MODULI SPACES OF REPRESENTATIONS OF SPECIAL BISERIAL ALGEBRAS

ANDREW T. CARROLL, CALIN CHINDRIS, RYAN KINSER, AND JERZY WEYMAN

ABSTRACT. We show that the irreducible components of any moduli space of semistable representations of a special biserial algebra are always isomorphic to products of projective spaces of various dimensions. This is done by showing that certain varieties of representations of special biserial algebras are isomorphic to products of varieties of circular complexes, and therefore normal, allowing us to apply recent results of the second and third authors on moduli spaces.

CONTENTS

1. INTRODUCTION

Throughout, K denotes an algebraically closed field of characteristic zero. Unless otherwise specified, all quivers are assumed to be finite and connected, and all algebras are assumed to be bound quiver algebras.

In this paper, we study representations of algebras within the general framework of Geometric Invariant Theory (GIT). This interaction between representations of algebras and GIT leads to the construction of moduli spaces of representations as solutions to the classification problem of semi-stable representations, up to S-equivalence. We point out that these moduli spaces can be arbitrarily complicated; indeed, arbitrary projective varieties can arise as moduli spaces of representations of algebras [Hil96, HZ98].

Our goal in this paper is to understand these moduli spaces for special biserial algebras. The results we obtain here are, in fact, part of a program aimed at finding geometric characterizations of the representation type of bound quiver algebras. This line of research has attracted a lot of attention, see for example [BC09, BCHZ15, Bob08, BS99, Bob14, Bob12, CW13, Chi09, Chi11, CKW15, CC15a, CK16, Dom11, GS03, Rie04, RZ04, RZ08, SW00].

Special biserial algebras play a prominent role in the representation theory of algebras and related areas. Their indecomposable representations can be nicely described, however the number of 1-parameter families needed to parametrize the n-dimensional

2010 *Mathematics Subject Classification.* 16G20, 14D20.

Key words and phrases. gentle algebras, moduli spaces, representations, regular irreducible components, special biserial algebras, varieties of circular complexes.

C.C. was supported by NSA grant H98230-15-1-0022 and J.W. by NSF grant DMS-1400740.

indecomposables can grow faster than any polynomial in n. Furthermore, gentle algebras and Brauer graph algebras, which are particular cases of special biserial algebras, have recently played an important role in the study of Jacobian and cluster algebras, see for example [ABCJP10, GLFS16, MS14].

Our main result is the following theorem which classifies the irreducible components of moduli spaces for special biserial algebras.

Theorem 1. *Let A be a special biserial algebra. Then any irreducible component of a moduli space $\mathcal{M}(A, \mathbf{d})^{ss}_{\theta}$ is isomorphic to a product of projective spaces.*

The isomorphism of the theorem results from a general decomposition theorem for moduli spaces proved in [CK17]. The key geometric condition needed to apply this theorem is that certain representation varieties are normal. In this paper, we show in Proposition 9 that this condition holds in all cases relevant to special biserial algebra by reducing the consideration to varieties of circular complexes (see Sections 3 and 4).

Acknowledgements. This project began during a visit of the first three authors to the University of Connecticut. The authors would like to acknowledge the generous support of the Stuart and Joan Sidney Professorship of Mathematics endowment for making the visit possible. We also thank Amelie Schreiber for participating in discussions about the project and Corrado de Concini for inspiring conversations.

2. Representation varieties and moduli spaces

2.1. **Representation varieties.** According to Gabriel's theorem, any finite-dimensional unital, associative K-algebra A can be viewed as a bound quiver algebra, up to Morita equivalence; that is there exist a quiver Q, uniquely determined by A, and an admissible ideal I of KQ such that $A \simeq KQ/I$. Throughout, we will adopt the language of representations of bound quivers. In particular, by abuse of terminology, we refer to a representation of Q satisfying the relations in I as a representation of A. Whenever we work with a set of generators $\mathcal{R}$ for I, we will always assume each generator is a linear combination of paths with the same source and target vertex.

We write Q_0 for the set of vertices of Q, and Q_1 for its set of arrows. For a dimension vector $\mathbf{d} \in \mathbb{Z}^{Q_0}_{\geq 0}$, the affine *representation variety* $\operatorname{rep}(A, \mathbf{d})$ parametrizes the $\mathbf{d}$-dimensional representations of $(Q, \mathcal{R})$ along with a fixed basis. Writing ta and ha for the tail and head of an arrow $a \in Q_1$, we have:

$$\operatorname{rep}(A, \mathbf{d}) := \{M \in \prod_{a \in Q_1} \operatorname{Mat}_{\mathbf{d}(ha) \times \mathbf{d}(ta)}(K) \mid M(r) = 0, \text{ for all } r \in \mathcal{R}\}.$$

Under the action of the change of base group $\operatorname{GL}(\mathbf{d}) := \prod_{x \in Q_0} \operatorname{GL}(\mathbf{d}(x), K)$, the orbits in $\operatorname{rep}(A, \mathbf{d})$ are in one-to-one correspondence with the isomorphism classes of $\mathbf{d}$-dimensional representations of $(Q, \mathcal{R})$. For more background on representation varieties, see [Bon98, Zwa11].

In general, $\operatorname{rep}(A, \mathbf{d})$ does not have to be irreducible. Let C be an irreducible component of $\operatorname{rep}(A, \mathbf{d})$. We say that C is *indecomposable* if C has a non-empty open subset of indecomposable representations. We say that C is a *Schur component* if C contains a Schur representation, in which case C has a non-empty open subset of Schur representations; in particular, any Schur component is indecomposable. As shown by de la Peña in [dlP91] and Crawley-Boevey and Schröer in [CBS02, Theorem 1.1] any irreducible component $C \subseteq \mathrm{rep}(A, \mathbf{d})$ satisfies a Krull-Schmidt type decomposition

$$C = \overline{C_1 \oplus \ldots \oplus C_l}$$

for some indecomposable irreducible components $C_i \subseteq \mathrm{rep}(A, \mathbf{d}_i)$ with $\sum \mathbf{d}_i = \mathbf{d}$. Moreover, $C_1, \ldots, C_l$ are uniquely determined by this property.

2.2. **Semi-Invariants.** Let $A = KQ/I$ be an algebra and $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$ a dimension vector of A. We are interested in the action of $\mathrm{SL}(\mathbf{d}) := \prod_{x \in Q_0} \mathrm{SL}(\mathbf{d}(x), K)$ on the representation variety $\mathrm{rep}(A, \mathbf{d})$. The resulting *ring of semi-invariants* $\mathrm{SI}(A, \mathbf{d}) := K[\mathrm{rep}(A, \mathbf{d})]^{\mathrm{SL}(\mathbf{d})}$ has a weight space decomposition over the group $X^\star(\mathrm{GL}(\mathbf{d}))$ of rational characters of $\mathrm{GL}(\mathbf{d})$:

$$\mathrm{SI}(A, \mathbf{d}) = \bigoplus_{\chi \in X^\star(\mathrm{GL}(\mathbf{d}))} \mathrm{SI}(A, \mathbf{d})_\chi.$$

For each character $\chi \in X^\star(\mathrm{GL}(\mathbf{d}))$,

$$\mathrm{SI}(A, \mathbf{d})_\chi = \{f \in K[\mathrm{rep}(A, \mathbf{d})] \mid g \cdot f = \chi(g) f \text{ for all } g \in \mathrm{GL}(\mathbf{d})\}$$

is called the *space of semi-invariants* on $\mathrm{rep}(A, \mathbf{d})$ of *weight* χ.

