QUIVERS, LONG EXACT SEQUENCES AND HORN TYPE INEQUALITIES

CALIN CHINDRIS

Abstract. We give necessary and sufficient inequalities for the existence of long exact sequences of m finite abelian p-groups with fixed isomorphism types. This problem is related to some generalized Littlewood-Richardson coefficients that we define in this paper. We also show how this problem is related to eigenvalues of Hermitian matrices satisfying certain (in)equalities. When $m = 3$, we recover the Horn type inequalities that solve the saturation conjecture for Littlewood-Richardson coefficients and Horn's conjecture.

1. Introduction

1.1. **Motivation.** Our main motivation in this paper goes back to the celebrated conjecture of Horn [9] on the possible eigenvalues of a sum of two Hermitian matrices. As explained in Fulton's paper [7], there are problems in other areas of mathematics that have the exact same solution as the eigenvalues of sums of two Hermitian matrices problem. Two of them are the problem concerning the existence of *short* exact sequences of finite abelian p-groups and that of the non-vanishing of the Littlewood-Richardson coefficients. To state these problems, we recall some definitions first. For every partition $\lambda = (\lambda_1, \dots, \lambda_r)$ and a (fixed) prime number p, one can construct a finite abelian p-group $M_\lambda = \mathbb{Z}/p^{\lambda_1} \times \cdots \times \mathbb{Z}/p^{\lambda_r}$. It is known that every finite abelian p-group is isomorphic to M_λ for a unique λ. We will say that such a group is an abelian p-group of type λ.

Let V be a complex vector space of dimension n. If $\lambda = (\lambda_1, \dots, \lambda_n)$ is a weakly decreasing sequence of n integers, we denote by $S^\lambda(V)$ the irreducible rational representation of $\mathrm{GL}(V)$ with highest weight λ. Given three weakly decreasing sequences $\lambda(1),\ \lambda(2),\ \lambda(3)$ of n integers, we define the Littlewood-Richardson coefficient $c^{\lambda(2)}_{\lambda(1),\lambda(3)}$ to be the multiplicity of $S^{\lambda(2)}(V)$ in $S^{\lambda(1)}(V) \otimes S^{\lambda(3)}(V)$, i.e.,

$$c^{\lambda(2)}_{\lambda(1),\lambda(3)} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}(V)}(S^{\lambda(2)}(V), S^{\lambda(1)}(V) \otimes S^{\lambda(3)}(V)).$$

An $n \times n$ complex matrix H is said to be Hermitian if $H = \overline{H}^t$. It is a basic fact that all the eigenvalues of a Hermitian matrix are real numbers. We always write the eigenvalues of a Hermitian matrix in weakly decreasing order.

Now, we can state the three problems mentioned above.

P1. Short exact sequences. For which partitions $\lambda(1), \lambda(2)$, and $\lambda(3)$ with at most n parts, does there exist a short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

where M_i is a finite abelian p-group of type $\lambda(i)$ for every $1 \le i \le 3$.

Date: 19 October, 2004; Revised: December 10, 2007.

2000 *Mathematics Subject Classification.* Primary 16G20; Secondary 05E15.

Key words and phrases. Horn type inequalities, Littlewood-Richardson coefficients, semi-invariants, quivers.

P2. Littlewood-Richardson coefficients. For which weakly decreasing sequences $\lambda(1), \lambda(2)$, and $\lambda(3)$ of n integers, do we have that

$$c^{\lambda(2)}_{\lambda(1),\lambda(3)} \neq 0.$$

P3. Eigenvalues of a sum. For which weakly decreasing sequences $\lambda(1), \lambda(2)$, and $\lambda(3)$ of n real numbers, do there exist $n \times n$ complex Hermitian matrices $H(1), H(2)$, and $H(3)$ with eigenvalues $\lambda(1), \lambda(2)$, and $\lambda(3)$, respectively and

$$H(2) = H(1) + H(3).$$

The equivalence of Problems **P1** and **P2** is due to Klein [12]. In [9], Horn conjectured that the set of solutions to Problem **P3** consists of triples of n-tuples of real numbers arranged in decreasing order satisfying certain linear homogeneous inequalities. In fact, the following result has been proved (we refer to the Notation paragraph at the end of this section for basic definitions and notations).

Theorem 1.1 (**Horn's conjecture**). *Let $\lambda(i) = (\lambda_1(i), \dots, \lambda_n(i))$, $i \in \{1, 2, 3\}$ be three weakly decreasing sequences of n real numbers. Then the following are equivalent:*

(1) *there exist $n \times n$ complex Hermitian matrices $H(1), H(2)$, and $H(3)$ with eigenvalues $\lambda(1), \lambda(2)$, and $\lambda(3)$ respectively, and*

$$H(2) = H(1) + H(3);$$

(2) *the numbers $\lambda_j(i)$ satisfy*

$$|\lambda(2)| = |\lambda(1)| + |\lambda(3)|,$$

together with

$$\sum_{j \in I_2} \lambda_j(2) \leq \sum_{j \in I_1} \lambda_j(1) + \sum_{j \in I_3} \lambda_j(3)$$

for every triple (I_1, I_2, I_3) of subsets of $\{1, \dots, n\}$ of the same cardinality r with $r < n$ and $c^{\lambda(I_2)}_{\lambda(I_1),\lambda(I_3)} \neq 0$.

Assume that $\lambda(i)$ are weakly decreasing sequences of n integers. Then (1) and (2) are equivalent to:

(3) *the Littlewood-Richardson coefficient $c^{\lambda(2)}_{\lambda(1),\lambda(3)}$ is not zero.*

Assume that $\lambda(i)$ are partitions with at most n parts. Then $(1)-(3)$ are equivalent to:

(4) *there exists a short exact sequence*

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

where M_i is a finite abelian p-group of type $\lambda(i)$ for every $1 \leq i \leq 3$.

