by Kac that the dimension vectors occurring in the canonical decomposition of α must be Schur roots.

Recall that a root of Q is just the dimension vector of an indecomposable representation of Q. We say that a root α is *real* if $\langle\alpha,\alpha\rangle = 1$. If $\langle\alpha,\alpha\rangle = 0$, α is said to be an *isotropic* root. Finally, we say that α is a *non-isotropic imaginary* root if $\langle\alpha,\alpha\rangle < 0$.

It is important to know how to obtain the canonical decomposition of a multiple of α from that of α. Schofield's theorem gives an answer to this question:

Theorem 3.1. [28, Theorem 3.8] *Let $\alpha = \alpha_1 \oplus \dots \oplus \alpha_l$ be the canonical decomposition of α and let $m \geq 1$. Then, the canonical decomposition of $m\alpha$ is*

$$m\alpha = [m\alpha_1] \oplus \dots \oplus [m\alpha_l],$$

where

$$[m\alpha_i] = \begin{cases} \alpha_i^{\oplus m} := \alpha_i \oplus \dots \oplus \alpha_i & \text{if } \alpha_i \text{ is a real or isotropic Schur root,} \\ m\alpha_i & \text{if } \alpha_i \text{ is a non-isotropic imaginary Schur root.} \end{cases}$$

Now, we are ready to prove:

Proposition 3.2. *Let Q be a Dynkin or Euclidean quiver. Then the orbit semigroup $S(W)$ is saturated for every representation W.*

Proof. Assume that Q is a Dynkin or Euclidean quiver and let $W \in \mathrm{Rep}(Q,\beta)$. We can clearly assume that β is a sincere dimension vector. Let $\sigma = \langle\alpha,\cdot\rangle \in \mathbb{Z}^{Q_0}$ be a weight with $\alpha \in \mathbb{Z}^{Q_0}$. Assume $n\sigma \in S(W)$ for some $n \geq 1$. From Theorem 2.3, it follows that α is a dimension vector. Now, consider the canonical decomposition of α:

$$\alpha = \alpha_1 \oplus \dots \oplus \alpha_l.$$

Since Q is a Dynkin or Euclidean quiver, the Schur roots α_i are either real or isotropic. Using Theorem 3.1, we obtain that the canonical decomposition of $n\alpha$ is:

$$n\alpha = \alpha_1^{\oplus n} \oplus \cdots \oplus \alpha_l^{\oplus n}.$$

It is well-known that the functions $\dim_k \operatorname{Ext}^1_Q(\cdot, W)$ and $\dim_k \operatorname{Hom}_Q(\cdot, W)$ are upper semi-continuous. This fact combined with Proposition 2.4 allows us to find a representation $V \in \operatorname{Rep}(Q, n\alpha)$ such that $V \perp W$ and V has a direct summand of the form $\bigoplus_{i=1}^{l} V_i$ with V_i an indecomposable representation of dimension vector α_i, $1 \leq i \leq l$. Set $\widetilde{V} = \bigoplus_{i=1}^{l} V_i$. Then, $\widetilde{V}$ is an α-dimensional representation with $\widetilde{V} \perp W$ and hence $\sigma \in S(W)$ by Proposition 2.4. $\square$

4. Reflection functors, the shrinking method, and exceptional sequences

In this section, we describe three reduction methods that behave nicely with respect to orbit semigroups. This will be particularly useful when dealing with wild quivers.

A quiver Q is said to satisfy *property* (S) provided the following is true: For every representation W of Q, the orbit semigroup $S(W)$ is saturated. It is clear that if Q has property (S) then any (not necessarily full) subquiver has it.

4.1. **Reflection functors.** Let Q be a quiver and $x \in Q_0$ a vertex. Define $s_x(Q)$ to be the quiver obtained from Q by reversing all arrows incident to x. The reflection transformation $s_x : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ at vertex x is defined by

$$s_x(\alpha)(y) = \begin{cases} \alpha(y) & \text{if } y \neq x, \\ \sum_{ha=x} \alpha(ta) + \sum_{ta=x} \alpha(ha) - \alpha(x) & \text{if } y = x. \end{cases}$$

Note that s_x is the reflection in the plane orthogonal to the simple root e_x.

Now, let us assume that x is a sink. The Bernstein-Gelfand-Ponomarev reflection functor at x is defined as follows:

$$\begin{aligned} C_x^+ : \operatorname{Rep}(Q) &\to \operatorname{Rep}(s_x(Q)) \\ V &\mapsto W = C_x^+(V), \end{aligned}$$

where $W(y) = V(y)$ for $y \neq x$ and $W(x) = \ker(\bigoplus_{ha=x} V(ta) \to V(x))$.