For a $\mathrm{GL}(\mathbf{d})$-invariant closed subvariety $C \subseteq \mathrm{rep}(A, \mathbf{d})$, we similarly define the ring of semi-invariants $\mathrm{SI}(C) := K[C]^{\mathrm{SL}(\mathbf{d})}$, and the space $\mathrm{SI}(C)_\chi$ of semi-invariants of weight $\chi \in X^*(\mathrm{GL}(\mathbf{d}))$.

Note that any $\theta \in \mathbb{Z}^{Q_0}$ defines a rational character $\chi_\theta : \mathrm{GL}(\mathbf{d}) \to K^*$ by

$$\chi_\theta((g(x))_{x \in Q_0}) = \prod_{x \in Q_0} \det g(x)^{\theta(x)}. \tag{1}$$

In this way, we get a natural epimorphism $\mathbb{Z}^{Q_0} \to X^\star(\mathrm{GL}(\mathbf{d}))$; we refer to the rational characters of $\mathrm{GL}(\mathbf{d})$ as integral weights of Q (or A). In case $\mathbf{d}$ is a sincere dimension vector, this epimorphism is an isomorphism which allows us to identify $\mathbb{Z}^{Q_0}$ with $X^\star(\mathrm{GL}(\mathbf{d}))$.

2.3. **Moduli spaces of representations.** Let $(Q, \mathcal{R})$ be a bound quiver, and $\theta \in \mathbb{Z}^{Q_0}$ an integral weight of Q. Following King [Kin94], a representation M of $(Q, \mathcal{R})$ is said to be *θ-semi-stable* if $\theta(\mathbf{dim}\, M) = 0$ and $\theta(\mathbf{dim}\, M') \leq 0$ for all subrepresentations $M' \leq M$. We say that M is *θ-stable* if M is non-zero, $\theta(\mathbf{dim}\, M) = 0$, and $\theta(\mathbf{dim}\, M') < 0$ for all subrepresentations $0 \neq M' < M$. Finally, we call M a *θ-polystable* representation if M is a direct sum of θ-stable representations.

Now, let $\mathbf{d}$ be a dimension vector of $(Q, \mathcal{R})$ and consider the (possibly empty) open subsets

$$\mathrm{rep}(A, \mathbf{d})^{ss}_\theta = \{M \in \mathrm{rep}(A, \mathbf{d}) \mid M \text{ is } \theta\text{-semi-stable}\}$$

and

$$\mathrm{rep}(A, \mathbf{d})^{s}_\theta = \{M \in \mathrm{rep}(A, \mathbf{d}) \mid M \text{ is } \theta\text{-stable}\}$$

of $\mathbf{d}$-dimensional θ(-semi)-stable representations of $(Q, \mathcal{R})$. Using methods from Geometric Invariant Theory, King shows in [Kin94] that the projective variety

$$\mathcal{M}(A, \mathbf{d})^{ss}_\theta := \mathrm{Proj}\left(\bigoplus_{n \geq 0} \mathrm{SI}(A, \mathbf{d})_{n\theta}\right)$$

is a GIT-quotient of $\operatorname{rep}(A,\mathbf{d})^{ss}_\theta$ by the action of $\operatorname{PGL}(\mathbf{d})$ where $\operatorname{PGL}(\mathbf{d}) = \operatorname{GL}(\mathbf{d})/T_1$ and $T_1 = \{(\lambda \operatorname{Id}_{\mathbf{d}(x)})_{x\in Q_0} \mid \lambda \in k^*\} \leq \operatorname{GL}(\mathbf{d})$. Moreover, there is a (possibly empty) open subset $\mathcal{M}(A,\mathbf{d})^s_\theta$ of $\mathcal{M}(A,\mathbf{d})^{ss}_\theta$ which is a geometric quotient of $\operatorname{rep}(A,\mathbf{d})^s_\theta$ by $\operatorname{PGL}(\mathbf{d})$. We say that $\mathbf{d}$ is a *θ-(semi-)stable dimension vector* of A if $\operatorname{rep}(A,\mathbf{d})^{(s)s}_\theta \neq \emptyset$.

For a given $\operatorname{GL}(\mathbf{d})$-invariant closed subvariety C of $\operatorname{rep}(A,\mathbf{d})$, we similarly define C^{ss}_θ, C^s_θ, $\mathcal{M}(C)^{ss}_\theta$, and $\mathcal{M}(C)^s_\theta$. We say that C is a *θ-(semi-)stable subvariety* if $C^{(s)s} \neq \emptyset$.

From now on, let us assume that the character $\chi_\theta \in X^*(\operatorname{GL}(\mathbf{d}))$ induced by θ is not trivial, i.e. the restriction of θ to the support of $\mathbf{d}$ is not zero, and denote by G_θ the kernel of χ_θ. Let C be a θ-semi-stable $\operatorname{GL}(\mathbf{d})$-invariant, irreducible, closed subvariety of $\operatorname{rep}(A,\mathbf{d})$. Then we have that:

$$K[C]^{G_\theta} = \bigoplus_{n\geq 0} \operatorname{SI}(C)_{n\theta}.$$

The restriction homomorphism $K[\operatorname{rep}(A,\mathbf{d})] \to K[C]$ remains surjective after taking G_θ-invariants since G_θ is linearly reductive in characteristic zero. This surjective homomorphism $K[\operatorname{rep}(A,\mathbf{d})]^{G_\theta} \to K[C]^{G_\theta}$ of graded algebras gives rise to a closed embedding $\mathcal{M}(C)^{ss}_\theta \hookrightarrow \mathcal{M}(A,\mathbf{d})^{ss}_\theta$. In fact, the image of this embedding is precisely $\pi(C^{ss}_\theta)$, where $\pi : \operatorname{rep}(A,\mathbf{d})^{ss}_\theta \to \mathcal{M}(A,\mathbf{d})^{ss}_\theta$ is the quotient morphism.

The points of $\mathcal{M}(C)^{ss}_\theta$ correspond bijectively to the (isomorphism classes of) θ-polystable representations in C. Indeed, each fiber of $\pi : C^{ss}_\theta \to \mathcal{M}(C)^{ss}_\theta$ contains a unique closed $\operatorname{GL}(\mathbf{d})$-orbit in C^{ss}_θ. On the other hand, as proved by King in [Kin94, Proposition 3.2(i)], these orbits are precisely the isomorphism classes of θ-polystable representation in C. In fact, for any $M \in C^{ss}_\theta$, there exists a 1-psg $\lambda \in X_\star(G_\theta)$ such that $\widetilde{M} := \lim_{t\to 0} \lambda(t)M$ exists and is the unique, up to isomorphism, polystable representation in $\overline{\operatorname{GL}(\mathbf{d})M} \cap C^{ss}_\theta$.

The goal now is to explain how to decompose a given irreducible component of a moduli space of representations into smaller spaces which are easier to handle. The following definition is from [CK17].

