QUIVERS, LONG EXACT SEQUENCES AND HORN TYPE INEQUALITIES II

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Abstract. We study the set of all m-tuples $(\lambda(1),\dots,\lambda(m))$ of possible types of finite abelian p-groups $M_{\lambda(1)},\dots,M_{\lambda(m)}$ for which there exists a long exact sequence $M_{\lambda(1)} \to \cdots \to M_{\lambda(m)}$. When $m=3$, we recover Fulton's [6] results on the possible eigenvalues of majorized Hermitian matrices.

1. Introduction

In [5], Friedland asked for a description of the possible eigenvalues of Hermitian matrices A, B, and C such that $B \leq A+C$ (i.e., $A+C-B$ is positive semi-definite). A complete answer to this majorization problem was obtained by Fulton in [6] who showed that the eigenvalues of A, B, and C are given by the same inequalities as in Klyachko's theorem [9] for the case when $B = A+C$, except that the equality $\mathrm{Tr}(B) = \mathrm{Tr}(A) + \mathrm{Tr}(C)$ is replaced by the linear homogeneous inequality $\mathrm{Tr}(B) \leq \mathrm{Tr}(A) + \mathrm{Tr}(C)$. As explained in [6], the problem about the existence of short exact sequences of finite abelian p-groups *without* zeros at the ends has the exact same answer as the majorization problem above. In this paper, we find necessary and sufficient inequalities for the existence of *long* exact sequences, generalizing Fulton's result.

For every partition $\lambda = (\lambda_1,\dots,\lambda_n)$ and a (fixed) prime number p, one can construct a finite abelian p-group $M_\lambda = \mathbb{Z}/p^{\lambda_1} \times \cdots \times \mathbb{Z}/p^{\lambda_n}$. It is known that every finite abelian p-group is isomorphic to M_λ for a unique partition λ. Such a group M_λ is said to be of type λ.

For an integer $n \geq 1$, let

$$\mathcal{P}_n = \{(\lambda_1,\dots,\lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}$$

be the semigroup of all partitions with at most n non-zero parts. Let $m \geq 3$ be a positive integer. We are interested in the set

$$\Sigma(n,m) = \{(\lambda(1),\dots,\lambda(m)) \in \mathcal{P}_n^m \mid \exists\ M_{\lambda(1)} \to M_{\lambda(2)} \to \cdots \to M_{\lambda(m)}\}.$$

The convex cone (in $\mathbb{R}^{nm}$) generated by $\Sigma(n,m)$ is denoted by $\mathcal{C}(n,m)$. Now, we are ready to state our first result:

Theorem 1.1. *Let $m \geq 3$ and $n \geq 1$ be two integers.*

(1) *The set $\Sigma(n,m)$ is a finitely generated subsemigroup of $\mathbb{Z}^{nm}$ and is saturated, i.e., for every integer $r \geq 1$,*

$$(\lambda(1),\dots,\lambda(m)) \in \Sigma(n,m) \Longleftrightarrow (r\lambda(1),\dots,r\lambda(m)) \in \Sigma(n,m).$$

(2) *$C(n,m)$ is a rational convex polyhedral cone and*

$$\dim \mathcal{C}(n,m) = nm.$$

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When m is odd, we obtain a recursive method for describing the cone $\mathcal{C}(n,m)$. For this, we need to recall some of the terminology from [1]. Let $\lambda(i), 1 \le i \le m$, be m partitions. Then the *generalized Littlewood-Richardson coefficient* $f(\lambda(1),\dots,\lambda(m))$ is defined by

$$f(\lambda(1),\dots,\lambda(m)) = \sum c^{\lambda(2)}_{\lambda(1),\mu(1)} \cdot c^{\lambda(3)}_{\mu(1),\mu(2)} \cdots c^{\lambda(m-2)}_{\mu(m-4),\mu(m-3)} \cdot c^{\lambda(m-1)}_{\mu(m-3),\lambda(m)},$$

where the sum is taken over all partitions $\mu(1),\dots,\mu(m-3)$. The convention is that when $m=3$, $f(\lambda(1),\lambda(2),\lambda(3))$ is the Littlewood-Richardson coefficient $c^{\lambda(2)}_{\lambda(1),\lambda(3)}$.

We refer to the notation paragraph at the end of this section for the details of our notations. Now, let $(I_1,\dots,I_m)$ be an m-tuple of subsets of $\{1,\dots,n\}$ such that at least one of them has cardinality at most $n-1$. We define the following weakly decreasing sequences of integers (using conjugate partitions):

$$\underline{\lambda}(I_1) = \lambda'(I_1), \qquad \underline{\lambda}(I_m) = \lambda'(I_m)$$

and for $2 \le i \le m-1$

$$\underline{\lambda}(I_i) = \begin{cases} \lambda'(I_i) & \text{if } i \text{ is even} \\ \lambda'(I_i) - ((|I_i| - |I_{i+1}| - |I_{i-1}|)^{n-|I_i|}) & \text{if } i \text{ is odd.} \end{cases}$$

Let $\mathcal{S}(n,m)$ be the set of all m-tuples $(I_1,\dots,I_m)$ for which:

(1) at least one of the I_i has cardinality at most $n-1$;
(2) $|I_1| = |I_2|$, $|I_{m-1}| = |I_m|$;
(3) $\underline{\lambda}(I_1),\dots,\underline{\lambda}(I_m)$ are partitions;
(4) the generalized Littlewood-Richardson coefficient

$$f(\underline{\lambda}(1),\dots,\underline{\lambda}(m)) = 1.$$

For example, if $m=3$ then $\mathcal{S}(n,3)$ consists of all those triples (I_1,I_2,I_3) of subsets of $\{1,\dots,n\}$ of the same cardinality r with $r<n$ and

$$c^{\lambda(I_2)}_{\lambda(I_1),\lambda(I_3)} = 1.$$

The set $\mathcal{S}(n,m)$ has been used in [1] to construct necessary and sufficient Horn type inequalities for the existence of long exact sequences of finite abelian p-groups with zeros at the ends. As we are going to see, the same set can be used to describe $\mathcal{C}(n,m)$:

Theorem 1.2. *Assume that $m \ge 3$ is odd and let $\lambda(1),\dots,\lambda(m)$ be m weakly decreasing sequences of n non-negative real numbers. Then the following are equivalent:*

(1) $(\lambda(1),\dots,\lambda(m)) \in \mathcal{C}(n,m)$;
(2) *the numbers $\lambda(i)_j$ satisfy*

$$\sum_{i \text{ even}} |\lambda(i)| \le \sum_{i \text{ odd}} |\lambda(i)|,$$

and

$$\sum_{i \text{ even}} \left(\sum_{j \in I_i} \lambda(i)_j \right) \le \sum_{i \text{ odd}} \left(\sum_{j \in I_i} \lambda(i)_j \right)$$

for every $(I_1,\dots,I_m) \in \mathcal{S}(n,m)$; if $m>3$ we also have

$$(\lambda(2),\dots,\lambda(m-1)) \in \mathcal{C}(n,m-2).$$

We should point out that the above theorem fails if m is even (see Example 5.5). Nonetheless, for arbitrary m, a similar description of the cone $C(n,m)$ can be found in Theorem 4.4.

The strategy for proving the main results of this paper is to show first that the existence of long exact sequences of finite abelian p-groups without zeros at the ends is equivalent to the existence of non-zero semi-invariants for a certain quiver. Next, we use methods from quiver invariant theory developed by Derksen and Weyman [2], [3] to prove Theorem 1.1 and to find the Horn type inequalities of Theorem 1.2 and Theorem 4.4.