If x is a source, we define C_x^- by

$$\begin{aligned} C_x^- : \operatorname{Rep}(Q) &\to \operatorname{Rep}(s_x(Q)) \\ V &\mapsto W = C_x^-(V), \end{aligned}$$

where $W(y) = V(y)$ for $y \neq x$ and $W(x) = \operatorname{coker}(V(x) \to \bigoplus_{ta=x} V(ha))$.

The following theorem, proved by Bernstein-Gelfand-Ponomarev, is one of the fundamental results about reflection functors.

Theorem 4.1. [5] *Let Q be a quiver and $x \in Q_0$ a sink.*

1. *If $V = S_x$ then $C_x^+(V) = 0$.*
2. *If $V \neq S_x$ is indecomposable then $C_x^+(V)$ is indecomposable, too. Furthermore, $C_x^- C_x^+(V) = V$ and*

$$\underline{d}_{C_x^+(V)} = s_x(\underline{d}_V).$$

The analogous result for C_x^- with x a source is also true.

Using the First Fundamental Theorem for special linear groups (see for example [10]), Kac [20, Sections 2 and 3] showed that reflection functors behave nicely with respect to semi-invariants of quivers. Other geometric properties that are preserved by reflection functors can be found in [7, Section 6]. We are going to show that property (S) defined above is also preserved by reflection functors. The following proposition will be very useful for us:

Proposition 4.2. [2] *Let V and W be two representations of Q. Assume that x is a sink and S_x is not a direct summand of V or W. Then:*

$$\dim_k \operatorname{Hom}_Q(V,W) = \dim_k \operatorname{Hom}_{s_x(Q)}(C_x^+(V), C_x^+(W)),$$

and

$$\dim_k \operatorname{Ext}^1_Q(V,W) = \dim_k \operatorname{Ext}^1_{s_x(Q)}(C_x^+(V), C_x^+(W)).$$

Consequently, one has

$$V \perp W \Longleftrightarrow C_x^+(V) \perp C_x^+(W).$$

The same is true when x is a source and C_x^+ is replaced by C_x^-.

Next, we show that when checking property (S), we can actually avoid those representations that have direct summands isomorphic to simple representations. We write $S_x \nmid W$ if W does not have a direct summand isomorphic to S_x. A representation W whose dimension vector is sincere is called a *sincere representation.*

Lemma 4.3. (1) *Let $W = W_1 \oplus W_2$ be a representation with $S(W_1)$ and $S(W_2)$ saturated. Then, $S(W)$ is saturated, too.*
(2) *Let x be a vertex of Q. Then, Q has property (S) if and only if $S(W)$ is saturated for every representation W such that $S_x \nmid W$.*

Proof. (1) We can clearly assume that W is a sincere representation. From Proposition 2.4, it follows that

$$S(W) \subseteq S(W_1) \cap S(W_2).$$

Now, let $\sigma \in \mathbb{Z}^{Q_0}$ be so that $n\sigma \in S(W)$ for some positive integer n. Then, $\sigma = \langle \alpha, \cdot \rangle$ for a unique dimension vector α, and since $S(W_i)$ are assumed to be saturated, we obtain that $\sigma \in S(W_1) \cap S(W_2)$. This is equivalent to the existence of $V_i \in \operatorname{Rep}(Q,\alpha)$ such that

$$\operatorname{Ext}^1_Q(V_i, W_i) = \operatorname{Hom}_Q(V_i, W_i) = 0,$$

for $i \in \{1,2\}$. Using the upper semi-continuity of the functions $\dim_k \operatorname{Ext}^1_Q(\cdot, W_i)$ and $\dim_k \operatorname{Hom}_Q(\cdot, W_i)$, $i \in \{1,2\}$, we can find a representation $V \in \operatorname{Rep}(Q,\alpha)$ such that $\operatorname{Ext}^1_Q(V, W_i) = \operatorname{Hom}_Q(V, W_i) = 0, \forall i \in \{1,2\}$. Applying Proposition 2.4 again, we get $\sigma \in S(W)$.

(2) The proof follows from part (1) and the fact that the orbit semigroup of S_x is clearly saturated. $\square$

For our purposes, we actually need to strengthen Lemma 4.3(2):

Lemma 4.4. *Let $W \in \operatorname{Rep}(Q,\beta)$ and $x \in Q_0$. Consider the set*

$$S_x(W) = \{\sigma = \langle \alpha, \cdot \rangle \in \mathbb{Z}^{Q_0} \mid \exists V \in \operatorname{Rep}(Q,\alpha) \text{ such that } V \perp W \text{ and } S_x \nmid V\}.$$

If $S_x(W)$ is saturated then so is $S(W)$.