Definition 2. Let C be a $\operatorname{GL}(\mathbf{d})$-invariant, irreducible, closed subvariety of $\operatorname{rep}(A,\mathbf{d})$, and assume C is θ-semistable. Consider a collection $(C_i \subseteq \operatorname{rep}(A,\mathbf{d}_i))_i$ of θ-stable irreducible components such that $C_i \neq C_j$ for $i \neq j$, along with a collection of multiplicities $(m_i \in \mathbb{Z}_{>0})_i$, and set $\widetilde{C} = \overline{C_1^{\oplus m_1} \oplus \cdots \oplus C_r^{\oplus m_r}}$. We say that $(C_i, m_i)_i$ is a *θ-stable decomposition of C* if, for a general representation $M \in C^{ss}_\theta$, its corresponding θ-polystable representation $\widetilde{M}$ is in $\widetilde{C}$, and write

$$C = m_1 C_1 \dotplus \ldots \dotplus m_r C_r. \tag{2}$$

Any $\operatorname{GL}(\mathbf{d})$-invariant, irreducible, closed subvariety C of $\operatorname{rep}(A,\mathbf{d})$ with $C^{ss}_\theta \neq \emptyset$ admits a θ-stable decomposition [CK17, Proposition 3]. This decomposition controls the geometry of irreducible components of moduli spaces in the following sense.

Theorem 3. [CK17, Theorem 1] *Let A be a finite-dimensional algebra and let $C \subseteq \operatorname{rep}(A,\mathbf{d})^{ss}_\theta$ be a $\operatorname{GL}(\mathbf{d})$-invariant, irreducible, closed subvariety. Let $C = m_1C_1 \dotplus \ldots \dotplus m_rC_r$ be a θ-stable decomposition of C where $C_i \subseteq \operatorname{rep}(A,\mathbf{d}_i)$, $1 \leq i \leq r$, are pairwise distinct θ-stable irreducible components, and define $\widetilde{C} = \overline{C_1^{\oplus m_1} \oplus \cdots \oplus C_r^{\oplus m_r}}$.*

(a) If $\mathcal{M}(C)^{ss}_\theta$ is an irreducible component of $\mathcal{M}(A,\mathbf{d})^{ss}_\theta$, then

$$\mathcal{M}(\widetilde{C})^{ss}_\theta = \mathcal{M}(C)^{ss}_\theta.$$

(b) If C_1 is an orbit closure, then

$$\mathcal{M}(\overline{C_1^{\oplus m_1} \oplus \cdots \oplus C_r^{\oplus m_r}})^{ss}_\theta \simeq \mathcal{M}(\overline{C_2^{\oplus m_2} \oplus \cdots \oplus C_r^{\oplus m_r}})^{ss}_\theta.$$

(c) Assume now that none of the C_i are orbit closures. Then there is a natural morphism

$$\Psi \colon S^{m_1}(\mathcal{M}(C_1)^{ss}_\theta) \times \ldots \times S^{m_r}(\mathcal{M}(C_r)^{ss}_\theta) \to \mathcal{M}(\widetilde{C})^{ss}_\theta$$

which is finite, and birational. In particular, if $\mathcal{M}(\widetilde{C})^{ss}_\theta$ is normal then Ψ is an isomorphism.

Note that given any (non-empty) moduli space $\mathcal{M}(A, \mathbf{d})^{ss}_\theta$, its irreducible components are all of the form $\mathcal{M}(C)^{ss}_\theta$ with C a θ-semistable irreducible component of $\operatorname{rep}(A, \mathbf{d})$. Thus, the theorem covers all the irreducible components of $\mathcal{M}(A, \mathbf{d})^{ss}_\theta$ and not just those of some special form.

We also remark that for a Schur-tame algebra, each $\mathcal{M}(C_i)^{ss}_\theta$ appearing in the theorem has dimension 0 if C_i is an orbit closure, and dimension 1 otherwise. Therefore, the dimension of $\mathcal{M}(C)^{ss}_\theta$ is precisely the sum of the multiplicities of the components which are not orbit closures.

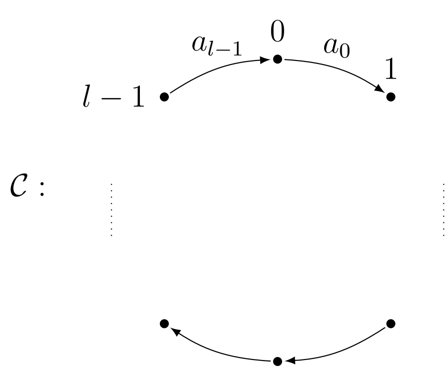
3. Varieties of circular complexes

3.1. **Definition.** Fix a positive integer l and an l-tuple of positive integers $\mathbf{n} = (n_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$ (for convenience in indices, we denote the residue class of an integer i modulo $l\mathbb{Z}$ by the same letter l). We are interested in the variety

$$\operatorname{Comp}(\mathbf{n}) := \{(A_i)_{i \in \mathbb{Z}/l\mathbb{Z}} \in \prod_{i \in \mathbb{Z}/l\mathbb{Z}} \operatorname{Mat}_{n_{i+1} \times n_i}(K) \mid A_{i+1}A_i = 0, \forall i \in \mathbb{Z}/l\mathbb{Z}\},$$

called the *variety of circular complexes associated to* $\mathbf{n}$. By convention, if $l = 1$, we get the variety of matrices A of size $n_0 \times n_0$ with $A^2 = 0$.

Our goal in this section is to describe the irreducible components of $\operatorname{Comp}(\mathbf{n})$. First, we explain how to "label" the irreducible components of $\operatorname{Comp}(\mathbf{n})$ by maximal rank sequences. For this initial step, we follow the same strategy as that for dealing with varieties of (non-circular) complexes (see for example the presentation in [CW13, Section 3]). It is useful to view $\operatorname{Comp}(\mathbf{n})$ as a representation variety for the following bound quiver. Consider the oriented cycle $\mathcal{C}$ with vertex set $\mathbb{Z}/l\mathbb{Z}$:



together with the admissible set of relations $\mathcal{R} := \{a_{i+1}a_i \mid i \in \mathbb{Z}/l\mathbb{Z}\}$. Viewing $\mathbf{n}$ as a dimension vector of $\mathcal{C}$, $\operatorname{Comp}(\mathbf{n})$ is precisely the representation variety $\operatorname{rep}(K\mathcal{C}/\langle\mathcal{R}\rangle, \mathbf{n})$. Furthermore, $K\mathcal{C}/\langle\mathcal{R}\rangle$ is a representation-finite algebra whose indecomposable representations are:

(1) the simples S_i, $i \in \mathbb{Z}/l\mathbb{Z}$;
(2) for each $i \in \mathbb{Z}/l\mathbb{Z}$, the representation $E_{i,i+1}$ defined to be K at vertices $i, i+1$, the identity map along the arrow a_i, and zero at all the other vertices and arrows.

By convention, in case $l = 1$, $\mathcal{C}$ is just the one-loop quiver with $\mathcal{R} = \{a^2\}$ where a denotes the loop of $\mathcal{C}$. The indecomposable representations in this case are the simple S_0 at vertex 0 of $\mathcal{C}$ and the 2-dimensional representation $J_{2,0}$, given by the 2×2 nilpotent Jordan block along the arrow a.