The first step in solving Horn's conjecture was taken by Klyachko [13] who proved the equivalence of (1) and (2) in Theorem 1.1. In the same paper, Klyachko made the connection between his solution to the eigenvalue problem and the Littlewood-Richardson coefficients. The next step was taken by Knutson and Tao [14] who proved what is now known as the *Saturation Conjecture* for the Littlewood-Richardson coefficients. Their proof is based on some combinatorial gadgets called honeycombs. Derksen and Weyman [4] proved the Saturation Conjecture in the more general context of quiver theory. The set of solutions to Problem **P3** forms a rational convex polyhedral cone $\mathcal{K}(n, 3)$ in $\mathbb{R}^{3n}$, known as the Klyachko's cone. In a subsequent paper [15], Knutson, Tao and Woodward have described all the facets of the Klyachko's cone. This way, they have obtained a minimal list of Horn type inequalities defining the Klyachko's cone:

Theorem 1.2. [15] *The Klyachko's cone $\mathcal{K}(n,3)$ consists of triples $(\lambda(1),\lambda(2),\lambda(3))$ of weakly decreasing sequences of n real numbers for which*

$$|\lambda(2)| = |\lambda(1)| + |\lambda(3)|$$

and

$$\sum_{j\in I_2} \lambda_j(2) \le \sum_{j\in I_1} \lambda_j(1) + \sum_{j\in I_3} \lambda_j(3)$$

for every triple (I_1, I_2, I_3) of subsets of $\{1,\dots,n\}$ of the same cardinality r with $r < n$ and $c^{\lambda(I_2)}_{\lambda(I_1),\lambda(I_3)} = 1$; furthermore, this is now a minimal list.

As shown in [1], [2], [3], and [5] most of the above results proved by Klyachko, Knutson, Tao and Woodward can be naturally obtained using quiver theory.

1.2. **The generalized problems.** When focusing on the existence of *short* exact sequences, it seems natural to extend Problem **P1** to the case of *long* exact sequences with zeros at the ends of finite abelian p-groups. Since a long exact sequence breaks into short exact sequences, we replace the Littlewood-Richardson coefficient in Problem **P2** with a sum of products of Littlewood-Richardson coefficients.

Let $m \ge 3$ and $n \ge 1$ be two integers.

Definition 1.3. Given m weakly decreasing sequences $\lambda(1),\dots,\lambda(m)$ of n integers, the *generalized Littlewood-Richardson coefficient* $f(\lambda(1),\dots,\lambda(m))$ is defined as follows:

$$f(\lambda(1),\dots,\lambda(m)) = \sum c^{\lambda(2)}_{\lambda(1),\mu(1)} \cdot c^{\lambda(3)}_{\mu(1),\mu(2)} \cdots c^{\lambda(m-2)}_{\mu(m-4),\mu(m-3)} \cdot c^{\lambda(m-1)}_{\mu(m-3),\lambda(m)},$$

where the sum is taken over all partitions $\mu(1),\dots,\mu(m-3)$ with at most n parts.

The convention is that when $m = 3$, $f(\lambda(1),\lambda(2),\lambda(3))$ is the Littlewood-Richardson coefficient $c^{\lambda(2)}_{\lambda(1),\lambda(3)}$. As it turns out, the generalized Littlewood-Richardson coefficients are also related with parabolic affine Kazhdan-Lusztig polynomials and decomposition numbers for q-Schur algebras. This will be explained in Section 8.

Now, we are ready to state our generalized problems.

Q1. Long exact sequences. For which partitions $\lambda(1),\dots,\lambda(m)$ with at most n parts, does there exist a long exact sequence

$$0 \to M_1 \to M_2 \to \cdots \to M_m \to 0,$$

where M_i is a finite abelian p-group of type $\lambda(i)$ for every $1 \le i \le m$.

Q2. Generalized Littlewood-Richardson coefficients. For which weakly decreasing sequences $\lambda(1),\dots,\lambda(m)$ of n integers, do we have that

$$f(\lambda(1),\dots,\lambda(m)) \neq 0.$$

Q3. Generalized eigenvalue problems. For which weakly decreasing sequences $\lambda(1),\dots,\lambda(m)$ of n real numbers, do there exist $n\times n$ complex Hermitian matrices $H(1),\dots,H(m)$ with eigenvalues $\lambda(1),\dots,\lambda(m)$ and

$$\sum_{i\ even} H(i) = \sum_{i\ odd} H(i);$$

if $m > 3$ we also require

$$\sum_{1\le j\le i} (-1)^{i+j} H(j) \text{ to have non-negative eigenvalues,}$$

for every $2 \leq i \leq m-2$.

We should point out that Problem **Q3** is quite different than Problem **P3** due to the condition on the eigenvalues of the alternating partial sums of Hermitian matrices. Our goal in this paper is to show that the three generalized problems have the exact same answer, generalizing in this way Theorem 1.1

1.3. **Statement of the results.** Our first result is the following saturation property of the generalized Littlewood-Richardson coefficients:

Theorem 1.4 (**Saturation property**). *Let $\lambda(1), \ldots, \lambda(m)$ be m weakly decreasing sequences of n integers. Then for every integer $r \geq 1$ we have*

$$f(\lambda(1), \ldots, \lambda(m)) \neq 0 \Longleftrightarrow f(r\lambda(1), \ldots, r\lambda(m)) \neq 0.$$

Next, we relate the generalized Littlewood-Richardson coefficients with the generalized spectral problem above.