Proof. We distinguish two cases:

Case 1: W is a sincere representation. Let $\sigma = \langle \alpha, \cdot \rangle \in \mathbb{Z}^{Q_0}$ be a weight and assume that there exists an integer $n > 1$ such that $n\sigma \in S(W)$.

We know that α has to be a dimension vector by Theorem 2.3. Consider the canonical decomposition of α:

$$\alpha = \alpha_1 \oplus \cdots \oplus \alpha_l.$$

Without loss of generality, let us assume that $\alpha_1, \dots, \alpha_m$ are the non-isotropic, imaginary Schur roots in the decomposition above (of course, we allow m to be zero). From Theorem 3.1, we obtain that

$$n\alpha = n\alpha_1 \oplus \cdots \oplus n\alpha_m \oplus \alpha_{m+1}^{\oplus n} \oplus \cdots \oplus \alpha_l^{\oplus n}$$

is the canonical decomposition of $n\alpha$.

Now, choose an $n\alpha$-dimensional representation V such that $V \perp W$ and

$$V = \bigoplus_{i=1}^{m} V_i' \oplus \bigoplus_{\substack{m+1 \le j \le l \\ 1 \le k \le n}} V_{jk}$$

where the V_i' are indecomposables of dimension vectors $n\alpha_i$, $1 \le i \le m$, and the V_{jk} are indecomposables of dimension vectors α_j, $1 \le k \le n, m+1 \le j \le l$.

Note that $V_i' \perp W$ and $S_x \nmid V_i'$ for every $1 \le i \le m$. Since $S_x(W)$ is assumed to be saturated, it follows that $\langle \alpha_i, \cdot \rangle \in S_x(W)$. So, we can choose α_i-dimensional representations V_i such that $V_i \perp W$ for all $1 \le i \le m$.

Set $U = \bigoplus_{i=1}^{m} V_i \oplus \bigoplus_{j=m+1}^{l} V_{j1}$. Then, U is an α-dimensional representation with $U \perp W$ and this finishes the proof in the sincere case.

Case 2: W is not necessarily a sincere representation. Let Q' be the full subquiver of Q whose vertex set is $\operatorname{supp}(W) = \{x \in Q_0 \mid W(x) \neq \{0\}\}$. Denote by W' the restriction of W to Q'. Then, W' is a sincere representation of Q' and we clearly have that $S(W)_Q$ is saturated if and only if $S(W')_{Q'}$ is saturated.

Now, consider the following linear transformation:

$$I : \mathbb{Z}^{Q_0'} \to \mathbb{Z}^{Q_0}$$
$$\sigma' = \langle \alpha', \cdot \rangle_{Q'} \to \langle \alpha', \cdot \rangle_Q,$$

where any $\alpha' \in \mathbb{Z}^{Q_0'}$ is viewed as an element of $\mathbb{Z}^{Q_0}$ in a natural way. Given a weight $\sigma' \in \mathbb{Z}^{Q_0'}$, we clearly have $\sigma' \in S_x(W')_{Q'}$ if and only if $I(\sigma') \in S_x(W)_Q$. Hence, $S_x(W')_{Q'}$ must be saturated. But now, this implies that $S(W)_Q$ is saturated, as well. $\square$

Now, we are ready to prove:

Proposition 4.5. *Let Q be a quiver without oriented cycles and let x be either a source or a sink. Then, Q satisfies property (S) if and only if so does $s_x(Q)$.*

Proof. It is enough to show that Q satisfies property (S) assuming that x is a source and $\widetilde{Q} = s_x(Q)$ satisfies (S). Let $W \neq 0$ be a representation of Q. By Lemma 4.3, we can assume that $S_x \nmid W$ and let us denote $C_x^-(W)$ by $\widetilde{W}$. Then, it follows from Theorem 4.1 that $S_x \nmid \widetilde{W}$ and $C_x^+(\widetilde{W}) = W$.

Now, let α be a dimension vector of Q. Using Theorem 4.1 again and then Proposition 4.2, we deduce that $\langle \alpha, \cdot \rangle_Q \in S_x(W)_Q$ if and only if $\langle s_x(\alpha), \cdot \rangle_{\widetilde{Q}} \in S_x(\widetilde{W})_{\widetilde{Q}}$. But this latter set is saturated by assumption and hence $S_x(W)$ must be saturated. The proof follows now from Lemma 4.4. $\square$

4.2. **The shrinking method.** Using the First Fundamental Theorem for general linear groups, it is often possible to shrink paths to just one arrow and still preserve weight spaces of semi-invariants. The following shrinking procedure appears in some form in [11], [13], [30], and [31]. We include a proof since it is straight forward.