The paper is organized as follows. In Section 2, we recall some well-known facts about semi-invariants of quivers and introduce the cone of effective weights of quivers without oriented cycles. The quiver setting corresponding to our problem is defined in Section 3 where we prove Theorem 1.1. In Section 4, we give a first description of the cone $\mathcal{C}(n,m)$ and prove Theorem 4.4. The proof of Theorem 1.2 is given in Section 5.

Notations. For a partition λ, we denote by λ' the partition conjugate to λ, i.e., the Young diagram of λ' is the Young diagram of λ reflected in its main diagonal. We will often refer to partitions as Young diagrams. If $\lambda=(\lambda_1,\dots,\lambda_N)$ is a weakly decreasing sequence then we define $r\lambda$ by $r\lambda=(r\lambda_1,\dots,r\lambda_N)$. Let $\lambda=(\lambda_1,\dots,\lambda_N)$ and $\mu=(\mu_1,\dots,\mu_M)$ be two sequences of integers. Then we define the sum $\lambda+\mu$ by first extending λ or μ with zero parts (if necessary) and then we add them componentwise. If $I=\{z_1<\dots<z_r\}$ is an r-tuple of integers then $\lambda(I)$ is defined by $\lambda(I)=(z_r-r,\dots,z_1-1)$. For $r\geq 0$ and a two integers, we denote the r-tuple $(a,\dots,a)$ by (a^r). A composition $\underline{a}$ is just a sequence $\underline{a}=(a_1,\dots,a_n)$ of non-negative integers. For a weakly decreasing sequence μ of n integers, $S^{\mu}(V)$ denotes the irreducible rational representation of $\mathrm{GL}(V)$ with highest weight μ, where V is an n-dimensional complex vector space. Let $\lambda(i)=(\lambda(i)_1,\dots,\lambda(i)_n), 1\leq i\leq 3$, be three weakly decreasing sequences of n integers. Then we define the Littlewood-Richardson coefficient $c^{\lambda(2)}_{\lambda(1),\lambda(3)}$ to be the multiplicity of $S^{\lambda(2)}(\mathbb{C}^n)$ in $S^{\lambda(1)}(\mathbb{C}^n)\otimes S^{\lambda(3)}(\mathbb{C}^n)$, i.e.

$$c^{\lambda(2)}_{\lambda(1),\lambda(3)}=\dim_{\mathbb{C}}\operatorname{Hom}_{\mathrm{GL}_n(\mathbb{C})}(S^{\lambda(2)}(\mathbb{C}^n),S^{\lambda(1)}(\mathbb{C}^n)\otimes S^{\lambda(3)}(\mathbb{C}^n)).$$

If $\underline{a}=(a_1,\dots,a_n)$ is a composition and $\lambda=(\lambda_1,\dots,\lambda_n)$ is a partition with at most n non-zero parts, we define the Kostka number $K_{\underline{a},\lambda}$ to be

$$K_{\underline{a},\lambda}=\dim_{\mathbb{C}}\operatorname{Hom}_{\mathrm{GL}_n(\mathbb{C})}(S^{\lambda}(\mathbb{C}^n),S^{a_1}(\mathbb{C}^n)\otimes\dots\otimes S^{a_n}(\mathbb{C}^n)).$$

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2. Preliminaries

2.1. **Generalities.** A quiver $Q=(Q_0,Q_1,t,h)$ consists of a finite set of vertices Q_0, a finite set of arrows Q_1, and two functions $t,h:Q_1\to Q_0$ that assign to each arrow a its tail ta and its head ha, respectively. We write $ta\xrightarrow{a}ha$ for each arrow $a\in Q_1$.

For simplicity, we will be working over the field $\mathbb{C}$ of complex numbers. A representation V of Q over $\mathbb{C}$ is a family of finite dimensional $\mathbb{C}$-vector spaces $\{V(x)\mid x\in Q_0\}$ together with a family $\{V(a):V(ta)\to V(ha)\mid a\in Q_1\}$ of $\mathbb{C}$-linear maps. If V is a representation of Q, we define its dimension vector $\underline{d}_V$ by $\underline{d}_V(x)=\dim_{\mathbb{C}}V(x)$ for every $x\in Q_0$. Thus the dimension vectors of representations of Q lie in $\Gamma=\mathbb{Z}^{Q_0}$, the set of all integer-valued functions on Q_0. For every vertex x, the dimension vector of the simple representation corresponding to x is denoted by e_x, i.e., $e_x(y)=\delta_{x,y},\forall y\in Q_0$, where $\delta_{x,y}$ is the Kronecker symbol.

Given two representations V and W of Q, we define a morphism $\phi : V \to W$ to be a collection of linear maps $\{\phi(x) : V(x) \to W(x) \mid x \in Q_0\}$ such that

$$\phi(ha)V(a) = W(a)\phi(ta),$$

for every arrow $a \in Q_1$. We denote by $\mathrm{Hom}_Q(V, W)$ the $\mathbb{C}$-vector space of all morphisms from V to W. Let W and V be two representations of Q. We say that V is a subrepresentation of W if $V(x)$ is a subspace of $W(x)$ for all vertices $x \in Q_0$ and $V(a)$ is the restriction of $W(a)$ to $V(ta)$ for all arrows $a \in Q_1$. In this way, we obtain the abelian category $\mathrm{Rep}(Q)$ of all quiver representations of Q. A dimension vector β is said to be a *Schur* root if there exists a β-dimensional representation W such that $\mathrm{End}_Q(W) = \mathbb{C}$.

If α, β are two elements of Γ, we define the Euler form by

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha). \tag{1}$$

2.2. **Semi-invariants for quivers.** Let β be a dimension vector of Q. The representation space of β-dimensional representations of Q is defined by

$$\mathrm{Rep}(Q, \beta) = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\beta(ta)}, \mathbb{C}^{\beta(ha)}).$$

If $\mathrm{GL}(\beta) = \prod_{x \in Q_0} \mathrm{GL}(\beta(x))$ then $\mathrm{GL}(\beta)$ acts algebraically on $\mathrm{Rep}(Q, \beta)$ by simultaneous conjugation, i.e., for $g = (g(x))_{x \in Q_0} \in \mathrm{GL}(\beta)$ and $V = (V(a))_{a \in Q_1} \in \mathrm{Rep}(Q, \beta)$, we define $g \cdot V$ by

$$(g \cdot V)(a) = g(ha)V(a)g(ta)^{-1} \text{ for every } a \in Q_1.$$

Note that $\mathrm{Rep}(Q, \beta)$ is a rational representation of the linearly reductive group $\mathrm{GL}(\beta)$ and the $\mathrm{GL}(\beta)-$orbits in $\mathrm{Rep}(Q, \beta)$ are in one-to-one correspondence with the isomorphism classes of $\beta-$dimensional representations of Q.

From now on, we will assume that our quivers are without oriented cycles. Under this assumption, one can show that there is only one closed $\mathrm{GL}(\beta)-$orbit in $\mathrm{Rep}(Q, \beta)$ and hence the invariant ring $\mathrm{I}(Q, \beta) = \mathbb{C}[\mathrm{Rep}(Q, \beta)]^{\mathrm{GL}(\beta)}$ is exactly the base field $\mathbb{C}$.

Now, consider the subgroup $\mathrm{SL}(\beta) \subseteq \mathrm{GL}(\beta)$ defined by

$$\mathrm{SL}(\beta) = \prod_{x \in Q_0} \mathrm{SL}(\beta(x)).$$

Although there are only constant $\mathrm{GL}(\beta)-$invariant polynomial functions on $\mathrm{Rep}(Q, \beta)$, the action of $\mathrm{SL}(\beta)$ on $\mathrm{Rep}(Q, \beta)$ provides us with a highly non-trivial ring of semi-invariants.