Definition 1.5. Let $\mathcal{K}(n,m) \subseteq \mathbb{R}^{nm}$ be the solution set to Problem **Q3,** i.e, $\mathcal{K}(n,m)$ is the set of all m-tuples $(\lambda(1), \ldots, \lambda(m))$ of weakly decreasing sequences of n reals for which there exist $n \times n$ complex Hermitian matrices $H(i),\ i \in \{1, \ldots, m\}$ satisfying the conditions of Problem **Q3.** We call $\mathcal{K}(n,m)$ the *generalized Klyachko's cone.*

To describe the generalized Klyachko's cone, we need to introduce some notation. Let $(I_1, \ldots, I_m)$ be an m-tuple of subsets of $\{1, \ldots, n\}$ such that at least one of them has cardinality at most $n-1$. We define the following weakly decreasing sequences of integers (using conjugate partitions):

$$\underline{\lambda}(I_1) = \lambda'(I_1), \qquad \underline{\lambda}(I_m) = \begin{cases} \lambda'(I_m) & \text{if } m \text{ is odd} \\ \lambda'(I_m \setminus \{n\}) & \text{if } m \text{ is even,} \end{cases}$$

and for $2 \leq i \leq m-1$

$$\underline{\lambda}(I_i) = \begin{cases} \lambda'(I_i) & \text{if } i \text{ is even} \\ \lambda'(I_i) - ((|I_i| - |I_{i+1}| - |I_{i-1}|)^{n-|I_i|}) & \text{if } i \leq m-2 \text{ is odd} \\ \lambda'(I_i) - ((|I_{m-1}| - |I_{m-2}| - |I_m \setminus \{n\}|)^{n-|I_i|}) & \text{if } i = m-1 \text{ is odd.} \end{cases}$$

Now, we can state our **generalization of Horn's conjecture:**

Theorem 1.6. *Let $\lambda(i) = (\lambda_1(i), \ldots, \lambda_n(i)),\ i \in \{1, \ldots, m\}$ be m weakly decreasing sequences of n real numbers. Then the following are equivalent:*

(1) $(\lambda(1), \ldots, \lambda(m)) \in \mathcal{K}(n,m)$;

(2) *the numbers $\lambda_j(i)$ satisfy*

$$\sum_{i \text{ even}} |\lambda(i)| = \sum_{i \text{ odd}} |\lambda(i)|$$

together with

$$(*) \qquad \sum_{i \text{ even}} \left(\sum_{j \in I_i} \lambda_j(i) \right) \leq \sum_{i \text{ odd}} \left(\sum_{j \in I_i} \lambda_j(i) \right)$$

for every m-tuple $(I_1, \ldots, I_m)$ for which $|I_1| = |I_2|$, $|I_{m-1}| = |I_m|$, $\underline{\lambda}(I_i),\ 1 \leq i \leq m$ are partitions and

$$f(\underline{\lambda}(I_1), \ldots, \underline{\lambda}(I_m)) \neq 0.$$

Assume that $\lambda(i)$ are sequences of integers. Then $(1)-(2)$ are equivalent to:

(3) $f(\lambda(1), \dots, \lambda(m)) \neq 0$.
Assume that $\lambda(i)$ are partitions. Then $(1) - (3)$ are equivalent to:
(4) *there exists a long exact sequence of the form*

$$0 \to M_1 \to M_2 \to \cdots \to M_m \to 0,$$

where M_i is a finite abelian p-group of type $\lambda(i)$ for every $1 \leq i \leq m$.

Note that the theorem above gives a recursive method for finding all non-zero generalized Littlewood-Richardson coefficients. It turns out that one can shorten the list of Horn type inequalities of Theorem 1.6(2):

Proposition 1.7. *The following statements are true.*

(1) *We have*

$$\dim \mathcal{K}(n, m) = mn - 1.$$

(2) *The cone $\mathcal{K}(n, m)$ consists of all m-tuples $(\lambda(1), \dots, \lambda(m))$ of weakly decreasing sequences of n reals for which*

$$\sum_{i \text{ even}} |\lambda(i)| = \sum_{i \text{ odd}} |\lambda(i)|$$

and $(*)$ holds for every m-tuple $(I_1, \dots, I_m)$ for which $|I_1| = |I_2|$, $|I_{m-1}| = |I_m|$, $\underline{\lambda}(I_i)$, $1 \leq i \leq m$ are partitions and

$$f(\underline{\lambda}(I_1), \dots, \underline{\lambda}(I_m)) = 1.$$

We want to point out that our results do not depend on the work of Klyachko, Knutson and Tao. In fact, our strategy is to show first that the non-vanishing of the generalized Littlewood-Richardson coefficients is equivalent to the existence of non-zero semi-invariants for the generalized flag quiver setting. Once we have switched to quiver invariant theory, our main tool is Derksen-Weyman's [5] description of the facets of the cone of effective weights for quivers without oriented cycles.

The paper is organized as follows. In Section 2, we recall the saturation theorem for effective weights of quivers which is due to Derksen and Weyman [4]. The generalized flag quiver setting is defined in Section 3 where we also prove the saturation property for the generalized Littlewood-Richardson coefficients. A more detailed description of the so called cone of effective weights for arbitrary quivers (without oriented cycles) is given in Section 4. In Section 5, we find the facets of the cone of effective weights associated to the generalized flag quiver setting. The Horn type inequalities and the m-tuples $(I_1, \dots, I_m)$ occurring in Theorem 1.6(2) are obtained in Section 6. In Section 7, we give a moment map description of the cone associated to the generalized flag quiver setting and prove Theorem 1.6 and Proposition 1.7. In Section 8, we discuss two representation theoretic interpretations of the generalized Littlewood-Richardson coefficients. First, we explain how the generalized Littlewood-Richardson coefficients are related to some parabolic affine Kazhdan-Lusztig polynomials and decomposition numbers for q-Schur algebras. We also show how our coefficients can be viewed as multiplicities of irreducible representations of a product of general linear groups.