Lemma 4.6. *Let Q be a quiver and v_0 a vertex such that near v_0, Q looks like:*

$$\begin{tikzcd}
v_1 \arrow[dr, "a_1"] \arrow[dd, dotted, no head] & & \\
& v_0 \arrow[r, "b"] & w \\
v_l \arrow[ur, "a_l"'] & &
\end{tikzcd}$$

Suppose that β is a dimension vector and σ is a weight such that

$$\beta(v_0) \geq \beta(w) \text{ and } \sigma(v_0) = 0.$$

Let $\overline{Q}$ be the quiver defined by $\overline{Q}_0 = Q_0 \setminus \{v_0\}$ and $\overline{Q}_1 = (Q_1 \setminus \{b, a_1, \dots, a_l\}) \cup \{ba_1, \dots, ba_l\}$. If $\overline{\beta} = \beta|_{\overline{Q}}$ and $\overline{\sigma} = \sigma|_{\overline{Q}}$ are the restrictions of β and σ to $\overline{Q}$ then

$$\mathrm{SI}(Q, \beta)_\sigma \cong \mathrm{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}.$$

Proof. Consider the reduction map

$$\pi : \mathrm{Rep}(Q, \beta) \to \mathrm{Rep}(\overline{Q}, \overline{\beta}),$$

defined by taking compositions of linear maps along the new arrows of $\overline{Q}$. As $\beta(v_0) \geq \beta(w)$, we know that π is a surjective morphism and hence the induced comorphism $\pi^\star$ is injective. Using the First Fundamental Theorem for $\mathrm{GL}(\beta(v_0))$ (see for example [10]), we obtain

$$\pi^\star(k[\mathrm{Rep}(\overline{Q}, \overline{\beta})]) = k[\mathrm{Rep}(Q, \beta)]^{\mathrm{GL}(\beta(v_0))}.$$

Since $\pi^\star$ is a $\mathrm{GL}(\overline{\beta})$-equivariant surjective linear map, $\mathrm{GL}(\overline{\beta})$ is linearly reductive, and $\sigma(v_0) = 0$, we have that $\pi^\star$ induces a surjective map from $\mathrm{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$ onto $\mathrm{SI}(Q, \beta)_\sigma$. (Note that at this point we need the assumption that the base field is of characteristic zero.) So, $\pi^\star$ defines an isomorphism from $\mathrm{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}$ into $\mathrm{SI}(Q, \beta)_\sigma$ and we are done. $\square$

Remark 4.7. Note that the lemma above remains true when we reverse the orientation of the arrows a_i and b.

Keeping the same notations as above, we have:

Proposition 4.8. *If Q satisfies property (S) then so does $\overline{Q}$.*

4.3. **Exceptional sequences.** A dimension vector β is called a *real Schur* root if there exists a representation $W \in \mathrm{Rep}(Q, \beta)$ such that $\mathrm{End}_Q(W) \simeq k$ and $\mathrm{Ext}^1_Q(W, W) = 0$ (we call such a representation *exceptional*). Note that if β is a real Schur root then there exists a unique, up to isomorphism, exceptional β-dimensional representation.

For α and β two dimension vectors, consider the generic ext and hom:

$$\mathrm{ext}_Q(\alpha, \beta) = \min\{\dim_k \mathrm{Ext}^1_Q(V, W) \mid (V, W) \in \mathrm{Rep}(Q, \alpha) \times \mathrm{Rep}(Q, \beta)\},$$

and

$$\hom_Q(\alpha, \beta) = \min\{\dim_k \mathrm{Hom}_Q(V, W) \mid (V, W) \in \mathrm{Rep}(Q, \alpha) \times \mathrm{Rep}(Q, \beta)\}.$$

Given two dimension vectors α and β, we write $\alpha \perp \beta$ provided that $\mathrm{ext}_Q(\alpha, \beta) = \hom_Q(\alpha, \beta) = 0$. If W is a representation, we define ${}^\perp W$ to be the full subcategory of $\mathrm{Rep}(Q)$ consisting of all representations V such that $V \perp W$.

Definition 4.9. We say that $(\varepsilon_1, \dots, \varepsilon_r)$ is an *exceptional sequence* if

(1) ε_i are real Schur roots;
(2) $\varepsilon_i \perp \varepsilon_j$ for all $1 \leq i < j \leq l$.

Following [12], a sequence $(\varepsilon_1, \dots, \varepsilon_r)$ is called a *quiver exceptional sequence* if it is exceptional and $\langle \varepsilon_j, \varepsilon_i \rangle \leq 0$ for all $1 \leq i < j \leq l$.