Let $\mathrm{SI}(Q, \beta) = \mathbb{C}[\mathrm{Rep}(Q, \beta)]^{\mathrm{SL}(\beta)}$ be the ring of semi-invariants. As $\mathrm{SL}(\beta)$ is the commutator subgroup of $\mathrm{GL}(\beta)$ and $\mathrm{GL}(\beta)$ is linearly reductive, we have that

$$\mathrm{SI}(Q, \beta) = \bigoplus_{\sigma \in X^\star(\mathrm{GL}(\beta))} \mathrm{SI}(Q, \beta)_\sigma,$$

where $X^\star(\mathrm{GL}(\beta))$ is the group of rational characters of $\mathrm{GL}(\beta)$ and

$$\mathrm{SI}(Q, \beta)_\sigma = \{f \in \mathbb{C}[\mathrm{Rep}(Q, \beta)] \mid gf = \sigma(g)f, \forall g \in \mathrm{GL}(\beta)\}$$

is the space of semi-invariants of weight σ. Note that any $\sigma \in \mathbb{Z}^{Q_0}$ defines a rational character of $\mathrm{GL}(\beta)$ by

$$\{g(x) \mid x \in Q_0\} \in \mathrm{GL}(\beta) \mapsto \prod_{x \in Q_0} (\det g(x))^{\sigma(x)}.$$

In this way, we can identify $\Gamma = \mathbb{Z}^{Q_0}$ with the group $X^\star(\mathrm{GL}(\beta))$ of rational characters of $\mathrm{GL}(\beta)$, assuming that β is a sincere dimension vector (i.e. $\beta(x) > 0$ for all vertices $x \in Q_0$). We also refer to the rational characters of $\mathrm{GL}(\beta)$ as weights.

If $\alpha \in \mathbb{Z}^{Q_0}$, we define the weight $\sigma = \langle \alpha, \cdot \rangle$ by

$$\sigma(x) = \langle \alpha, e_x \rangle \ , \ \forall x \in Q_0.$$

Conversely, it is easy to see that for any weight $\sigma \in \mathbb{Z}^{Q_0}$ there is a unique $\alpha \in \mathbb{Z}^{Q_0}$ (not necessarily a dimension vector) such that $\sigma = \langle \alpha, \cdot \rangle$. Similarly, one can define $\mu = \langle \cdot, \alpha \rangle$.

2.3. **Derksen-Weyman saturation.** We write $\beta_1 \hookrightarrow \beta$ if every β-dimensional representation has a subrepresentation of dimension vector β_1. If $\sigma \in \mathbb{R}^{Q_0}$ and $\beta \in \mathbb{Z}^{Q_0}$ we define $\sigma(\beta)$ to be

$$\sigma(\beta) = \sum_{x \in Q_0} \sigma(x)\beta(x).$$

The *cone of effective weights* associated to (Q, β) is defined by

$$C(Q, \beta) = \{\sigma \in \mathbb{R}^{Q_0} \mid \sigma(\beta) = 0 \text{ and } \sigma(\beta_1) \leq 0 \text{ for all } \beta_1 \hookrightarrow \beta\}.$$

Now, let

$$\Sigma(Q, \beta) = C(Q, \beta) \bigcap \mathbb{Z}^{Q_0}$$

be the semigroup of lattice points of $C(Q, \beta)$. By construction $C(Q, \beta)$ is a rational convex polyhedral cone and hence $\Sigma(Q, \beta)$ is saturated and finitely generated.

In [10], Schofield constructed semi-invariants of quivers with remarkable properties. We should point out that these Schofield semi-invariants have weights of the form $\langle \alpha, \cdot \rangle$, with α dimension vectors. A fundamental result due to Derksen and Weyman [2] (see also [12]) states that each weight space of semi-invariants is spanned by Schofield semi-invariants. An important consequence of this spanning theorem is the following description of $\Sigma(Q, \beta)$ (see [2]):

Theorem 2.1 (Derksen-Weyman saturation)**.** *Let Q be a quiver and let β be a sincere dimension vector. If $\sigma = \langle \alpha, \cdot \rangle \in \mathbb{Z}^{Q_0}$ is a weight with $\alpha \in \mathbb{Z}^{Q_0}$ then the following statements are equivalent:*

(1) $\sigma \in \Sigma(Q, \beta)$;
(2) $\dim \mathrm{SI}(Q, \beta)_\sigma \neq 0$;
(3) *α must be a dimension vector, $\sigma(\beta) = 0$ and $\alpha \hookrightarrow \alpha + \beta$.*

In particular, the dimensions of the weight spaces of semi-invariants are saturated, i.e., if $n \geq 1$ then

$$\dim \mathrm{SI}(Q, \beta)_\sigma \neq 0 \Longleftrightarrow \dim \mathrm{SI}(Q, \beta)_{n\sigma} \neq 0.$$

We also have the following reciprocity property:

Lemma 2.2. [2, Corollary 1] *Let α and β be two dimension vectors. Then*

$$\dim \mathrm{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} = \dim \mathrm{SI}(Q, \alpha)_{-\langle \cdot, \beta \rangle}.$$

Now, we can define $(\alpha \circ \beta)$ by

$$(\alpha \circ \beta) = \dim \mathrm{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} = \dim \mathrm{SI}(Q, \alpha)_{-\langle \cdot, \beta \rangle}.$$

In case β is a Schur root, we have the following refinement of Theorem 2.1 which is also due to Derksen and Weyman [3, Corollary 5.2]:

Proposition 2.3. *Let Q be a quiver with N vertices and let β be a Schur root. Then*

(1) $\dim C(Q,\beta) = N-1$.
(2) $\sigma \in C(Q,\beta)$ *if and only if* $\sigma(\beta) = 0$ *and* $\sigma(\beta_1) \le 0$ *for every decomposition* $\beta = c_1\beta_1 + c_2\beta_2$ *with* β_1, β_2 *Schur roots,* $\beta_1 \circ \beta_2 = 1$ *and* $c_i = 1$ *whenever* $\langle \beta_i, \beta_i \rangle < 0$.

Finally, we record a theorem of Schofield on Schur roots which will be used in the proof of Lemma 4.1.

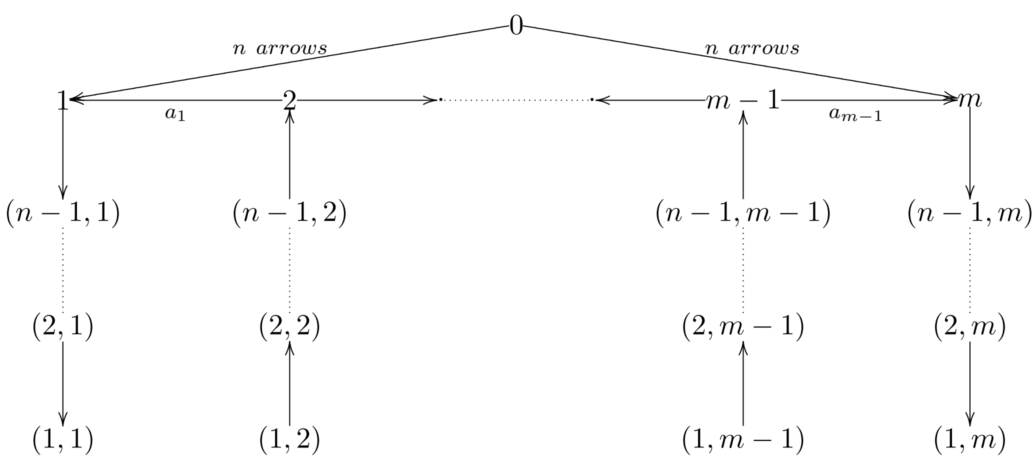
Theorem 2.4. [11, Theorem 6.1] *Let Q be a quiver and let β be a dimension vector. Then the following are equivalent:*

(1) β *is a Schur root;*
(2) $\sigma_\beta(\beta') < 0, \forall\ \beta' \hookrightarrow \beta,\ \beta' \neq 0, \beta,$ *where* $\sigma_\beta = \langle \beta, \cdot \rangle - \langle \cdot, \beta \rangle$.