Consequently, if $l > 1$, any $\mathbf{n}$-dimensional representation of $(\mathcal{C}, \mathcal{R})$, M can be written as:

$$M \simeq \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} E_{i,i+1}^{t_i} \oplus \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} S_i^{s_i},$$

where the non-negative integers t_i and s_i, $i \in \mathbb{Z}/l\mathbb{Z}$, satisfy the following conditions:

$$t_{i-1} + t_i + s_i = n_i \text{ and } t_i = \operatorname{rank} M(a_i), \forall i \in \mathbb{Z}/l\mathbb{Z}.$$

If $l = 1$, these equations become $2t_0 + s_0 = n_0$ and $t_0 = \operatorname{rank} M(a)$ where $M \simeq J_{2,0}^{t_0} \oplus S_0^{s_0}$. In either case, we can see that M is uniquely determined, up to isomorphism, by its dimension vector and the rank sequence $(\operatorname{rank} M(a_i))_{i \in \mathbb{Z}/l\mathbb{Z}}$.

In what follows, by a *rank sequence for* $\mathbf{n}$, we mean a sequence $\mathbf{r} = (r_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$ such that there exists an $M \in \operatorname{rep}(K\mathcal{C}/\langle \mathcal{R} \rangle, \mathbf{n})$ with $r_i = \operatorname{rank} M(a_i), \forall i \in \mathbb{Z}/l\mathbb{Z}$; in particular, such an $\mathbf{r}$ must satisfy $r_{i-1} + r_i \leq n_i$ for all $i \in \mathbb{Z}/l\mathbb{Z}$.

3.2. **Properties of irreducible components.** Now, for a rank sequence $\mathbf{r}$ for $\mathbf{n}$, consider the closed subvariety:

$$\operatorname{Comp}(\mathbf{n}, \mathbf{r}) := \{(A_i)_{i \in \mathbb{Z}/l\mathbb{Z}} \in \operatorname{Comp}(\mathbf{n}) \mid \operatorname{rank} A_i \leq r_i, \forall i \in \mathbb{Z}/l\mathbb{Z}\}.$$

From the discussion above, we get that

$$\operatorname{Comp}^0(\mathbf{n}, \mathbf{r}) := \{(A_i)_{i \in \mathbb{Z}/l\mathbb{Z}} \in \operatorname{Comp}(\mathbf{n}) \mid \operatorname{rank} A_i = r_i, \forall i \in \mathbb{Z}/l\mathbb{Z}\}$$

is the $\operatorname{GL}(\mathbf{n})$-orbit in $\operatorname{Comp}(\mathbf{n})$ of

$$M^0(\mathbf{n}, \mathbf{r}) := \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} E_{i,i+1}^{r_i} \oplus \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} S_i^{n_i - r_i - r_{i-1}}.$$

A priori, the rank conditions defining $\operatorname{Comp}(\mathbf{n}, \mathbf{r})$ could lead to a variety with more than one irreducible component; we will now show that this does not happen assuming the rank conditions are maximal.

Lemma 4. *The irreducible components of* $\operatorname{Comp}(\mathbf{n})$ *are precisely those* $\operatorname{Comp}(\mathbf{n}, \mathbf{r})$ *with* $\mathbf{r}$ *a maximal (with respect to the coordinate-wise order) rank sequence for* $\mathbf{n}$*. Moreover, they are normal varieties.*

Proof. Since every irreducible component must have a dense orbit, it is enough to show that

$$\operatorname{Comp}(\mathbf{n}, \mathbf{r}) = \overline{\operatorname{Comp}^0(\mathbf{n}, \mathbf{r})}$$

for all dimension vectors $\mathbf{n}$ and rank sequences $\mathbf{r}$. The containment $\supseteq$ is immediate from semi-continuity of rank; to show the opposite containment, we take an arbitrary point of $\operatorname{Comp}(\mathbf{n}, \mathbf{r})$ and produce an explicit degeneration from $\operatorname{Comp}^0(\mathbf{n}, \mathbf{r})$ to that point. Indeed, let $M \in \mathrm{Comp}(\mathbf{n}, \mathbf{r})$, and set $r'_i := \mathrm{rank}\, M(a_i)$ and $\epsilon_i := r_i - r'_i$ for all $i \in \mathbb{Z}/l\mathbb{Z}$. Then M belongs to the $\mathrm{GL}(\mathbf{n})$-orbit of

$$M^0(\mathbf{n}, \mathbf{r}') := \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} E_{i,i+1}^{r'_i} \oplus \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} S_i^{n_i - r'_i - r'_{i-1}}.$$

Next, for each $\lambda \in K$, consider the representation

$$\bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} E_{i,i+1}^{r'_i} \oplus \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} S_i^{n_i - r_i - r_{i-1}} \oplus \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} E_{i,i+1}(\lambda)^{\epsilon_i},$$

where $E_{i,i+1}$ is K at vertices i and $i+1$, λ along the arrow a_i, and zero elsewhere. This representation is isomorphic to $M^0(\mathbf{n}, \mathbf{r})$ for $\lambda \neq 0$, and to $M^0(\mathbf{n}, \mathbf{r}')$ when $\lambda = 0$. So, we get that $M \in \overline{\mathrm{GL}(\mathbf{n}) M^0(\mathbf{n}, \mathbf{r})} = \overline{\mathrm{Comp}^0(\mathbf{n}, \mathbf{r})}$. This proves our claim that $\mathrm{Comp}(\mathbf{n}, \mathbf{r}) = \overline{\mathrm{Comp}^0(\mathbf{n}, \mathbf{r})}$. In particular, $\mathrm{Comp}(\mathbf{n}, \mathbf{r})$ is an irreducible closed subvariety of $\mathrm{Comp}(\mathbf{n})$ for any rank sequence $\mathbf{r}$ for $\mathbf{n}$.

To see normality, every irreducible component of $\mathrm{Comp}(\mathbf{n})$ is an orbit closure since $K\mathcal{C}/\langle \mathcal{R} \rangle$ is representation-finite. Orbit closures in varieties of circular complexes are examples of orbit closures of nilpotent representations of cyclicly-oriented type $\tilde{A}$ quivers, so [Lus90, Theorem 11.3] gives that irreducible components of $\mathrm{Comp}(\mathbf{n})$ are locally isomorphic to an affine Schubert variety of type A. These varieties are known to be normal, for example by [Fal03, Theorem 8]. □

We need a dimension bound for irreducible components of varieties of circular complexes.

Lemma 5. *The dimension of* $\mathrm{Comp}(\mathbf{n}, \mathbf{r})$ *is less than or equal to* $\frac{1}{2} \sum_{i=0}^{l-1} n_i^2$.