A sequence $(E_1, \dots, E_r)$ of exceptional representations is said to be a (quiver) exceptional sequence if the sequence of their dimension vectors $(\underline{d}_{E_1}, \dots, \underline{d}_{E_r})$ is a (quiver) exceptional sequence.

Now, let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ be a quiver exceptional sequence and let $E_i \in \mathrm{Rep}(Q, \varepsilon_i)$ be exceptional representations. Construct a new quiver $Q(\varepsilon)$ with vertex set $\{1, \dots, r\}$ and $-\langle \varepsilon_j, \varepsilon_i \rangle$ arrows from j to i. Define $\mathcal{C}(\varepsilon)$ to be the smallest full subcategory of $\mathrm{Rep}(Q)$ which contains $E_1, \dots, E_r$ and is closed under extensions, kernels of epimorphisms, and cokernels of monomorphisms.

For the remaining of this section, we assume that $r \leq N - 1$, where N is the number of vertices of Q. We recall a very useful result from [12, Section 2.7] in a form that is convenient for us:

Proposition 4.10. [12] *The category $\mathcal{C}(\varepsilon)$ is naturally equivalent to $\mathrm{Rep}(Q(\varepsilon))$ with $E_1, \dots, E_r$ being the simple objects of $\mathcal{C}(\varepsilon)$. Furthermore, the inverse functor from $\mathrm{Rep}(Q(\varepsilon))$ to $\mathcal{C}(\varepsilon)$ is a full exact embedding into $\mathrm{Rep}(Q)$.*

Proof. As the ε_i are Schur roots and $\mathrm{ext}_Q(\varepsilon_i, \varepsilon_j) = 0, \forall i < j$, we know that either $\hom_Q(\varepsilon_j, \varepsilon_i) = 0$ or $\mathrm{ext}_Q(\varepsilon_j, \varepsilon_i) = 0$ by [28, Theorem 4.1]. This fact combined with $\langle \varepsilon_j, \varepsilon_i \rangle \leq 0$ implies $\hom_Q(\varepsilon_j, \varepsilon_i) = 0$. But this is equivalent to $\mathrm{Hom}_Q(E_j, E_i) = 0, \forall i < j$ as the representations E_i have dense orbits in $\mathrm{Rep}(Q, \varepsilon_i)$. From [12, Lemma 2.36], it follows that $E_1, \dots, E_r$ are precisely the simple objects of $\mathcal{C}(\varepsilon)$.

Using [27, Theorem 2.3] (see also [12]), we can (naturally) extend $E_1, \dots, E_r$ to an exceptional sequence of representations of the form $E_1, \dots, E_r, E_{r+1}, \dots, E_N$. Now, we have the equality

$$\mathcal{C}(\varepsilon) = {}^{\perp}E_{r+1} \cap \cdots \cap {}^{\perp}E_N,$$

by [9, Lemma 4]. Applying [27, Theorem 2.3] again, we deduce that $\mathcal{C}(\varepsilon)$ is naturally equivalent to the category of representations of a quiver Q' without oriented cycles and r vertices. Furthermore, the inverse functor from $\mathrm{Rep}(Q')$ to $\mathcal{C}(\varepsilon)$ is a full exact embedding into $\mathrm{Rep}(Q)$. But now, it is clear that Q' must be precisely $Q(\varepsilon)$. □

Remark 4.11. Let us point out that for a complete quiver exceptional sequence ε, the corresponding exceptional representations E_i are precisely the simple representations of Q as it was shown by Ringel [26, Theorem 3].

From Proposition 2.4 and Proposition 4.10, we deduce:

Proposition 4.12. *Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ be a quiver exceptional sequence for Q. If property (S) fails for $Q(\varepsilon)$ then the same is true for Q.*

5. Wild quivers

In this section, we show that for every wild quiver Q without oriented cycles there is a representation W such that $S(W)$ is not saturated. Our strategy is very similar to the one from [31, Section 6]. More precisely, we use reflection functors and the shrinking method to reduce the list of wild quivers to just seven quivers. Then, we use exceptional sequences to further reduce this list to the generalized Kronecker quiver with three arrows.

Example 5.1 (Generalized Kronecker quivers)**.** Let $\theta(3)$ be the generalized Kronecker quiver with two vertices and three arrows:

$$\cdot \Rrightarrow \cdot$$

Label the three arrows by a, b, and c. Now, choose $W \in \operatorname{Rep}(\theta(3), (3,3))$ so that $W(a), W(b)$, and $W(c)$ are linearly independent skew-symmetric 3×3 matrices. If $\sigma = (1, -1)$ then it is easy to see that $\sigma(\underline{d}_{W'}) \leq 0$ for every subrepresentation W' of W (i.e., W is σ-semi-stable). By King semi-stability criterion [21], this is means that $n\sigma \in S(W)$ for some integer $n \geq 1$.