3. Long exact sequences from semi-invariants

In this section, we show that the existence of long exact sequences of finite abelian p-groups without zeros at the ends is equivalent to the existence of semi-invariants of a certain quiver. To be more precise, let (Q,β) be the following quiver setting:

(1) the quiver Q has $m+1$ central vertices denoted by 0, $1 = (n,1)$, $2 = (n,2)$, $\dots$, $m = (n,m)$ such that at vertices $1, 2, \dots, m$ we attach m equioriented type $\mathbb{A}_n$ quivers (call them flags or arms) $\mathcal{F}(1), \dots, \mathcal{F}(m)$ with $\mathcal{F}(i)$ going in the central vertex i if i is even and going out from the central vertex i if i is odd; there are $m-1$ main arrows $a_1, \dots, a_{m-1}$ connecting the central vertices such that $i+1 \xrightarrow{a_i} i$ if i is odd and $i \xrightarrow{a_i} i+1$ if i is even. Furthermore, there are n arrows going from vertex 0 to vertex 1 and there are n arrows going from 0 to m if m is odd; the n arrows go from m to 0 if m is even. For example, if m is odd, then the quiver Q looks like:



(2) the dimension vector β is given by $\beta(j,i) = j$ for all $j \in \{1, \dots, n\}$, $i \in \{1, \dots, m\}$, and $\beta(0) = 1$, i.e., β is equal to

$$\begin{array}{ccccc} & & 1 & & \\ n & n & \cdots & n & n \\ n-1 & n-1 & \cdots & n-1 & n-1 \\ \vdots & \vdots & & \vdots & \vdots \\ 2 & 2 & \cdots & 2 & 2 \\ 1 & 1 & \cdots & 1 & 1 \end{array}$$

Let $\lambda(1), \ldots, \lambda(m)$ be m sequences of n real numbers. Then we define the weight σ_λ by

$$\sigma_\lambda(j,i) = (-1)^i(\lambda(i)_j - \lambda(i)_{j+1}), \forall 1 \leq j \leq n, \forall 1 \leq i \leq m, \tag{2}$$

with the convention that $\lambda(i)_{n+1} = 0$ and

$$\sigma_\lambda(0) = -\sum_{1\leq i\leq m}\sum_{1\leq j\leq n} \sigma_\lambda(j,i)j = \sum_{i \text{ odd}} |\lambda(i)| - \sum_{i \text{ even}} |\lambda(i)|. \tag{3}$$

Note that (3) is equivalent to $\sigma_\lambda(\beta) = 0$.

Lemma 3.1. *Let $\lambda(1), \ldots, \lambda(m)$ be m partitions with at most n non-zero parts. Then*

$$\dim \mathrm{SI}(Q,\beta)_{\sigma_\lambda} \neq 0 \Longleftrightarrow (\lambda(1), \ldots, \lambda(m)) \in \Sigma(n,m).$$

Proof. First, we compute the space of semi-invariants $\mathrm{SI}(Q,\beta)_{\sigma_\lambda}$. This is a standard computation involving Schur functors. For simplicity, let us define $V_j(i) = \mathbb{C}^{\beta(j,i)}$. Using the same arguments as in [1, Lemma 3.1], one can show that each flag $\mathcal{F}(l)$ going out from the central vertex (n,l) contributes to $\mathrm{SI}(Q,\beta)_{\sigma_\lambda}$ with

$$S^{\gamma^{n-1}(l)}V_n(l),$$

where

$$\gamma^{n-1}(l) = ((n-1)^{-\sigma_\lambda(n-1,l)}, \ldots, 1^{-\sigma_\lambda(1,l)})'.$$

Now, it is easy to see that

$$\gamma^{n-1}(l) = (\lambda(l)_1 - \lambda(l)_n, \ldots, \lambda(l)_{n-1} - \lambda(l)_n).$$

Similarly, if $\mathcal{F}(i)$ is a flag going in the central vertex (n,i), then its contribution to $\mathrm{SI}(Q,\beta)_{\sigma_\lambda}$ is

$$S^{\gamma^{n-1}(i)}V_n^*(i),$$

where

$$\gamma^{n-1}(i) = (\lambda(i)_1 - \lambda(i)_n, \ldots, \lambda(i)_{n-1} - \lambda(i)_n).$$

So far, we have found those spaces of semi-invariants coming from the vertices of the m flags, except for the central vertices i, where $i \in \{0, 1, \ldots, m\}$. Taking into account the weights attached to the central vertices, one can easily see that:

$$\dim \mathrm{SI}(Q,\beta)_{\sigma_\lambda} = \sum K_{\underline{a},\mu(0)} \cdot c^{\lambda(1)}_{\mu(0),\mu(1)} \cdot c^{\lambda(2)}_{\mu(1),\mu(2)} \cdot \cdots \cdot c^{\lambda(m)}_{\mu(m-1),\mu(m)} \cdot K_{\underline{b},\mu(m)},$$

where the sum is over all partitions $\mu(0), \ldots, \mu(m)$ and compositions $\underline{a}, \underline{b}$ with $|\underline{a}| + (-1)^{m+1}|\underline{b}| = |\lambda(1)| - |\lambda(2)| + \cdots + (-1)^{m+1}|\lambda(m)|$.

Now let us prove the implication " $\Rightarrow$ ". If $\dim \mathrm{SI}(Q,\beta)_{\sigma_\lambda} \neq 0$ then there exist partitions $\mu(0), \ldots, \mu(m)$ such that

$$f(\mu(0), \lambda(1), \ldots, \lambda(m), \mu(m)) \neq 0.$$

This together with Klein's theorem [8] imply the existence of a long exact sequence without zeros at the ends of finite abelian p-groups of types $\lambda(1), \ldots, \lambda(m)$, i.e., $(\lambda(1), \ldots, \lambda(m)) \in \Sigma(n,m)$.

For the other implication " $\Leftarrow$ ", we extend the given exact sequence to a long exact sequence with zeros at the ends by taking the kernel (say, of type $\mu(0)$) of the first morphism and the cokernel (say, of type $\mu(m)$) of the last morphism of our long exact sequence. Now, let us break this long exact sequence with zeros at the ends in short exact sequences by taking cokernels:

$$0 \to M_{\mu(0)} \to M_{\lambda(1)} \to M_{\mu(1)} \to 0,$$

$$0 \to M_{\mu(1)} \to M_{\lambda(2)} \to M_{\mu(2)} \to 0,$$
$$\cdots$$
$$0 \to M_{\mu(m-1)} \to M_{\lambda(m)} \to M_{\mu(m)} \to 0.$$

Using Klein's theorem [8], this is equivalent to

$$K_{\underline{a},\mu(0)} \cdot c^{\lambda(1)}_{\mu(0),\mu(1)} \cdot c^{\lambda(2)}_{\mu(1),\mu(2)} \cdots c^{\lambda(m)}_{\mu(m-1),\mu(m)} \cdot K_{\underline{b},\mu(m)} \neq 0,$$

where $\underline{a} = \mu(0)$ and $\underline{b} = \mu(m)$. This implies $\dim \mathrm{SI}(Q,\beta)_{\sigma_\lambda} \neq 0$. □

Remark 3.2. Note the lemma above remains true when we work with the quiver obtained from Q by reversing all arrows. Of course, in this case the new weight is just $-\sigma_\lambda$. This observation is particular useful when proving Lemma 5.4.