Proof. Consider the product of flag varieties

$$Fl := \prod_{i \in \mathbb{Z}/l\mathbb{Z}} Flag(r_{i-1}, n_i - r_i, V_i)$$

where $Flag(r_{i-1}, n_i - r_i, V_i)$ denotes a two step flag variety (which becomes a Grassmannian if $r_{i-1} + r_i = n_i$) for each $i \in \mathbb{Z}/l\mathbb{Z}$. Now consider the incidence variety:

$$Z(\mathbf{n}, \mathbf{r}) := \{(A_i, (R_i^1, R_i^2))_{i \in \mathbb{Z}/l\mathbb{Z}} \in \mathrm{Comp}(\mathbf{n}, \mathbf{r}) \times Fl \mid \mathrm{Im}(A_{i-1}) \subseteq R_i^1 \subseteq R_i^2 \subseteq \mathrm{Ker}(A_i), \forall i \in \mathbb{Z}/l\mathbb{Z}\}.$$

We have the two projections:

$$\begin{array}{ccc} & Z(\mathbf{n}, \mathbf{r}) & \\ {}^{p}\swarrow & & \searrow^{q} \\ Fl & & \mathrm{Comp}(\mathbf{n}, \mathbf{r}) \end{array}$$

The projection p makes $Z(\mathbf{n}, \mathbf{r})$ a vector bundle over $\prod_{i \in \mathbb{Z}/l\mathbb{Z}} Flag(r_{i-1}, n_i - r_i, V_i)$, so $Z(\mathbf{n}, \mathbf{r})$ is nonsingular, and the map q is a birational isomorphism, since it is an isomorphism over $\mathrm{Comp}^0(\mathbf{n}, \mathbf{r})$. In particular,

$$\dim \mathrm{Comp}(\mathbf{n}, \mathbf{r}) = \dim Z(\mathbf{n}, \mathbf{r}) = \dim Fl + \dim p^{-1}((R_i^1, R_i^2)_{i \in \mathbb{Z}/l\mathbb{Z}})$$

where $(R_i^1, R_i^2)_{i\in\mathbb{Z}/l\mathbb{Z}}$ is an arbitrary flag in Fl. For such a fixed flag, $p^{-1}((R_i^1, R_i^2)_{i\in\mathbb{Z}/l\mathbb{Z}}$ is isomorphic to $\prod_{i\in\mathbb{Z}/l\mathbb{Z}} \mathrm{Hom}_K(V_i/R_i^2, R_{i+1}^1)$, which has dimension $\sum_{i\in\mathbb{Z}/l\mathbb{Z}} r_i^2$. Meanwhile, the formula for the dimension of a flag variety (see for example [Bri05, §1.2]) in this case gives

$$\dim Fl = \sum_{i\in\mathbb{Z}/l\mathbb{Z}} (r_{i-1}+r_i)(n_i - r_{i-1} - r_i) + r_{i-1}r_i.$$

Therefore,

$$\begin{aligned}\dim \mathrm{Comp}(\mathbf{n},\mathbf{r}) &= \sum_{i\in\mathbb{Z}/l\mathbb{Z}} (r_{i-1}+r_i)(n_i - r_{i-1} - r_i) + r_{i-1}r_i + r_i^2\\ &= \sum_{i\in\mathbb{Z}/l\mathbb{Z}} (r_{i-1}+r_i)(n_i - r_{i-1}).\end{aligned}$$

Let $k_i = n_i - r_i - r_{i-1}$. Note that with this notation, $\sum n_i^2 = \sum (r_i + r_{i-1})^2 + 2k_i(r_i + r_{i-1}) + k_i^2$, while $\dim \mathrm{Comp}(\mathbf{n},\mathbf{r}) = \sum (r_{i-1}+r_i)(r_i + k_i)$. Thus, we compute (suppressing the index of summation where convenient):

$$\begin{aligned}\sum n_i^2 - 2\dim \mathrm{Comp}(\mathbf{n},\mathbf{r}) &= \sum (r_i + r_{i-1})^2 + 2k_i(r_i + r_{i-1}) + k_i^2 - 2(r_{i-1}+r_i)(r_i+k_i)\\ &= \sum (r_i + r_{i-1})(r_i + r_{i-1} + 2k_i - 2r_i - 2k_i) + k_i^2\\ &= \sum (r_i + r_{i-1})(r_{i-1} - r_i) + \sum k_i^2\\ &= \sum r_{i-1}^2 - \sum r_i^2 + \sum k_i^2.\end{aligned}$$

Note that the first two sums are equal, since indices are taken modulo l, and the remaining sum is patently positive. Hence, the result follows. □

4. Representation varieties of special biserial algebras and proof of the main result

Our main goal in this section is to check the normality condition in Theorem 3(c), when the algebra in question is special biserial. We do this by reducing the considerations to varieties of circular complexes, whose irreducible components we already know are normal varieties (see Section 3).

4.1. **Special biserial and complete gentle algebras.** We begin by quickly recalling the definition of a special biserial bound quiver algebra (see [SW83]). A bound quiver $(Q,\mathcal{R})$ is called a special biserial bound quiver if:

- (SB1) for each vertex $v \in Q_0$ there are at most two arrows with head v, and at most two arrows with tail v;
- (SB2) for every arrow $a \in Q_1$, there exists at most arrow $b \in Q_1$ such that $ab \notin \langle\mathcal{R}\rangle$, and there exists at most one arrow $c \in Q_1$ such that $ca \notin \langle\mathcal{R}\rangle$.

A special biserial bound quiver $(Q,\mathcal{R})$ is called *gentle* if the following additional properties hold (see [ASS06, §IX.6]):

- (SB3) if a_1 and a_2 are two arrows with the same tail v then, for any arrow b with head v, precisely one of the a_1b and a_2b belongs to $\mathcal{R}$;

(SB4) if b_1 and b_2 are two arrows with the same head v then, for any arrow a with tail v, precisely one of the ab_1 and ab_2 belongs to $\mathcal{R}$;
(SB5) $\mathcal{R}$ consists of paths of length two.

If $(Q, \mathcal{R})$ is gentle, we call $KQ/\langle\mathcal{R}\rangle$ a gentle bound quiver algebra. Any finite-dimensional algebra isomorphic to a gentle bound quiver algebra is called a gentle algebra. An algebra obtained from a gentle algebra by adding only mononial relations is known as a string algebra, and by adding arbitrary relations is a special biserial algebra. The finite-dimensional indecomposable representation for these algebras are well-known. Specifically, an indecomposable representation is either a projective, or string, or band representation (see [BR87], [Rin75]).

As explained by Ringel in [Rin11], a special biserial algebra can be viewed as a quotient of a rather special infinite-dimensional gentle algebra, called a complete gentle algebra.

Definition 6. Let Q^* be a quiver and $\mathcal{R}^*$ a finite set of monomial relations of length two. We say that $(Q^*, \mathcal{R}^*)$ is a *complete gentle quiver with relations* if for every vertex $i \in Q_0^*$, there are precisely two arrows ending at i and precisely two arrows starting at i, and for every arrow $a \in Q_1^*$, there is precisely one arrow $a' \in Q_1^*$ and precisely one arrow $a'' \in Q_1^*$ such that aa' and $a''a$ belong to $\mathcal{R}^*$.

A *complete gentle algebra* is an algebra isomorphic to $KQ^*/\langle\mathcal{R}^*\rangle$ with $(Q^*, \mathcal{R}^*)$ a complete gentle quiver with relations. Note that a complete gentle algebra is infinite-dimensional.

To describe the finite-dimensional indecomposable representations of complete gentle algebras, one uses the recipe developed for dealing with finite-dimensional gentle/string algebras. In particular, the finite-dimensional indecomposable representations are given again by bands and strings. This is due to the work of Ringel in [Rin75], and of Crawley-Boevey in [Cra13] where the more general case of finitely controlled or pointwise artinian indecomposable representations over infinite-dimensional string algebras is discussed.