Notation. A partition λ of length N is a sequence of N positive integers $\lambda = (\lambda_1, \dots, \lambda_N)$ with $\lambda_1 \geq \cdots \geq \lambda_N \geq 1$. We say that λ is a partition with at most N (non-zero) parts if $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N$ with $\lambda_1 \geq \cdots \geq \lambda_N \geq 0$. If $\lambda = (\lambda_1, \dots, \lambda_N)$ is a weakly decreasing sequence then we define $r\lambda$ by $r\lambda = (r\lambda_1, \dots, r\lambda_N)$. Let $\lambda = (\lambda_1, \dots, \lambda_N)$ and $\mu = (\mu_1, \dots, \mu_M)$ be two weakly decreasing sequences of integers. Then we define the sum $\lambda + \mu$ by first extending

λ or μ with zero parts (if necessary) and then we add them componentwise. For a partition λ, we denote by λ' the partition conjugate to λ, i.e., the Young diagram of λ' is the Young diagram of λ reflected with respect to its main diagonal. We will often refer to partitions as Young diagrams. If $I = \{z_1 < \cdots < z_r\}$ is an r-tuple of integers then $\lambda(I)$ is defined by $\lambda(I) = (z_r - r, \ldots, z_1 - 1)$. For $r \geq 0$ and a two integers, we denote the r-tuple $(a, \ldots, a)$ by (a^r). If $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a sequence of real numbers, we define $|\lambda| = \sum_{i=1}^{N} \lambda_i$.

Acknowledgment. I would like to thank William Fulton and Jerzy Weyman for helpful comments on a preliminary version of this work. I am especially grateful to my advisor, Harm Derksen, for drawing my attention to these problems, for his continuing support and many enlightening discussions on the subject.

2. Preliminaries

2.1. **Generalities.** A finite quiver $Q = (Q_0, Q_1, t, h)$ consists of a finite set of vertices Q_0, a finite set of arrows Q_1 and two functions $t, h : Q_1 \to Q_0$ that assign to each arrow a its tail ta and its head ha, respectively. We write $ta \xrightarrow{a} ha$ for each arrow $a \in Q_1$.

For simplicity, we will be working over the field of complex numbers $\mathbb{C}$. A representation V of Q over $\mathbb{C}$ is a family of finite dimensional $\mathbb{C}$-vector spaces $\{V(x) \mid x \in Q_0\}$ together with a family $\{V(a) : V(ta) \to V(ha) \mid a \in Q_1\}$ of $\mathbb{C}$-linear maps. If V is a representation of Q, we define its dimension vector $\underline{d}_V$ by $\underline{d}_V(x) = \dim_{\mathbb{C}} V(x)$ for every $x \in Q_0$. Thus the dimension vectors of representations of Q lie in $\Gamma = \mathbb{Z}^{Q_0}$, the set of all integer-valued functions on Q_0. For each vertex x, we denote by ε_x the simple dimension vector corresponding to x, i.e., $\varepsilon_x(y) = \delta_{x,y}$, $\forall\, y \in Q_0$, where $\delta_{x,y}$ is the Kronecker symbol.

Given two representations V and W of Q, we define a morphism $\phi : V \to W$ to be a collection of linear maps $\{\phi(x) : V(x) \to W(x) \mid x \in Q_0\}$ such that for every arrow $a \in Q_1$, we have $\phi(ha)V(a) = W(a)\phi(ta)$. We denote by $\mathrm{Hom}_Q(V, W)$ the $\mathbb{C}$-vector space of all morphisms from V to W. In this way, we obtain the abelian category $\mathrm{Rep}(Q)$ of all quiver representations of Q. Let W and V be two representations of Q. We say that V is a subrepresentation of W if $V(x)$ is a subspace of $W(x)$ for all vertices $x \in Q_0$ and $V(a)$ is the restriction of $W(a)$ to $V(ta)$ for all arrows $a \in Q_1$.

If α, β are two elements of Γ, we define the Euler form by

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha). \tag{1}$$

2.2. **Semi-invariants for quivers.** Let β be a dimension vector of Q. The representation space of $\beta-$dimensional representations of Q is defined by

$$\mathrm{Rep}(Q, \beta) = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\beta(ta)}, \mathbb{C}^{\beta(ha)}).$$

If $\mathrm{GL}(\beta) = \prod_{x \in Q_0} \mathrm{GL}(\beta(x))$ then $\mathrm{GL}(\beta)$ acts algebraically on $\mathrm{Rep}(Q, \beta)$ by simultaneous conjugation, i.e., for $g = (g(x))_{x \in Q_0} \in \mathrm{GL}(\beta)$ and $V = \{V(a)\}_{a \in Q_1} \in \mathrm{Rep}(Q, \beta)$, we define $g \cdot V$ by

$$(g \cdot V)(a) = g(ha)V(a)g(ta)^{-1} \text{ for each } a \in Q_1.$$

In this way, $\mathrm{Rep}(Q, \beta)$ becomes a rational representation of the linearly reductive group $\mathrm{GL}(\beta)$ and the $\mathrm{GL}(\beta)-$orbits in $\mathrm{Rep}(Q, \beta)$ are in one-to-one correspondence with the isomorphism classes of $\beta-$dimensional representations of Q.