Lemma 3.3. *The map*

$$\mathcal{C}(n,m) \longrightarrow C(Q,\beta)$$
$$\lambda = (\lambda(1),\dots,\lambda(m)) \longrightarrow \sigma_\lambda,$$

is an isomorphism of cones that restricts to an isomorphism between the semigroups of the lattice points.

Proof. The map is well-defined because of Lemma 3.1 and the fact that

$$\sigma_{\alpha\lambda+\beta\gamma} = \alpha\sigma_\lambda + \beta\sigma_\gamma,$$

for all α, β (non-negative) real numbers. Note also that the map is injective. To complete the proof, we only need to show that the map is surjective.

Let $\sigma \in \Sigma(Q,\beta)$. For $1 \leq j \leq n$ and $1 \leq i \leq m$, define

$$\beta_1 = \begin{cases} \beta - e_{(j,i)} & \text{if } i \text{ is even} \\ e_{(j,i)} & \text{if } i \text{ is odd} \end{cases}$$

Then it is easy to see that $\beta_1 \hookrightarrow \beta$ and $\sigma(\beta_1) = (-1)^{i+1}\sigma(j,i)$. So, σ must satisfy the so called chamber inequalities, i.e.,

$$(-1)^i\sigma(j,i) \geq 0,$$

for all $1 \leq j \leq n$ and $1 \leq i \leq m$. Now, define $\lambda(i) = (\lambda(i)_1,\dots,\lambda(i)_n)$ by

$$\lambda(i)_j = (-1)^i \sum_{j \leq k \leq n} \sigma(k,i), \forall 1 \leq i \leq m, 1 \leq j \leq n.$$

Then the $\lambda(i)$ are partitions with at most n non-zero parts and $\sigma = \sigma_\lambda$. Hence $(\lambda(1),\dots,\lambda(m)) \in \Sigma(n,m)$ by Lemma 3.1 and this finishes the proof.

□

Lemma 3.4. *The dimension vector β is a Schur root of Q.*

Proof. The dimension vector β is in the fundamental region and the greatest common divisor of its coordinates is one. Then it follows from [7, Theorem B(d)] that β must be a Schur root. □

Proof of Theorem 1.1. (1) This follows from Derksen-Weyman Saturation Theorem 2.1 and Lemma 3.3.

(2) As β is a Schur root, we know that $\dim C(Q,\beta)$ is the number of vertices of Q minus one and so $\dim \mathcal{C}(n,m) = nm$. □

4. Horn type inequalities

We work with the quiver set up (Q, β) introduced in the previous section.

Lemma 4.1. *Let $\lambda(1), \dots, \lambda(m)$ be weakly decreasing sequences of n real numbers. Then*

$$\sigma_\lambda \in C(Q, \beta) \iff \sigma_\lambda(\beta_1) \leq 0,$$

for every dimension vector $\beta_1 \neq \beta$ with $\beta_1 \circ (\beta - \beta_1) = 1$ and β_1 weakly increasing with jumps of at most one along the m flags (from bottom to top).

Proof. The implication " $\implies$ " follows from Theorem 2.1(3).

Now, let us prove " $\impliedby$ ". Using Proposition 2.3 and $\sigma_\lambda(\beta) = 0$, we only need to show

$$\sigma_\lambda(\beta_1) \leq 0,$$

for every decomposition $\beta = c_1\beta_1 + c_2\beta_2$ with $\beta_1,\ \beta_2$ Schur roots and $\beta_1 \circ \beta_2 = 1$.

If β_1 is either $\beta_1 = e_{(j,i)}$ for some $1 \leq j \leq n-1$ and i odd or $\beta_1 = \beta - e_{(j,i)}$ for some $1 \leq j \leq n-1$ and i even then $\sigma_\lambda(\beta_1) \leq 0$ is equivalent to $\lambda(1), \dots, \lambda(m)$ being weakly decreasing sequences.

Now, let us assume β_1 is not of the above form. We are going to show that $c_1 = c_2 = 1$ and that β_1, β_2 are weakly decreasing with jumps of at most one along the m flags (from bottom to top). Let us denote $c_1\beta_1 = \beta', c_2\beta_2 = \beta''$. Since $\beta' \circ \beta'' \neq 0$ it follows from Theorem 2.1 that any representation of dimension vector β has a subrepresentation of dimension vector β'. Therefore, β' must be weakly increasing along each flag going in and it has jumps of at most one along each flag going out.

Next, we will show that β' has jumps of at most one along each flag $\mathcal{F}(i)$ going in a central vertex and β' is weakly increasing along each flag $\mathcal{F}(i)$ going out of a central vertex. For simplicity, let us write

$$\mathcal{F}(i) : 1 \longrightarrow 2 \qquad \cdots \qquad n-1 \longrightarrow n,$$

for a flag going in its central vertex (n, i) (i.e., i is even). Assume to the contrary that there is an $l \in \{1, \dots, n-1\}$ such that $\beta'(l+1) > \beta'(l) + 1$. Then $\beta''(l+1) < \beta''(l)$ which implies that $e_l \hookrightarrow \beta''$. Since β'' is $\langle \beta', \cdot \rangle$-semi-stable it follows that $\langle \beta', e_l \rangle \leq 0$. So, $\beta'(l) \leq \beta'(l-1)$ and hence $\beta'(l) = \beta'(l-1)$ or $\beta''(l) = \beta''(l-1) + 1$. This shows that $c_2 = 1$ and $\beta'' - e_l \hookrightarrow \beta''$. From the fact that $\beta''(= \beta_2)$ is a Schur root and Theorem 2.4 we obtain that β'' is $\sigma_{\beta''}$-stable. Since $e_l \hookrightarrow \beta''$, $\beta'' - e_l \hookrightarrow \beta''$ and $\beta'' \neq e_l$ it follows $\langle \beta'', e_l \rangle - \langle e_l, \beta'' \rangle < 0$ and $\langle \beta'', \beta'' - e_l \rangle - \langle \beta'' - e_l, \beta'' \rangle < 0$. But this is a contradiction. We have just proved that β' has jumps of at most one along each flag going in. Similarly, one can show that β' has to be weakly increasing along each flag going out.

Now, let us show that $c_1 = c_2 = 1$. Since $\beta' = c_1\beta_1$ has jumps of at most one along each flag, we obtain $0 \leq c_1(\beta_1(l+1, i) - \beta_1(l, i)) \leq 1$ for all $l \in \{1, \dots, n-1\}$ and $i \in \{1, \dots, m\}$. If there are l, i such that $\beta_1(l+1, i) - \beta_1(l, i) \neq 0$ then $c_1 = 1$. Otherwise, there is an i such that $\beta'(1, i) = 1$ and so $c_1 = 1$. Similarly, one can show $c_2 = 1$.

In conclusion, $\beta = \beta_1 + \beta_2$ with β_1 weakly increasing with jumps of at most one along the m flags. So, we have $\sigma_\lambda(\beta_1) \leq 0$ and we are done. □

Remark 4.2. We want to point out that some of the inequalities obtained in Lemma 4.1 are redundant. The reason for the redundancy is that some of the β_1 or $\beta_2 = \beta - \beta_1$ above might not be Schur roots.