4.2. **Representation varieties of complete gentle algebras.** Let $(Q^*, \mathcal{R}^*)$ be a complete gentle quiver with relations, $\Lambda = KQ^*/\langle\mathcal{R}^*\rangle$ its complete gentle algebra, and $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0^*}$ a dimension vector. In what follows, by an *effective oriented cycle* of $(Q^*, \mathcal{R}^*)$, we mean an oriented cycle $\mathcal{C} = a_1 \dots a_n$ of Q^* such that $a_i \neq a_j$ for $i \neq j$, and $a_1a_2, \dots, a_{n-1}a_n,\ a_na_1 \in \mathcal{R}^*$. (If $n = 1$, we say that $\mathcal{C} = a_1$ is an effective oriented cycle if $a_1^2 \in \mathcal{R}^*$).

Since each arrow belongs to a unique effective oriented cycle, Q_1 can be written as a disjoint union of subsets of the form $\{a_i\}_{i=1}^n$ (n varying with the subset) where $\mathcal{C} = a_1 \dots a_n$ is an effective oriented cycle. Therefore, the representation variety $\operatorname{rep}(\Lambda, \mathbf{d})$ is a product of varieties of circular complexes. Hence, the irreducible components of $\operatorname{rep}(\Lambda, \mathbf{d})$ are normal varieties by Lemma 4. This is, in fact, one of the key advantages of working with complete gentle algebras.

To describe the irreducible components in more concrete terms, let us recall that a sequence $\mathbf{r} = (r_a)_{a \in Q_1^*}$ of non-negative integers is called a *rank sequence for* $\mathbf{d}$ if there exists an $M \in \operatorname{rep}(\Lambda, \mathbf{d})$ with $r_a = \operatorname{rank} M(a), \forall a \in Q_1^*$. Note that this condition implies that $r_a + r_b \leq \mathbf{d}(ta)$ for any two arrows a, b with $ab \in \mathcal{R}^*$. A rank sequence for $\mathbf{d}$ which is maximal with respect to the coordinate-wise order is called a *maximal rank sequence for* $\mathbf{d}$.

It follows from Section 3 that for any rank sequence $\mathbf{r}$ for $\mathbf{d}$, the set

$$\operatorname{rep}(\Lambda, \mathbf{d}, \mathbf{r}) := \{M \in \operatorname{rep}(\Lambda, \mathbf{d}) \mid \operatorname{rank} M(a) \leq r_a, \forall a \in Q_1^*\}$$

is an irreducible closed subvariety of $\operatorname{rep}(\Lambda, \mathbf{d})$. Moreover, by Lemma 4, the irreducible components of $\operatorname{rep}(\Lambda, \mathbf{d})$ are precisely those $\operatorname{rep}(\Lambda, \mathbf{d}, \mathbf{r})$ with $\mathbf{r}$ a maximal rank sequence for $\mathbf{d}$.

Lemma 7. *Let Λ be a complete gentle algebra and $\mathbf{r}$ a rank sequence for a dimension vector $\mathbf{d}$. Then* $\dim \operatorname{rep}(\Lambda, \mathbf{d}, \mathbf{r}) \leq \sum_{i \in Q_0^*} \mathbf{d}(i)^2 = \dim \operatorname{GL}(\mathbf{d})$.

Proof. We have noted above that $\operatorname{rep}(\Lambda, \mathbf{d}, \mathbf{r})$ is isomorphic to a product of varieties of complexes, say $\prod_j \operatorname{Comp}(\mathbf{n}^j, \mathbf{r}^j)$. Then we have by Lemma 5 that

$$\dim \operatorname{rep}(\Lambda, \mathbf{d}, \mathbf{r}) = \sum_j \dim \operatorname{Comp}(\mathbf{n}^j, \mathbf{r}^j) \leq \sum_j \frac{1}{2} \sum_i (n_i^j)^2.$$

Now for each vertex $i \in Q_0^*$, the value $\mathbf{d}(i)$ appears exactly twice among the values (n_i^j), since each vertex of Q_0^* is a vertex for precisely two varieties of complexes (or the same one twice). So the last double sum simplifies to $\sum_{i \in Q_0^*} \mathbf{d}(i)^2$. □

We also need the concept of a regular irreducible component (for Λ or any special biserial algebra). By this, we mean an irreducible component C that contains a regular representation (i.e. a representation that breaks into a direct sum of only band representations).

We have the following very useful result, which is used in the proof of Proposition 9.

Lemma 8. *Let Λ be a complete gentle algebra, $\mathbf{d}$ a dimension vector, and $\mathbf{r}$ a rank sequence. Assume that $\operatorname{rep}(\Lambda, \mathbf{d}, \mathbf{r})$ is a regular irreducible component of $\operatorname{rep}(\Lambda, \mathbf{d})$. Then:*

$$r_a + r_b = \mathbf{d}(ta), \forall a, b \in Q_1^* \textit{ with } ab \in \mathcal{R}^*.$$

Proof. Choose a regular representation $M \in \operatorname{rep}(\Lambda, \mathbf{d}, \mathbf{r})$. We claim that $\operatorname{rank} M(a) = r_a, \forall a \in Q_1^*$. Indeed, recall that for any finite-dimensional representation X:

$$\sum_{a \in Q_1^*} \operatorname{rank} X(a) = \dim_K X - s,$$

where s is the number of strings occurring in a direct sum decomposition of X into indecomposables. (This follows immediately from the construction of the string and band representations.) Consequently, if $X \in \operatorname{rep}(\Lambda, \mathbf{d}, \mathbf{r})$ is so that $\operatorname{rank} X(a) = r_a$ for all $a \in Q_1^*$, then

$$\sum_{i \in Q_0^*} \mathbf{d}(i) = \sum_{a \in Q_1^*} \operatorname{rank} M(a) \leq \sum_{a \in Q_1^*} \operatorname{rank} X(a) \leq \sum_{i \in Q_0^*} \mathbf{d}(i),$$

and so $\operatorname{rank} M(a) = \operatorname{rank} X(a) = r_a$ for all $a \in Q_1^*$.

Next, given an effective oriented cycle $\mathcal{C}$, we have:

$$\sum_{a \in \mathcal{C}_1} \operatorname{rank} M(a) \leq \frac{1}{2} \sum_{i \in \mathcal{C}_0'} \mathbf{d}(i) + \sum_{j \in \mathcal{C}_0''} \mathbf{d}(j),$$

where $\mathcal{C}_1$ is the set of arrows of $\mathcal{C}$, $\mathcal{C}_0'$ is the set of vertices of $\mathcal{C}$ where $\mathcal{C}$ does not cross itself, and $\mathcal{C}_0''$ is the set of vertices of $\mathcal{C}$ where $\mathcal{C}$ crosses itself twice. The equality holds if and only if $\operatorname{rank} M(a) + \operatorname{rank} M(b) = \mathbf{d}(ta)$ for any two arrows a and b of $\mathcal{C}$ with $ab \in \mathcal{R}$.