On the other hand, we claim that $\sigma \notin S(W)$. Indeed, the weight space $\mathrm{SI}(\theta(3), (3,3))_\sigma$ is spanned by the coefficients of the functional determinant:

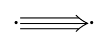
$$W \to \det(t_1 W(a) + t_2 W(b) + t_3 W(c))$$

as a polynomial in the variables t_1, t_2, and t_3. But 3×3 skew symmetric matrices have zero determinant, and hence, $f(W) = 0$ for every semi-invariant f of weight σ. This shows that $S(W)$ is not saturated.

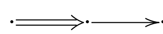
The following combinatorial result is essentially taken from [31, Proposition 49] (see also [1, Lemma 2.1, pp. 253]):

Proposition 5.2. *Let Q be a finite, connected, wild quiver without oriented cycles. Then Q contains a subquiver which can be reduced to one of the following seven quivers by applying reflection transformations and shrinking paths to arrows:*

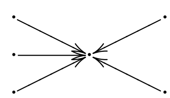
(a)



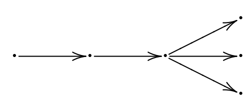
(b)



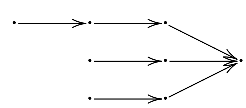
(c)



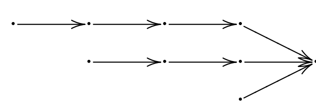
(d)



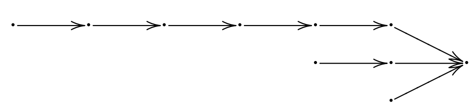
(e)



(f)



(g)



Proof. Similar to [31, Proposition 49]. □

Remark 5.3. Note that the first two quivers on our list differ than the first two from [31, Proposition 49]. This is because we are shrinking paths to arrows instead of shrinking arrows to loops (or identity) as in the aforementioned paper. Also, our subquiver Q' is not necessarily a full subquiver.

Proposition 5.4. *Let Q be one of the quivers from the list obtained in Proposition 5.2. Then, there exists a representation $W \in \operatorname{Rep}(Q, \beta)$ such that $S(W)$ is not saturated.*

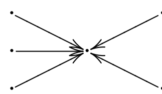
Proof. We have seen in Example 5.1 that our proposition is true for the quiver of type (a). Next, we use exceptional sequences to embed this generalized Kronecker quiver into each of the remaining six quivers.

For the quiver

$$\cdot \Longrightarrow \cdot \longrightarrow \cdot$$

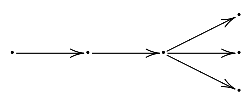
of type (b), we take $\varepsilon_1 = (0,0,1)$ and $\varepsilon_2 = (2,3,0)$.

For the quiver



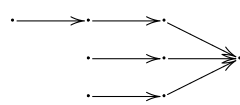
of type (c), we take $\varepsilon_1 = \begin{array}{ccc} 1 & & 1 \\ 1 & 3 & \\ 1 & & 0 \end{array}, \quad \varepsilon_2 = \begin{array}{ccc} 0 & & 0 \\ 0 & 0 & \\ 0 & & 1 \end{array}.$

For the quiver



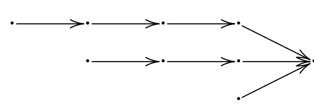
of type (d), take $\varepsilon_1 = \begin{array}{cccc} & & & 0 \\ 0 & 0 & 0 & 0, \\ & & & 1 \end{array} \quad \varepsilon_2 = \begin{array}{cccc} & & & 1 \\ 0 & 1 & 2 & 1, \\ & & & 0 \end{array} \quad \varepsilon_3 = \begin{array}{cccc} & & & 0 \\ 1 & 0 & 0 & 0. \\ & & & 0 \end{array}$ Note that for this quiver, the generalized Kronecker quiver embeds via the quiver of type (b).

For the quiver



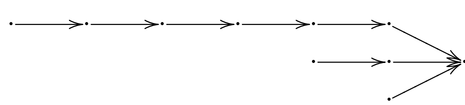
of type (e), we take $\varepsilon_1 = \begin{array}{cccc} 0 & 1 & 2 & \\ & 1 & 2 & 3, \\ & 0 & 2 & \end{array} \quad \varepsilon_2 = \begin{array}{cccc} 0 & 0 & 0 & \\ & 0 & 0 & 0, \\ & 1 & 0 & \end{array} \quad \varepsilon_3 = \begin{array}{cccc} 1 & 0 & 0 & \\ & 0 & 0 & 0. \\ & 0 & 0 & \end{array}$ Note that for this quiver, first we get an embedding of the quiver $\cdot \Longrightarrow \cdot \longleftarrow \cdot$ (call it of type (b')) and then we embed the generalized Kronecker quiver into the quiver of type (b') by using the sequence $((2,3,0),(0,0,1))$.