Example 4.3. Let $n = 1$ and $m \geq 3$. Let $\lambda(i) = (\lambda_i)$ with λ_i non-negative integers, $1 \leq i \leq m$. We show that there exists an exact sequence

$$\mathbb{Z}/p^{\lambda_1} \to \mathbb{Z}/p^{\lambda_2} \to \cdots \to \mathbb{Z}/p^{\lambda_m}$$

if and only if

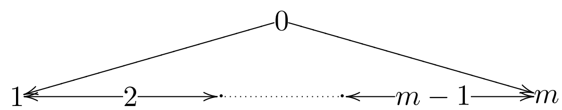
$$\lambda_i - \lambda_{i+1} + \cdots - \lambda_{j-1} + \lambda_j \geq 0,$$

for all even numbers i and j with $2 \leq i \leq j \leq m$ and

$$\lambda_{i'} - \lambda_{i'+1} + \cdots - \lambda_{j'-1} + \lambda_{j'} \geq 0,$$

for all odd numbers i' and j' with $1 \leq i' \leq j' \leq m$.

The quiver we work with in this case is of type $\tilde{\mathbb{A}}_m$. For example, if m is odd then the quiver looks like:



The dimension vector β is

$$\begin{matrix} & & 1 & & \\ n & n & \cdots & n & n. \end{matrix}$$

We want to find all Schur roots β_1 and β_2 such that $\beta_1 \neq \beta$ and $\beta_1 \circ \beta_2 = 1$.

Case 1. If $\beta_1(0) = 1$ then β_1 has to be of the form

$$\beta_1(v) = \begin{cases} 0 & \text{if } i \leq v \leq j, \\ 1 & \text{otherwise,} \end{cases}$$

for two even numbers i and j, $2 \leq i \leq j \leq m$. Conversely, any dimension vector β_1 of this form has the property that $\beta_1,\ \beta - \beta_1$ are Schur roots and $\beta_1 \circ (\beta - \beta_1) = 1$. In this case, we have

$$\sigma_\lambda(\beta_1) = \sum_{\substack{i \leq v \leq j \\ v \text{ odd}}} \lambda_v - \sum_{\substack{i \leq v \leq j \\ v \text{ even}}} \lambda_v.$$

Case 2. If $\beta_1(0) = 0$ then β_1 has to be of the form

$$\beta_1(v) = \begin{cases} 1 & \text{if } i' \leq v \leq j', \\ 0 & \text{otherwise,} \end{cases}$$

for two odd numbers i' and j', $1 \leq i' \leq j' \leq m$. Again, if β_1 is of this form then $\beta_1,\ \beta - \beta_1$ are Schur roots and $\beta_1 \circ (\beta - \beta_1) = 1$. In this case, we have

$$\sigma_\lambda(\beta_1) = \sum_{\substack{i' \leq v \leq j' \\ v \text{ even}}} \lambda_v - \sum_{\substack{i' \leq v \leq j' \\ v \text{ odd}}} \lambda_v.$$

In what follows, we find a closed form of those inequalities obtained in Lemma 4.1. Let β_1 be a dimension vector which is weakly increasing with jumps of at most one along the m flags of Q. Define the sets

$$I_i = \{l \mid \beta_1(l, i) > \beta_1(l-1, i)), 1 \leq l \leq n\}$$

with the convention that $\beta_1(0, i) = 0$ for all $1 \leq i \leq m$. Then it is easy to see that $|I_i| = \beta_1(i), \forall 1 \leq i \leq m$.

Conversely, given an m-tuple $I = (I_1, \dots, I_m)$ of subsets of $\{1, \dots, n\}$, we can construct two dimension vectors β_I and β'_I as follows. If

$$I_i = \{z(i)_1 < \dots < z(i)_r\},$$

we define

$$\beta_I(k,i) = \beta'_I(k,i) = j-1, \forall z(i)_{j-1} \leq k < z(i)_j, \forall 1 \leq j \leq r+1,$$

with the convention that $z(i)_0 = 0$ and $z(i)_{r+1} = n+1$ for all $1 \leq i \leq m$. At vertex 0, we let $\beta_I(0) = 0$ and $\beta'_I(0) = 1$.

Theorem 4.4. *The cone $\mathcal{C}(n,m)$ consists of all m-tuples $(\lambda(1), \dots, \lambda(m))$ of weakly decreasing sequences of n real numbers for which:*

(1)

$$\sum_{i \ even} \left(\sum_{j \in I_i} \lambda(i)_j \right) \leq \sum_{i \ odd} \left(\sum_{j \in I_i} \lambda(i)_j \right),$$

for every m-tuple $I = (I_1, \dots, I_m)$ of subsets of $\{1, \dots, n\}$ with

$$\beta_I \circ (\beta - \beta_I) = 1;$$

(2)

$$\sum_{i \ odd} \left(\sum_{j \notin I_i} \lambda(i)_j \right) \leq \sum_{i \ even} \left(\sum_{j \notin I_i} \lambda(i)_j \right),$$

for every m-tuple $I = (I_1, \dots, I_m)$ of subsets of $\{1, \dots, n\}$ with

$$\beta'_I \circ (\beta - \beta'_I) = 1.$$

Proof. From Lemma 3.3 it follows that

$$(\lambda(1), \dots, \lambda(m)) \in \mathcal{C}(n,m) \iff \sigma_\lambda \in C(Q, \beta).$$

Now, let β_1 be a dimension vector which is weakly increasing with jumps of at most one along the m flags, $\beta_1 \neq \beta$ and $\beta_1 \circ (\beta - \beta_1) = 1$. Let $I = (I_1, \dots, I_m)$ be the jump sets. Then β_1 is β_I if $\beta_1(0) = 0$ or β_1 is β'_I if $\beta_1(0) = 1$. Moreover, we have that

$$\sigma_\lambda(\beta_I) = \sum_{i \ even} \left(\sum_{j \in I_i} \lambda(i)_j \right) - \sum_{i \ odd} \left(\sum_{j \in I_i} \lambda(i)_j \right),$$

$$\sigma_\lambda(\beta'_I) = \sum_{i \ odd} \left(\sum_{j \notin I_i} \lambda(i)_j \right) - \sum_{i \ even} \left(\sum_{j \notin I_i} \lambda(i)_j \right)$$

and, of course, $\sigma_\lambda(\beta) = 0$. The proof follows now from Lemma 4.1. □

Remark 4.5. It is easy to see that if $\lambda(i)$ are weakly decreasing sequences satisfying the conditions (1) and (2) of Theorem 4.4 then $\lambda(i)$ are sequences of non-negative real numbers. Of course, this non-negativity is automatically satisfied in $C(n,m)$.

5. A recursive description

First, we recall a reduction method that appears in [2], [4], [13], and [14].