So, we get that:

$$\sum_{i\in Q_0^*}\mathbf{d}(i)=\sum_{a\in Q_1^*}\operatorname{rank}M(a)=\sum_{\mathcal{C}}\sum_{a\in\mathcal{C}_1}\operatorname{rank}M(a)\leq\sum_{\mathcal{C}}\left(\frac{1}{2}\sum_{i\in\mathcal{C}_0'}\mathbf{d}(i)+\sum_{j\in\mathcal{C}_0''}\mathbf{d}(j)\right)=\sum_{i\in Q_0^*}\mathbf{d}(i),$$

where last two sums are over all effective oriented cycles in $(Q^*,\mathcal{R}^*)$. Hence, we must have $r_a+r_b=\mathbf{d}(ta)$ for all arrows a and b of Q^* with $ab\in\mathcal{R}^*$. □

4.3. **Proof of the main result.** For two given irreducible components $C\subseteq\operatorname{rep}(A,\mathbf{f})$ and $C'\subseteq\operatorname{rep}(A,\mathbf{f}')$, we set:

$$\hom_A(C,C')=\min\{\dim_K\operatorname{Hom}_A(X,Y)\mid(X,Y)\in C\times C'\}.$$

We are now ready to prove:

Proposition 9. *Let $A=KQ/I$ be an arbitrary special biserial bound quiver algebra. Let $C_i\subseteq\operatorname{rep}(A,\mathbf{d}_i)$, $1\leq i\leq m$, be irreducible components such that a general representation in each C_i is Schur, and that $\hom_A(C_i,C_j)=0$ for all $1\leq i,j\leq m$. Then $C:=\overline{C_1\oplus\ldots\oplus C_m}$ is a normal variety.*

Proof. Take a complete gentle quiver with relations $(Q^*,\mathcal{R}^*)$ such that $Q_0^*=Q_0$ and A is a quotient of $\Lambda:=KQ^*/\langle\mathcal{R}^*\rangle$ by an ideal generated by arrows and admissible relations. We will find a maximal rank sequence $\mathbf{r}$ for a dimension vector $\mathbf{d}$ for Λ such that $C=\operatorname{rep}(\Lambda,\mathbf{d},\mathbf{r})$, with the latter normal by Lemma 4.

Now, for each $1\leq i\leq m$, we have that $\dim C_i=\dim\operatorname{GL}(\mathbf{d}_i)$ by the same arguments in [CC15b, Lemma 3], since C_i is not an orbit closure. But, we can also view $C_i\subseteq\operatorname{rep}(\Lambda,\mathbf{d}_i)$, where the maximal dimension of an irreducible component is $\dim\operatorname{GL}(\mathbf{d}_i)$ by Lemma 7. Therefore, C_i has to be an irreducible component of $\operatorname{rep}(\Lambda,\mathbf{d}_i)$ for each $1\leq i\leq m$; in particular, each C_i is a normal variety.

Next, for each $1\leq i\leq m$, write $C_i=\operatorname{rep}(\Lambda,\mathbf{d}_i,\mathbf{r}^i)$ where $\mathbf{r}^i=(r_a^i)_{a\in Q_1^*}$ is a maximal rank sequence for $\mathbf{d}_i$ with $r_a^i=0,\forall a\in Q_1^*\setminus Q_1$. For $\mathbf{r}:=\mathbf{r}^1+\ldots+\mathbf{r}^m$, we have that $r_a+r_b=\mathbf{d}(ta)$ for all arrows a,b with $ab\in\mathcal{R}^*$ by Lemma 8. Consequently, $\mathbf{r}$ is a maximal rank sequence for $\mathbf{d}$, and so $\operatorname{rep}(\Lambda,\mathbf{d},\mathbf{r})$ is an irreducible component of $\operatorname{rep}(\Lambda,\mathbf{d})$. On the other hand, we have that $C=\overline{\bigoplus_{i=1}^m\operatorname{rep}(\Lambda,\mathbf{d}_i,\mathbf{r}^i)}\subseteq\operatorname{rep}(\Lambda,\mathbf{d},\mathbf{r})$, with the latter having dimension at most $\dim\operatorname{GL}(\mathbf{d})$ by Lemma 7. So we will show that $\dim C=\dim\operatorname{GL}(\mathbf{d})$ as well, forcing equality.

Let $M\in C$ be a general element, so that M is a direct sum of Schur representations with no nonzero morphisms between these summands. Thus $\dim_K\operatorname{End}_A(M)=m=\dim\operatorname{Stab}_{\operatorname{GL}(\mathbf{d})}(M)$. We also know by the general relation between dimensions of orbits and stabilizers that

$$\dim\operatorname{GL}(\mathbf{d})=\dim\operatorname{GL}(\mathbf{d})\cdot M+\dim\operatorname{Stab}_{\operatorname{GL}(\mathbf{d})}(M)=\dim\operatorname{GL}(\mathbf{d})\cdot M+m. \tag{3}$$

On the other hand, C has a dense m-parameter family of distinct orbits, so for a general $M\in C$ we have that

$$\dim C=\dim\operatorname{GL}(\mathbf{d})\cdot M+m. \tag{4}$$

Combining equations (3) and (4) then finishes the proof. □

Proof of Theorem 1. Let Y be an arbitrary irreducible component of $\mathcal{M}(A, \mathbf{d})^{ss}_{\theta}$. Then there exists an irreducible component C of $\operatorname{rep}(A, \mathbf{d})$ such that $Y = \mathcal{M}(C)^{ss}_{\theta}$. Consider the θ-stable decomposition

$$C = m_1 \cdot C_1 \dotplus \ldots \dotplus m_l \cdot C_l$$

as in Definition 2, so that by Theorem 3(a)(c) we have a morphism

$$\Psi \colon S^{m_1}(\mathcal{M}(C_1)^{ss}_{\theta}) \times \ldots \times S^{m_r}(\mathcal{M}(C_r)^{ss}_{\theta}) \to \mathcal{M}(C)^{ss}_{\theta}$$

which is surjective, finite, and birational. By Theorem 3(a) we can assume that $C = \overline{C_1^{m_1} \oplus \ldots \oplus C_l^{m_l}}$ and no C_i is an orbit closure, so each C_i must contain a dense family of band representations.

Next, we claim that $\hom_A(C_i, C_j) = 0$ for all $1 \leq i, j \leq l$. Indeed, for any $1 \leq i, j \leq l$, simply choose two non-isomorphic θ-stable representations X_i and Y_j from C_i and C_j, respectively; this is always possible since each C_i is not an orbit closure. Then $\operatorname{Hom}_A(X_i, Y_j) = 0$ and so $\hom_A(C_i, C_j) = 0$. It now follows from Proposition 9 that C (keeping in mind the reductions above) is normal.

Since A is tame, we already know that each $\mathcal{M}(C_i)^{ss}_{\theta}$ is a rational projective curve (see, for example, [CC15a, Proposition 12]). But $\mathcal{M}(C_i)^{ss}_{\theta}$ is also normal since C_i is normal by the $m = 1$ case in Proposition 9; hence $\mathcal{M}(C_i)^{ss}_{\theta} \simeq \mathbb{P}^1$ for all $1 \leq i \leq l$. We conclude that $\mathcal{M}(C)^{ss}_{\theta} \simeq \prod_{i=1}^{l} \mathbb{P}^{m_i}$. □

REFERENCES

[ABCJP10] I. Assem, T. Brüstle, G. Charbonneau-Jodoin, and P.-G. Plamondon, *Gentle algebras arising from surface triangulations*, Algebra Number Theory **4** (2010), no. 2, 201–229. MR 2592019

[ASS06] I. Assem, D. Simson, and A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1: Techniques of representation theory*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006. MR MR2197389 (2006j:16020)

[BC09] F. M. Bleher and T. Chinburg, *Pullback moduli spaces*, Comm. Algebra **37** (2009), no. 4, 1216–1239. MR 2510980

[BCHZ15] F. M. Bleher, T. Chinburg, and B. Huisgen-Zimmermann, *The geometry of finite dimensional algebras with vanishing radical square*, J. Algebra **425** (2015), 146–178. MR 3295982

[Bob08] G. Bobiński, *On the zero set of semi-invariants for regular modules over tame canonical algebras*, J. Pure Appl. Algebra **212** (2008), no. 6, 1457–1471. MR MR2391660

[Bob12] G. Bobinski, *Semi-invariants for concealed-canonical algebras*, Preprint available at arXiv:1212.4013 [math.RT], 2012.