For the quiver



of type (f), we take $\varepsilon_1 = \begin{array}{ccccc} 0 & 3 & 5 & 7 & \\ & 3 & 5 & 7 & 9, \\ & & & 5 & \end{array} \quad \varepsilon_2 = \begin{array}{ccccc} 1 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0. \\ & & & 0 & \end{array}$

Finally, for the quiver



of type (g), we take $\varepsilon_1 = \begin{array}{ccccccc} 0 & 3 & 5 & 7 & 9 & 11 & \\ & & & & 5 & 9 & 13, \\ & & & & & 7 & \end{array} \quad \varepsilon_2 = \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0. \\ & & & & & 0 & \end{array}$

Now, the proof follows from Proposition 4.12. $\square$

Proof of Theorem 1.1. The implication " $\Longrightarrow$ " is proved in Proposition 3.2. The other implication follows from Proposition 5.4 and Proposition 4.5, Proposition 4.8, Proposition 4.12. $\square$

Remark 5.5. We would like to end this section with some comments about the possibility of extending our theorem to other classes of algebras. First of all, it is obvious how to define orbit semigroups for finite dimensional modules over finite dimensional algebras. Furthermore, some of the main tools used in the proof of Theorem 1.1, such as Derksen-Weyman spanning theorem and Kac's canonical decomposition, are available for finite dimensional algebras as well. It is also useful to know if Schofield's theorem [28, Theorem 3.8] extends to other classes of algebras. This is clearly the case for regular dimension vectors for concealed-canonical algebras (for more details, see [14, pp. 382]). This opens the possibility of proving our theorem for this class of algebras. Finally, let us mention that for canonical algebras, the implication " $\Longleftarrow$ " of Theorem 1.1 follows from its validity for quivers. Indeed, from [24] we know that a canonical algebra Λ with underlying quiver Δ is tame if and only if $\Delta \setminus \{\infty\}$ is a Dynkin or Euclidean quiver. (Here, ∞ is the unique sink of Δ.) Now, we can see that by working with representations of Λ which are zero at the sink ∞, the implication " $\Longleftarrow$ " of Theorem 1.1 for canonical algebras follows.

6. The thin sincere case

In this section we look into the case when the dimension vector $\mathbf{1}$ is thin sincere, i.e., $\mathbf{1}(x) = 1, \forall x \in Q_0$. Let us fix some notation first. For an affine G-variety X, where G is a linear algebraic group, and $\sigma \in X^*(G)$ a rational character of G, we set

$$\mathrm{SI}(X,G)_\sigma := \{f \in k[X] \mid g \cdot f = \sigma(g) f, \forall g \in G\}.$$

For a given representation $W \in \mathrm{Rep}(Q, \mathbf{1})$, we have

$$S(W) = \{\sigma \in \mathbb{Z}^{Q_0} \mid \exists f \in \mathrm{SI}(\overline{\mathrm{GL}(\mathbf{1})W}, \mathrm{GL}(\mathbf{1}))_\sigma \text{ such that } f(W) \neq 0\}.$$

Consider the weight space decomposition:

$$k[\overline{\mathrm{GL}(\mathbf{1})W}] = k[\overline{\mathrm{GL}(\mathbf{1})W}]^{\mathrm{SL}(\mathbf{1})} = \bigoplus \mathrm{SI}(\overline{\mathrm{GL}(\mathbf{1})W}, \mathrm{GL}(\mathbf{1}))_\sigma,$$

where the sum is over all weights $\sigma \in S(W)$. As $\mathrm{GL}(\mathbf{1})$ acts with a dense orbit on the closure of the orbit of W, we must have $\dim_k \mathrm{SI}(\overline{\mathrm{GL}(\mathbf{1})W}, \mathrm{GL}(\mathbf{1}))_\sigma \leq 1$, and so,

$$k[\overline{\mathrm{GL}(\mathbf{1})W}] = k[S(W)].$$

From toric geometry, we deduce that if $S(W)$ is saturated then $\overline{\mathrm{GL}(\mathbf{1})W}$ is normal. We should point out that this last observation is not true for other dimension vectors as the following example, due to Zwara, shows.