Lemma 5.1. *Let Q be a quiver and v_0 a vertex such that near v_0, Q looks like:*

$$v_1 \xrightarrow{a} v_0 \xrightarrow{b} w_1.$$

Suppose that β is a dimension vector and σ is a weight such that

$$\beta(v_0) \geq \min\{\beta(w_1), \beta(v_1)\} \text{ and } \sigma(v_0) = 0.$$

Let $\overline{Q}$ be the quiver defined by $\overline{Q}_0 = Q_0 \setminus \{v_0\}$ and $\overline{Q}_1 = (Q_1 \setminus \{a, b\}) \cup \{ba\}$. If $\overline{\beta} = \beta|_{\overline{Q}}$ is the restriction of β and $\overline{\sigma} = \sigma|_{\overline{Q}}$ is the restriction of σ to $\overline{Q}$ then

$$\mathrm{SI}(Q, \beta)_\sigma \cong \mathrm{SI}(\overline{Q}, \overline{\beta})_{\overline{\sigma}}.$$

From now on we will assume that m is odd. Under this assumption, we are able to further describe $\beta_I \circ (\beta - \beta_I)$ and $\beta'_I \circ (\beta - \beta'_I)$. For the convenience of the reader, we recall some of the notations from Section 1. Let $(I_1, \dots, I_m)$ be an m-tuple of subsets of $\{1, \dots, n\}$ such that at least one of them has cardinality at most $n - 1$. We define the following weakly decreasing sequences of integers (using conjugate partitions):

$$\underline{\lambda}(I_1) = \lambda'(I_1),\ \underline{\lambda}(I_m) = \lambda'(I_m)$$

and for $2 \leq i \leq m - 1$

$$\underline{\lambda}(I_i) = \begin{cases} \lambda'(I_i) & \text{if } i \text{ is even} \\ \lambda'(I_i) - ((|I_i| - |I_{i+1}| - |I_{i-1}|)^{n-|I_i|}) & \text{if } i \text{ is odd} \end{cases}$$

Lemma 5.2. *Let $I = (I_1, \dots, I_m)$ be an m-tuple of subsets of $\{1, \dots, n\}$ as above and such that $|I_1| = |I_2|$ and $|I_{m-1}| = |I_m|$. If $\beta_I \circ (\beta - \beta_I) \neq 0$ then $\underline{\lambda}(I_i)$ are partitions and*

$$\beta_I \circ (\beta - \beta_I) = f(\underline{\lambda}(I_1), \dots, \underline{\lambda}(I_m)).$$

Consequently,

$$\beta_I \circ (\beta - \beta_I) = 1 \iff I \in \mathcal{S}(n, m).$$

Proof. Let us denote β_I by β_1 and $\beta - \beta_I$ by β_2. Then we have that

$$\beta_1 \circ \beta_2 = \dim \mathrm{SI}(Q, \beta_1)_{-\langle \cdot, \beta_2 \rangle}.$$

Since $\beta_1(0) = 0$, we can work with the quiver Q' obtained from Q by deleting the vertex 0 and all the arrows going out from this vertex. If β'_1 and β'_2 are the restrictions of β_1 and β_2 to Q', then the restriction of the weight $-\langle \cdot, \beta_2 \rangle$ to Q' is exactly $-\langle \cdot, \beta'_2 \rangle$ as the n arrows connecting vertex 0 and m point towards vertex m. Therefore, we have

$$\beta_1 \circ \beta_2 = \beta'_1 \circ \beta'_2.$$

Let us denote $\langle \beta'_1, \cdot \rangle$ by σ'_1. As $\beta'_1(1) = \beta'_1(2) = |I_1| = |I_2|$ and $\beta'_1(m-1) = \beta'_1(m) = |I_{m-1}| = |I_m|$ it follows that $\sigma'_1(1) = \sigma'_1(m) = 0$.

At this point, we can apply the reduction Lemma 5.1 to reduce Q' to the quiver Q'' obtained from Q' by removing the two vertices 1 and m. Again, it easy to check that if β''_1, β''_2 are the restriction of β'_1, β'_2 to Q'' then

$$\beta'_1 \circ \beta'_2 = \beta''_1 \circ \beta''_2.$$

On the other hand, this reduced quiver Q'' is exactly the generalized flag quiver from [1, Section 3]. It follows from ([1, Lemma 6.4]) that $\underline{\lambda}(i),\ 1 \le i \le m$ are partitions and

$$\beta''_1 \circ \beta''_2 = f(\underline{\lambda}(I_1), \dots, \underline{\lambda}(I_m)).$$

This finishes the proof. □

Remark 5.3. Let $\beta = \beta_1 + \beta_2$ with β_1 weakly increasing with jumps of at most one along the flags and $\beta_1 \circ \beta_2 \neq 0$. We claim that

$$\beta_1(0) = 1 \Rightarrow \beta_1 \text{ is } \beta \text{ along the flags } \mathcal{F}(1) \text{ and } \mathcal{F}(m).$$

Indeed, we have that $\beta_1 \hookrightarrow \beta$ by Theorem 2.1(3). Consider a representation $W \in \operatorname{Rep}(Q, \beta)$ with $\{\operatorname{Im} W(a_i)\}_{1 \le i \le n}$ linearly independent. Since W must have a β_1-dimensional subrepresentation, we obtain that $\beta_1(1) = n$ and so β_1 has to be β along $\mathcal{F}(1)$. Similarly, as m is odd, we have that $\beta_1(0) = 1$ implies that β_1 equals β along the flag $\mathcal{F}(m)$.

Lemma 5.4. *Let $I = (I_1, \dots, I_m)$ be an m-tuple of subsets of $\{1, \dots, n\}$ and let $\lambda(i), 1 \le i \le m$ be weakly decreasing sequences of n non-negative reals.*

(1) *If $\beta'_I \circ (\beta - \beta'_I) \neq 0$ and $(\lambda(2), \dots, \lambda(m-1)) \in \mathcal{C}(n, m-2)$ then*

$$\sigma_\lambda(\beta'_I) \le 0.$$

(2) *Suppose that at least one of the sets $I_1, \dots, I_m$ has cardinality at most $n-1$ and $\beta_I \circ (\beta - \beta_I) = 1$. Furthermore, assume that*

$$\sum_{i \ even} \left(\sum_{j \in J_i} \lambda(i)_j \right) \le \sum_{i \ odd} \left(\sum_{j \in J_i} \lambda(i)_j \right),$$

for every $(J_1, \dots, J_m) \in \mathcal{S}(n, m)$. Then

$$\sigma_\lambda(\beta_I) \le 0.$$

Proof. (1) Let us write $\beta_1 = \beta'_I$ and $\beta_2 = \beta - \beta'_I$. As $\beta_1(0) = 1$ and m is odd it follows from Remark 5.3 that β_1 has to be equal to β along the flags $\mathcal{F}(1)$ and $\mathcal{F}(m)$. In other words, β_2 is zero at vertex 0 and at all vertices of the flags $\mathcal{F}(1)$ and $\mathcal{F}(m)$.

Now, let Q' be the quiver obtained from Q by deleting the vertex 0, the flags $\mathcal{F}(1)$ and $\mathcal{F}(m)$ and all the arrows connected with these deleted vertices. If β'_i is the restriction of β_i to Q', $i \in \{1, 2\}$, then

$$\beta_1 \circ \beta_2 = \beta'_1 \circ \beta'_2.$$

Let Q'' be the quiver obtained from Q' by adding a new vertex 0, n arrows from vertex 2 to 0 and n arrows from vertex $m-1$ to 0. We denote by β''_1 and β''_2 the extensions of β'_1 and β'_2 to Q'' such that $\beta''_1(0) = 1$ and $\beta''_2(0) = 0$. Again, it easy to see that

$$\beta''_1 \circ \beta''_2 = \beta'_1 \circ \beta'_2.$$

Note that Q'' is the quiver corresponding to $\Sigma(n, m-2)$, except that all the arrows have the opposite orientation. So, let us define the weight σ''_λ for Q'' by

$$\sigma''_\lambda(j, i) = (-1)^i (\lambda(i)_j - \lambda(i)_{j+1}), \forall 1 \le j \le n, \forall 2 \le i \le m-1,$$

and $\sigma''_\lambda(0)$ is determined by $\sigma''_\lambda(\beta'') = 0$, where β'' is just the restriction of β to Q''_0.