From now on, we will assume that our quivers are without oriented cycles. As Q is a quiver without oriented cycles, one can show that there is only one closed $\mathrm{GL}(\beta)-$orbit in $\mathrm{Rep}(Q,\beta)$ and hence the invariant ring $\mathrm{I}(Q,\beta)=\mathbb{C}[\mathrm{Rep}(Q,\beta)]^{\mathrm{GL}(\beta)}$ is exactly the base field $\mathbb{C}$.

Now, consider the subgroup $\mathrm{SL}(\beta)\subseteq \mathrm{GL}(\beta)$ defined by

$$\mathrm{SL}(\beta)=\prod_{x\in Q_0}\mathrm{SL}(\beta(x)).$$

Although there are only constant $\mathrm{GL}(\beta)-$invariant polynomial functions on $\mathrm{Rep}(Q,\beta)$, the action of $\mathrm{SL}(\beta)$ on $\mathrm{Rep}(Q,\beta)$ provides us with a highly non-trivial ring of semi-invariants. Note that any $\sigma\in\mathbb{Z}^{Q_0}$ defines a rational character of $\mathrm{GL}(\beta)$ by

$$\{g(x)\mid x\in Q_0\}\in \mathrm{GL}(\beta)\mapsto \prod_{x\in Q_0}(\det g(x))^{\sigma(x)}.$$

In this way, we can identify $\Gamma=\mathbb{Z}^{Q_0}$ with the group $X^\star(\mathrm{GL}(\beta))$ of rational characters of $\mathrm{GL}(\beta)$, assuming that β is a sincere dimension vector (i.e. $\beta(x)>0$ for all vertices $x\in Q_0$). We also refer to the rational characters of $\mathrm{GL}(\beta)$ as weights.

Let $\mathrm{SI}(Q,\beta)=\mathbb{C}[\mathrm{Rep}(Q,\beta)]^{\mathrm{SL}(\beta)}$ be the ring of semi-invariants. As $\mathrm{SL}(\beta)$ is the commutator subgroup of $\mathrm{GL}(\beta)$ and $\mathrm{GL}(\beta)$ is linearly reductive, we have that

$$\mathrm{SI}(Q,\beta)=\bigoplus_{\sigma\in X^\star(\mathrm{GL}(\beta))}\mathrm{SI}(Q,\beta)_\sigma,$$

where $X^\star(\mathrm{GL}(\beta))$ is the group of rational characters of $\mathrm{GL}(\beta)$ and

$$\mathrm{SI}(Q,\beta)_\sigma=\{f\in\mathbb{C}[\mathrm{Rep}(Q,\beta)]\mid g\cdot f=\sigma(g)f,\text{for all }g\in\mathrm{GL}(\beta)\}$$

is the space of semi-invariants of weight σ.

If $\alpha\in\Gamma$, we define $\sigma=\langle\alpha,\cdot\rangle$ by

$$\sigma(x)=\langle\alpha,\varepsilon_x\rangle\ ,\ \forall x\in Q_0.$$

Conversely, it is easy to see that for any weight $\sigma\in\mathbb{Z}^{Q_0}$ there is a unique $\alpha\in\mathbb{Z}^{Q_0}$ (not necessarily a dimension vector) such that $\sigma=\langle\alpha,\cdot\rangle$. Similarly, one can define $\sigma=\langle\cdot,\alpha\rangle$.

Given a quiver Q and a dimension vector β, we define the set $\Sigma(Q,\beta)$ of *(integral) effective weights* by

$$\Sigma(Q,\beta)=\{\sigma\in\mathbb{Z}^{Q_0}\mid \mathrm{SI}(Q,\beta)_\sigma\neq 0\}.$$

In [17], Schofield constructed semi-invariants of quivers with remarkable properties. We should point out that Schofield's semi-invariants have weights of the form $\langle\alpha,\cdot\rangle$, with α dimension vectors. A fundamental result due to Derksen and Weyman [4] (see also [19]) states that each weight space of semi-invariants is spanned by such semi-invariants. An important consequence of this spanning theorem is the following saturation theorem.

Theorem 2.1. [4, Theorem 3] *If Q is a quiver and β is a dimension vector, then the set*

$$\Sigma(Q,\beta)=\{\sigma\in\mathbb{Z}^{Q_0}\mid \mathrm{SI}(Q,\beta)_\sigma\neq 0\},$$

is saturated, i.e., if σ is a weight and $r\geq 1$ is an integer,

$$\mathrm{SI}(Q,\beta)_\sigma\neq 0\Longleftrightarrow \mathrm{SI}(Q,\beta)_{r\sigma}\neq 0.$$