[Bob14] G. Bobiński, *On moduli spaces for quasitilted algebras*, Algebra Number Theory **8** (2014), no. 6, 1521–1538. MR 3267143

[Bon98] Klaus Bongartz, *Some geometric aspects of representation theory*, Algebras and modules, I (Trondheim, 1996), CMS Conf. Proc., vol. 23, Amer. Math. Soc., Providence, RI, 1998, pp. 1–27. MR 1648601 (99j:16005)

[BR87] M. C. R. Butler and C. M. Ringel, *Auslander-Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra **15** (1987), no. 1-2, 145–179. MR MR876976 (88a:16055)

[Bri05] Michel Brion, *Lectures on the geometry of flag varieties*, Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 33–85. MR 2143072

[BS99] G. Bobiński and A. Skowroński, *Geometry of modules over tame quasi-tilted algebras*, Colloq. Math. **79** (1999), no. 1, 85–118. MR MR1671811 (2000i:14067)

[CBS02] W. Crawley-Boevey and J. Schröer, *Irreducible components of varieties of modules*, J. Reine Angew. Math. **553** (2002), 201–220. MR MR1944812 (2004a:16020)

[CC15a] A. T. Carroll and C. Chindris, *Moduli spaces of modules of Schur-tame algebras*, Algebr. Represent. Theory **18** (2015), no. 4, 961–976. MR 3372127

[CC15b] ______, *On the invariant theory for acyclic gentle algebras*, Trans. Amer. Math. Soc. **367** (2015), no. 5, 3481–3508. MR 3314814

[Chi09] C. Chindris, *Orbit semigroups and the representation type of quivers*, J. Pure Appl. Algebra **213** (2009), no. 7, 1418–1429. MR 2497586 (2010a:16024)

[Chi11] ______, *Geometric characterizations of the representation type of hereditary algebras and of canonical algebras*, Adv. Math. **228** (2011), no. 3, 1405–1434. MR 2824559 (2012h:16033)

[CK16] C. Chindris and D Kline, *On locally semi-simple representations of quivers*, J. Algebra **467** (2016), 284–306. MR 3545962

[CK17] Calin Chindris and Ryan Kinser, *Decomposing moduli of representations of finite-dimensional algebras*, arxiv:1705.10255, 2017.

[CKW15] C. Chindris, R. Kinser, and J. Weyman, *Module varieties and representation type of finite-dimensional algebras*, Int. Math. Res. Not. IMRN (2015), no. 3, 631–650. MR 3340331

[Cra13] W. Crawley-Boevey, *Classification of modules for infinite-dimensional string algebras*, ArXiv e-prints (2013).

[CW13] A. T. Carroll and J. Weyman, *Semi-invariants for gentle algebras*, Noncommutative birational geometry, representations and combinatorics, Contemp. Math., vol. 592, Amer. Math. Soc., Providence, RI, 2013, pp. 111–136. MR 3087942

[dlP91] J. A. de la Peña, *On the dimension of the module-varieties of tame and wild algebras*, Comm. Algebra **19** (1991), no. 6, 1795–1807. MR MR1113958 (92i:16016)

[Dom11] M. Domokos, *On singularities of quiver moduli*, Glasg. Math. J. **53** (2011), no. 1, 131–139. MR 2747139 (2012a:16029)

[Fal03] G. Faltings, *Algebraic loop groups and moduli spaces of bundles*, J. Eur. Math. Soc. (JEMS) **5** (2003), no. 1, 41–68. MR 1961134

[GLFS16] Ch. Geiß, D. Labardini-Fragoso, and J. Schröer, *The representation type of Jacobian algebras*, Adv. Math. **290** (2016), 364–452. MR 3451928

[GS03] Ch. Geiss and J. Schröer, *Varieties of modules over tubular algebras*, Colloq. Math. **95** (2003), no. 2, 163–183. MR MR1967418 (2004d:16026)

[Hil96] L. Hille, *Tilting line bundles and moduli of thin sincere representations of quivers*, An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. **4** (1996), no. 2, 76–82, Representation theory of groups, algebras, and orders (Constanţa, 1995). MR 1428456 (97h:16017)

[HZ98] Birge Huisgen-Zimmermann, *The geometry of uniserial representations of finite-dimensional algebra. I*, J. Pure Appl. Algebra **127** (1998), no. 1, 39–72. MR 1609508

[Kin94] A.D. King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser.(2) **45** (1994), no. 180, 515–530.

[Lus90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498. MR 1035415 (90m:17023)

[MS14] R. J. Marsh and S. Schroll, *The geometry of Brauer graph algebras and cluster mutations*, J. Algebra **419** (2014), 141–166. MR 3253283

[Rie04] Ch. Riedtmann, *Tame quivers, semi-invariants, and complete intersections*, J. Algebra **279** (2004), no. 1, 362–382. MR 2078406 (2005j:16015)

[Rin75] C. M. Ringel, *The indecomposable representations of the dihedral 2-groups*, Math. Ann. **214** (1975), 19–34. MR 0364426

[Rin11] C. Ringel, *Representations of string algebras*, Lecture notes available at https://www.math.uni-bielefeld.de/~ sek/shanghai/2011-3.pdf, 2011.

[RZ04] Ch. Riedtmann and G. Zwara, *On the zero set of semi-invariants for tame quivers*, Comment. Math. Helv. **79** (2004), no. 2, 350–361. MR 2059437 (2005g:16024)

[RZ08] ______, *The zero set of semi-invariants for extended Dynkin quivers*, Trans. Amer. Math. Soc. **360** (2008), no. 12, 6251–6267. MR 2434286 (2009i:14064)

[SW83] Andrzej Skowroński and Josef Waschbüsch, *Representation-finite biserial algebras*, J. Reine Angew. Math. **345** (1983), 172–181. MR 717892

[SW00] A. Skowroński and J. Weyman, *The algebras of semi-invariants of quivers*, Transform. Groups **5** (2000), no. 4, 361–402. MR MR1800533 (2001m:16017)

[Zwa11] Grzegorz Zwara, *Singularities of orbit closures in module varieties*, Representations of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 661–725. MR 2931906

DEPAUL UNIVERSITY, DEPARTMENT OF MATHEMATICAL SCIENCES, CHICAGO, IL, USA
E-mail address, Andrew T. Carroll: acarro15@depaul.edu

UNIVERSITY OF MISSOURI-COLUMBIA, MATHEMATICS DEPARTMENT, COLUMBIA, MO, USA
E-mail address, Calin Chindris: chindrisc@missouri.edu

UNIVERSITY OF IOWA, DEPARTMENT OF MATHEMATICS, IOWA CITY, IA, USA
E-mail address, Ryan Kinser: ryan-kinser@uiowa.edu

UNIVERSITY OF CONNECTICUT, DEPARTMENT OF MATHEMATICS, STORRS, CT, USA
E-mail address, Jerzy Weyman: jerzy.weyman@uconn.edu