From Remark 3.2, we deduce that $\sigma''_\lambda \in C(Q'',\beta'')$ if and only if $(\lambda(2),\dots,\lambda(m-1)) \in \mathcal{C}(n,m-2)$. As $\beta''_1 \hookrightarrow \beta''$ and $\sigma''_\lambda \in C(Q'',\beta'')$ it follows that $\sigma''_\lambda(\beta''_1) \leq 0$, i.e.,

$$\sum_{\substack{2\leq i\leq m-1\\ i \text{ even}}} \left(\sum_{j\in I_i} \lambda(i)_j\right) - \sum_{\substack{2\leq i\leq m-1\\ i \text{ odd}}} \left(\sum_{j\in I_i} \lambda(i)_j\right) + \sum_{\substack{2\leq i\leq m-1\\ i \text{ odd}}} |\lambda(i)| - \sum_{\substack{2\leq i\leq m-1\\ i \text{ even}}} |\lambda(i)| \leq 0.$$

In other words, we have

$$\sigma_\lambda(\beta'_I) \leq 0.$$

(2) Let $\alpha_1 = \beta_I$ and $\alpha_2 = \beta - \beta_I$. Again, as $\alpha_1(0) = 0$, we can simplify our quiver by deleting the vertex 0 and all the arrows going out from this vertex. We denote the simplified quiver by $\widetilde{Q}$ and the restriction of the dimension vectors will be noted by $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, and $\widetilde{\beta}$.

Next, we compute the dimension

$$\beta_I \circ (\beta - \beta_I) = \widetilde{\alpha}_1 \circ \widetilde{\alpha}_2 = \dim \mathrm{SI}(\widetilde{Q}, \widetilde{\alpha}_2)_{\langle \widetilde{\alpha}_1, \cdot\rangle}$$

using the same arguments as in Lemma 5.2. Note that the weight $\widetilde{\sigma}_1 = \langle \widetilde{\alpha}_1, \cdot \rangle$ is equal to $\widetilde{\alpha}_1(1) - \widetilde{\alpha}_1(2)$ at vertex 1 and it is equal to $\widetilde{\alpha}_1(m) - \widetilde{\alpha}_1(m-1)$ at vertex m. Furthermore, as $\widetilde{\alpha}_1 \circ \widetilde{\alpha}_2 \neq 0$, we have $\widetilde{\alpha}_1(1) \geq \widetilde{\alpha}_1(2)$ and $\widetilde{\alpha}_1(m) \geq \widetilde{\alpha}_1(m-1)$. To see this, just take $\widetilde{W} \in \mathrm{Rep}(\widetilde{Q}, \widetilde{\beta})$ to be bijective along the main arrows a_1 and a_{m-1}.

Note that $I_1, \dots, I_m$ are the jump sets of $\widetilde{\alpha}_1$ along the m flags of Q. Let J_1 be the subset of I_1 consisting of the first $\widetilde{\alpha}_1(2)$ elements of I_1. Similarly, let J_m be the subset of I_m consisting of the first $\widetilde{\alpha}_1(m-1)$ elements of I_m. As $\widetilde{\alpha}_1 \circ \widetilde{\alpha}_2 \neq 0$, we know that $\underline{\lambda}(J_1)$, $\underline{\lambda}(J_m)$, $\underline{\lambda}(I_i)$ must be partitions for all $2 \leq i \leq m-1$ and

$$\widetilde{\alpha}_1 \circ \widetilde{\alpha}_2 = f(\underline{\lambda}(J_1), \underline{\lambda}(I_2), \dots, \underline{\lambda}(I_{m-1}), \underline{\lambda}(J_m)).$$

It is clear that at least one of the $J_1, I_2, \dots, I_{m-1}, J_m$ has cardinality at most $n-1$, and hence, $(J_1, I_2, \dots, I_{m-1}, J_m) \in S(n,m)$. Therefore, we have

$$\sum_{i \text{ even}} \left(\sum_{j\in I_i} \lambda(i)_j\right) \leq \sum_{j\in J_1} \lambda(1)_j + \sum_{j\in J_m} \lambda(m)_j + \sum_{\substack{2\leq i\leq m-1\\ i \text{ odd}}} \left(\sum_{j\in I_i} \lambda(i)_j\right).$$

As $\lambda(1)_j$ and $\lambda(m)_j$ are assumed to be non-negative for all $1 \leq j \leq n$ we obtain that $\sigma_\lambda(\beta_I) \leq 0$. □

Proof of Theorem 1.2. First, let us prove that (1) $\Rightarrow$ (2). If $I = (I_1, \dots, I_m)$ is an m-tuple in $\mathcal{S}(n,m)$ then $\beta_I \circ (\beta - \beta_I) \neq 0$, by Lemma 5.2 and so $\beta_I \hookrightarrow \beta$. As $\sigma_\lambda \in C(Q,\beta)$, we have that $\sigma_\lambda(\beta_I) \leq 0$ which is equivalent to

$$\sum_{i \text{ even}} \left(\sum_{j\in I_i} \lambda(i)_j\right) \leq \sum_{i \text{ odd}} \left(\sum_{j\in I_i} \lambda(i)_j\right).$$

To obtain the first inequality, we just note that $\beta - e_0 \hookrightarrow \beta$ (this is not true if m is even) and this clearly implies that

$$\sum_{i \text{ even}} |\lambda(i)| \leq \sum_{i \text{ odd}} |\lambda(i)|.$$

Next, it is clear that $(\lambda(1), \dots, \lambda(m)) \in \mathcal{C}(n,m)$ implies $(\lambda(2), \dots, \lambda(m-1)) \in \mathcal{C}(n, m-2)$.

For the other implication (1) $\Leftarrow$ (2), let $I = (I_1, \dots, I_m)$ be an m-tuple of subsets of $\{1, \dots, n\}$. If $|I_i| = n, \forall 1 \le i \le m$ then $\beta_I = \beta - e_0$ and $\beta'_I = \beta$. In this case, we have

$$\sigma_\lambda(\beta_I) = \sum_{i \ even} |\lambda(i)| - \sum_{i \ odd} |\lambda(i)| \le 0,$$

and $\sigma_\lambda(\beta'_I) = 0$.

Now, let us assume that at least one of the I_i has cardinality at most $n-1$. If $\beta'_I \circ (\beta - \beta'_I) = 1$ then $\sigma_\lambda(\beta'_I) \le 0$ by Lemma 5.4(1). If $\beta_I \circ (\beta - \beta_I) = 1$ then it follows from Lemma 5.4(2) that $\sigma_\lambda(\beta_I) \le 0$. The proof follows now from Theorem 4.4. □

Remark 5.5. Let us point out that Theorem 1.2 fails if m is even. For example, one can take $m = 4$, $n = 1$. Then $\lambda(1) = (3)$, $\lambda(2) = (3)$, $\lambda(3) = (1)$, $\lambda(4) = (2)$ give a counterexample to Theorem 1.2.

When $m = 3$ in Theorem 1.2, we recover Fulton's result [6]:

Corollary 5.6 (Majorization problem). *Let $\lambda(1), \lambda(2), \lambda(3)$ be three partitions with at most n non-zero parts. Then the following are equivalent:*

(1) *there exist a short exact sequence of the form*

$$M_{\lambda(1)} \to M_{\lambda(2)} \to M_{\lambda(3)},$$

where $M_{\lambda(i)}$ is a finite abelian p-group of type $\lambda(i)$;

(2) *the numbers $\lambda(i)_j$ satisfy*

$$|\lambda(2)| \le |\lambda(1)| + |\lambda(3)|$$

and

$$\sum_{j \in I_2} \lambda(2)_j \le \sum_{j \in I_1} \lambda(1)_j + \sum_{j \in I_3} \lambda(3)_j$$

for all triples (I_1, I_2, I_3) of subsets of $\{1, \dots, n\}$ of the same cardinality r with $r < n$ and $c^{\lambda(I_2)}_{\lambda(I_1), \lambda(I_3)} = 1$.

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