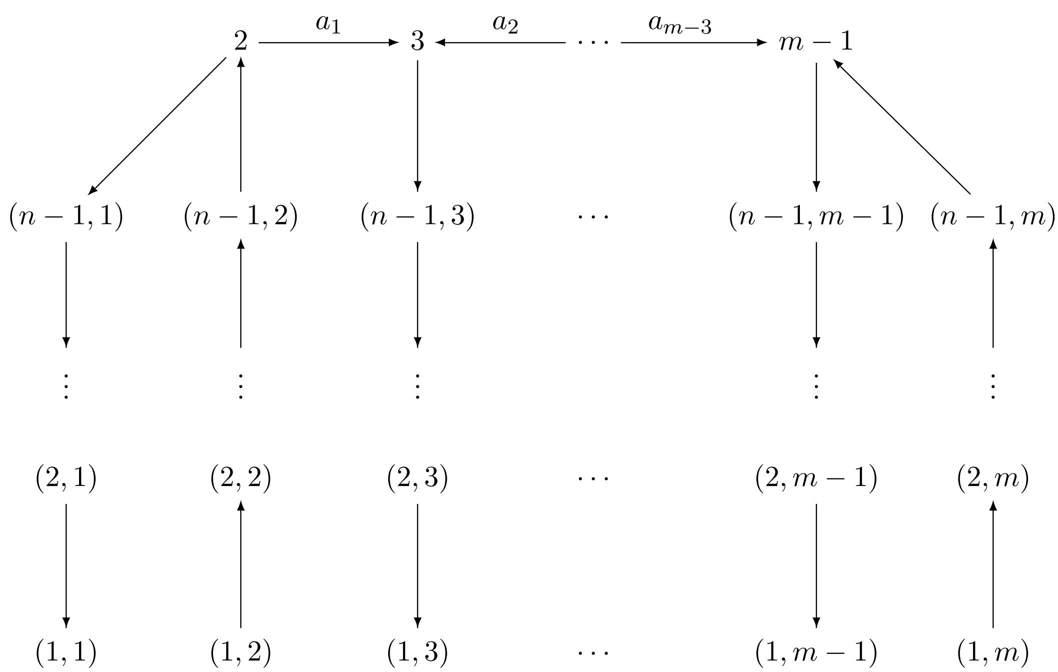
A detailed description of the set $\Sigma(Q,\beta)$ can be found in Section 4, Theorem 4.2 and Proposition 4.5.

3. The generalized flag quiver and the saturation property

In this section, we define the generalized flag quiver and show that the generalized Littlewood-Richardson coefficients can be viewed as dimensions of weight spaces of semi-invariants of this particular quiver.

Let $m \geq 3$ and $n \geq 1$ be two positive integers. The *generalized flag quiver setting* is defined as follows.

(a) The quiver Q has $m-2$ *central vertices* $2=(n,2)=(n,1), 3=(n,3), \dots, m-2=(n,m-2), m-1=(n,m-1)=(n,m)$ at which we attach m equioriented $\mathbb{A}_n$ quivers (or flags) $\mathcal{F}(1), \dots, \mathcal{F}(m)$ such that $\mathcal{F}(i)$ goes in the corresponding cental vertex (n,i) if i is even and it goes out from the corresponding cental vertex (n,i) if i is odd. Furthermore, there are $m-3$ main arrows $a_1, \dots, a_{m-3}$ connecting the central vertices such that $i+1 \xrightarrow{a_i} i+2$ if i is odd and $i+2 \xrightarrow{a_i} i+1$ if i is even. For example, if the number of flags m is even then our quiver Q looks like



(b) The dimension vector β is defined by $\beta(j,i)=j$ for all $j\in\{1,\dots,n\}$ and $i\in\{1,\dots,m\}$, i.e., β is equal to

$$\begin{array}{cccccc} & n & n & \cdots & n & \\ n-1 & n-1 & n-1 & \cdots & n-1 & n-1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{array}$$

The only quiver setting we will be working with in this section is the generalized flag quiver setting.

Lemma 3.1. *Let $\sigma\in\mathbb{Z}^{Q_0}$ be a weight. If $\dim \mathrm{SI}(Q,\beta)_\sigma\neq 0$ then:*

(1) *the weight σ must satisfy the inequalities*

$$(-1)^i\sigma(j,i)\geq 0,$$

for all $1\leq j\leq n,\ 2\leq i\leq m-1$ and

$$(-1)^i\sigma(j,i)\geq 0,$$

for all $1\leq j\leq n-1,\ i\in\{1,m\}$;

(2) *we have*

$$\dim \mathrm{SI}(Q,\beta)_\sigma=\sum_{\mu(1),\cdots,\mu(m-3)} c^{\gamma(2)}_{\gamma(1),\mu(1)}\cdot c^{\gamma(3)}_{\mu(1),\mu(2)}\cdots c^{\gamma(m)}_{\mu(m-3),\gamma(m-1)},$$

where

$$\begin{aligned}\gamma(1)&=((n-1)^{-\sigma(n-1,1)},\dots,1^{-\sigma(1,1)})',\\ \gamma(m)&=((n-1)^{(-1)^m\cdot\sigma(n-1,m)},\dots,1^{(-1)^m\cdot\sigma(1,m)})',\\ \gamma(i)&=(n^{(-1)^i\cdot\sigma(n,i)},\dots,1^{(-1)^i\cdot\sigma(1,i)})',\end{aligned}$$

for all $i\in\{2,\dots,m-1\}$.

Proof. The first part of our lemma follows as we compute $\mathrm{SI}(Q,\beta)_\sigma$. For simplicity, let us define $V_j(i)=\mathbb{C}^{\beta(j,i)}$. Using Cauchy's formula [6, page 121], we can decompose the affine coordinate ring $\mathbb{C}[\mathrm{Rep}(Q,\beta)]$ as a sum of tensor products of irreducible representations of the general linear groups $\mathrm{GL}(V_j(i))$. The idea is to identify those terms that will give us non-zero semi-invariants of weight σ. An arbitrary term in this decomposition is made up of tensor products of irreducible representations coming from the m flags. If $\mathcal{F}(i)$ is a flag going in the cental vertex (n,i), then the $n-1$ arrows of this flag contribute with

$$S^{\gamma^1(i)}V_1(i)\otimes\bigotimes_{j=2}^{n-1}\left(S^{\gamma^{j-1}(i)}V_j^*(i)\otimes S^{\gamma^j(i)}V_j(i)\right)\otimes S^{\gamma^{n-1}(i)}V_n^*(i),$$

for partitions $\gamma^1(i),\dots,\gamma^{n-1}(i)$.