Example 6.1. Consider the Kronecker quiver $\theta(2)$:

$$1 \rightrightarrows 2$$

with arrows labeled by a and b. Let $W \in \mathrm{Rep}(\theta(2), (3,3))$ be the representation given by $W(a) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $W(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It was proved by Zwara [32] that $\overline{\mathrm{GL}(\alpha)W}$ is not normal. On the other hand, Proposition 3.2 tells us that $S(W)$ is saturated. In fact, in this example it is not difficult to see that $S(W) = \{(0,0)\}$. Indeed, it was first proved by Happel [17] (see also [22]) that the algebra of semi-invariants $\mathrm{SI}(\theta(2),(3,3))$ is (a polynomial algebra) generated by the coefficients of the functional determinant:

$$W \to \det(t_1 W(a) + t_2 W(b))$$

as a polynomial in the variables t_1, t_2. But, for our choice of $W(a)$ and $W(b)$, $\det(t_1 W(a) + t_2 W(b)) = 0$ and hence $S(W) = \{(0,0)\}$.

In [6, Theorem 1.3], it was proved that $\overline{\mathrm{GL}(\mathbf{1})W}$ is normal when $W \in \mathrm{Rep}(Q, \mathbf{1})$ is just the identity along the arrows. We are going to see that this is the case for any representation W by showing that $S(W)$ is saturated. In the thin sincere case, it is rather easy to write down a k-basis for each weight space of semi-invariants. Consider the boundary map $I = I_Q : \mathbb{R}^{Q_1} \to \mathbb{R}^{Q_0}$ of Q; this is the function which assigns to every $\lambda = (\lambda(a))_{a \in Q_1}$, the vector $(I(\lambda)_x)_{x \in Q_0}$, where

$$I(\lambda)_x := \sum_{\substack{a \in Q_1 \\ ta = x}} \lambda(a) - \sum_{\substack{a \in Q_1 \\ ha = x}} \lambda(a),$$

for every vertex $x \in Q_0$. Let us record the following simple lemma:

Lemma 6.2. *Keep notation as above. Let $\sigma \in \mathbb{Z}^{Q_0}$ be a weight. Then*

$$\dim_k \mathrm{SI}(Q, \mathbf{1})_\sigma = |I^{-1}(\sigma) \bigcap \mathbb{Z}^{Q_1}_{\geq 0}|.$$

Proof. For convenience, denote $V(x) = k, \forall x \in Q_0$. If V is a vector space, we denote by $\det_V^l$, the l^{th} power of the determinant representation of $\mathrm{GL}(V)$; the symmetric algebra of V is denoted by $S(V)$. It is easy to see that

$$\begin{aligned} k[\mathrm{Rep}(Q, \mathbf{1})] &= \bigotimes_{a \in Q_1} S(V(ta) \otimes V(ha)^*) \\ &= \bigotimes_{a \in Q_1} \bigoplus_{\lambda(a) \in \mathbb{Z}_{\geq 0}} \det{}_{V(ta)}^{\lambda(a)} \otimes \det{}_{V(ha)}^{-\lambda(a)} \\ &= \bigoplus_{\lambda \in \mathbb{Z}^{Q_1}_{\geq 0}} \bigotimes_{x \in Q_0} \det{}_{V(x)}^{I(\lambda)_x} \end{aligned}$$

The proof now follows. □

For every $\lambda \in \mathbb{Z}^{Q_1}_{\geq 0}$, define

$$\begin{aligned} f_\lambda : &\mathrm{Rep}(Q, \mathbf{1}) \to k \\ &(t(a))_{a \in Q_1} \mapsto \prod_{a \in Q_1} t(a)^{\lambda(a)}. \end{aligned}$$

Now it is clear that $\{f_\lambda \mid \lambda \in I^{-1}(\sigma) \bigcap \mathbb{Z}^{Q_1}_{\geq 0}\}$ is a k-basis of $\mathrm{SI}(Q, \mathbf{1})_\sigma$.

Proposition 6.3. *For every $W \in \mathrm{Rep}(Q, \mathbf{1})$, the semigroup $S(W)$ is saturated.*

Proof. Write $W = (t(a))_{a \in Q_1}$, where $t(a) \in k, \forall a \in Q_1$. Note that for a weight $\sigma \in \mathbb{Z}^{Q_0}$, we have

$$\sigma \in S(W) \iff \exists \lambda \in I^{-1}(\sigma) \bigcap \mathbb{Z}^{Q_1}_{\geq 0} \text{ such that } \lambda(a) = 0 \text{ whenever } t(a) = 0.$$

To check that $S(W)$ is saturated, we can clearly assume that $t(a) \neq 0, \forall a \in Q_1$. Under this assumption, we deduce that $S(W) = \Sigma(Q, \mathbf{1})$ which is known to be saturated by Theorem 2.5. □

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