When computing semi-invariants, we see that $\left(S^{\gamma^1(i)}V_1(i)\right)^{\mathrm{SL}(V_1(i))}$ is non-zero if and only if it is one dimensional. In this case, $\gamma^1(i)$ is a $\beta(1,i)\times w$ rectangle and the space is spanned by a semi-invariant of weight w. So, $\left(S^{\gamma^1(i)}V_1(i)\right)^{\mathrm{SL}(V_1(i))}$ contains non-zero semi-invariants of weight $\sigma(1,i)$ if and only if $\sigma(1,i) \geq 0$ and $\gamma^1(i) = (\sigma(1,i)^{\beta(1,i)})$, i.e.,

$$\gamma^1(i) = (1^{\sigma(1,i)})'.$$

Next, we look at the space

$$\left(S^{\gamma^1(i)}V_2^*(i) \otimes S^{\gamma^2(i)}V_2(i)\right)^{\mathrm{SL}(V_2(i))}$$

which is canonically isomorphic to $\mathrm{Hom}_{\mathrm{SL}(V_2(i))}(S^{\gamma^1(i)}V_2(i), S^{\gamma^2(i)}V_2(i))$. Now, this space is non-zero if and only if it is one dimensional in which case $\gamma^2(i)$ is $\gamma^1(i)$ plus some extra columns of length $\beta(2,i)$ and the number of these extra columns is the weight of a semi-invariant spanning this space. Consequently, $\left(S^{\gamma^1(i)}V_2^*(i) \otimes S^{\gamma^2(i)}V_2(i)\right)^{\mathrm{SL}(V_2(i))}$ contains non-zero semi-invariants of weight $\sigma(2,i)$ if and only if $\sigma(2,i) \geq 0$ and $\gamma^2(i)$ is $\gamma^1(i)$ plus $\sigma(2,i)$ columns of length $\beta(2,i)$, i.e.,

$$\gamma^2(i) = (2^{\sigma(2,i)}, 1^{\sigma(1,i)})'.$$

Reasoning in this way, we see that the vertices of this flag $\mathcal{F}(i)$, except the central one (n,i), give non-zero spaces of semi-invariants (in which case they must be one dimensional) of weight $\sigma(1,i), \dots, \sigma(n-1,i)$ if and only if $\sigma(j,i) \geq 0$ for all $1 \leq j \leq n-1$, $\gamma^1(i)$ is a $\beta(1,i) \times \sigma(1,i)$ rectangle and $\gamma^j(i)$ is $\gamma^{j-1}(i)$ plus $\sigma(j,i)$ columns of length $\beta(j,i)$ for all $j \in \{2, \dots, n-1\}$, i.e.,

$$\gamma^{n-1}(i) = ((n-1)^{\sigma(n-1,i)}, \dots, 1^{\sigma(1,i)})'.$$

We have proved that a flag $\mathcal{F}(i)$ going in the central vertex (n,i) contributes to the space of semi-invariants $\mathrm{SI}(Q,\beta)_\sigma$ with

$$S^{\gamma^{n-1}(i)}V_n^*(i),$$

where $\gamma^{n-1}(i)$ is completely determined by the weight σ along the flag $\mathcal{F}(i)$.

Similarly, if $\mathcal{F}(l)$ is a flag going out of the central vertex (n,l), then $\sigma(j,l) \leq 0$ for all $1 \leq j \leq n-1$ and $\mathcal{F}(l)$ contributes to $\mathrm{SI}(Q,\beta)_\sigma$ with

$$S^{\gamma^{n-1}(l)}V_n(l),$$

where

$$\gamma^{n-1}(l) = ((n-1)^{-\sigma(n-1,l)}, \dots, 1^{-\sigma(1,l)})'.$$

Next, the main $m-3$ arrows of our quiver give us partitions $\mu(1), \dots, \mu(m-3)$, with at most n parts, and the central vertices give us the following spaces of semi-invariants:

$$\left(S^{\gamma^{n-1}(1)}V(2) \otimes S^{\mu(1)}V(2) \otimes S^{\gamma^{n-1}(2)}V^*(2)\right)^{\mathrm{SL}(V(2))}$$

coming from the vertex 2,

$$\left(S^{\gamma^{n-1}(3)}V(3) \otimes S^{\mu(1)}V^*(3) \otimes S^{\mu(2)}V^*(3)\right)^{\mathrm{SL}(V(3))}$$

coming from the vertex 3 and so on. Taking into account the weights at the central vertices, it is clear that the dimension of the space of semi-invariants $\mathrm{SI}(Q,\beta)_\sigma$ is the desired sum of products of Littlewood-Richardson coefficients. $\square$

Let $\lambda(1), \dots, \lambda(m)$ be weakly decreasing sequences of n integers. To show that $f(\lambda(1), \dots, \lambda(m))$ can be viewed as the dimension of a space of semi-invariants, we are going to apply Lemma 3.1. Let us define σ_λ by

$$\sigma_\lambda(j,i) = (-1)^i(\lambda_j(i) - \lambda_{j+1}(i)), \forall\ 1 \le j \le n-1, \forall\ 1 \le i \le m, \tag{2}$$

$$\sigma_\lambda(i) = (-1)^i \lambda_n(i), \forall\ i \ne 2,\ m-1, \tag{3}$$

$$\sigma_\lambda(2) = \lambda_n(2) - \lambda_n(1), \tag{4}$$

$$\sigma_\lambda(m-1) = (-1)^{m-1}(\lambda_n(m-1) - \lambda_n(m)). \tag{5}$$

If $m = 3$ then σ_λ at the central vertex becomes

$$\sigma_\lambda(2) = \lambda_n(2) - \lambda_n(1) - \lambda_n(3).$$

With these notations we have:

Lemma 3.2. *Let $\lambda(1), \dots, \lambda(m)$ be $m \ge 3$ weakly decreasing sequences of n integers. Then for every integer $r \ge 1$,*

$$f(r\lambda(1), \dots, r\lambda(m)) = \dim \mathrm{SI}(Q, \beta)_{r\sigma_\lambda}.$$

Proof. We prove this Lemma when $r = 1$, as the general case reduces to this one. First, let us consider the following transformations

$$\begin{aligned}
&\gamma(1) = \lambda(1) - (\lambda_n(1)^n),\\
&\gamma(2) = \lambda(2) - (\lambda_n(1)^n),\\
&\gamma(m-1) = \lambda(m-1) - (\lambda_n(m)^n),\\
&\gamma(m) = \lambda(m) - (\lambda_n(m)^n),\\
&\gamma(i) = \lambda(i), \forall\ i \notin \{1, 2, m-1, m\}.
\end{aligned}$$

If $m = 3$ then $\gamma(2)$ becomes $\gamma(2) = \lambda(2) - ((\lambda_n(1) + \lambda_n(3))^n)$. With this transformations, we have

$$\begin{aligned}
\gamma(1) &= ((n-1)^{-\sigma(n-1,1)}, \dots, 1^{-\sigma(1,1)})',\\
\gamma(m) &= ((n-1)^{(-1)^m \cdot \sigma(n-1,m)}, \dots, 1^{(-1)^m \cdot \sigma(1,m)})',\\
\gamma(i) &= (n^{(-1)^i \cdot \sigma(n,i)}, \dots, 1^{(-1)^i \cdot \sigma(1,i)})',
\end{aligned}$$

for all $i \in \{2, \dots, m-1\}$. Applying Lemma 3.1, we get that

$$f(\gamma(1), \dots, \gamma(m)) = \dim \mathrm{SI}(Q, \beta)_{\sigma_\lambda}.$$

On the other hand, we clearly have $f(\lambda(1), \dots, \lambda(m)) = f(\gamma(1), \dots, \gamma(m))$ and so the proof follows. □

Remark 3.3. Let us note that if $f(\lambda(1), \dots, \lambda(m))$ is non-zero then the first part of Lemma 3.1 tells us that $\lambda(i), i \notin \{1, 2, m-1, m\}$ are in fact partitions. Of course, this is also clear from the definition of $f(\lambda(1), \dots, \lambda(m))$.

Proof of Theorem 1.4. The proof follows from Theorem 2.1 and Lemma 3.2. □

4. The cone of effective weights for quivers

Let Q be a quiver without oriented cycles and β a dimension vector. In this section, we will further describe the rational convex polyhedral cone whose lattice points form the set of integral effective weights

$$\Sigma(Q,\beta)=\{\sigma\in\mathbb{Z}^{Q_0}\mid \mathrm{SI}(Q,\beta)_\sigma\neq 0\}.$$

If $\sigma\in\mathbb{R}^{Q_0}$ is a real valued function on the set of vertices and $\alpha\in\Gamma$, we define $\sigma(\alpha)$ by

$$\sigma(\alpha)=\sum_{x\in Q_0}\sigma(x)\alpha(x).$$

A necessary condition for a weight $\sigma\in\mathbb{Z}^{Q_0}$ to belong to $\Sigma(Q,\beta)$ is $\sigma(\beta)=0$. Indeed, the action of the one dimensional torus $\{(t\,\mathrm{Id}_{\beta(i)})_{i\in Q_0}\mid t\in K\setminus\{0\}\}$ on the representation space $\mathrm{Rep}(Q,\beta)$ is trivial. If f is a non-zero semi-invariant of weight σ and $g_t=(t\,\mathrm{Id}_{\beta(i)})_{i\in Q_0}\in\mathrm{GL}(\beta)$ then

$$g_t\cdot f=t^{\sigma(\beta)}\cdot f$$

clearly implies that $\sigma(\beta)=0$.

Lemma 4.1 (Reciprocity Property)**.** [4, Corollary 1] *Let α and β be two dimension vectors. Then:*

$$\dim\mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle}=\dim\mathrm{SI}(Q,\alpha)_{-\langle\cdot,\beta\rangle}.$$

Now, we can define $\alpha\circ\beta$ by

$$\alpha\circ\beta=\dim\mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle}=\dim\mathrm{SI}(Q,\alpha)_{-\langle\cdot,\beta\rangle}.$$

Following Schofield [18], we write $\alpha\hookrightarrow\beta$ if every representation of dimension vector β has a subrepresentation of dimension vector α.

Theorem 4.2. [4, Theorem 3] *Let Q be a quiver and β a sincere dimension vector. If $\sigma=\langle\alpha,\cdot\rangle\in\mathbb{Z}^{Q_0}$ is a weight with $\alpha\in\mathbb{Z}^{Q_0}$ then the following statements are equivalent:*

1. $\dim\mathrm{SI}(Q,\beta)_\sigma\neq 0$*;*
2. $\sigma(\beta)=0$ *and* $\sigma(\beta')\leq 0$*, for every* $\beta'\hookrightarrow\beta$
3. α *must be a dimension vector,* $\sigma(\beta)=0$ *and* $\alpha\hookrightarrow\alpha+\beta$.

Remark 4.3. It turns out that some of the linear homogeneous inequalities obtained in Theorem 4.2(2) are redundant. In the next subsection, we will explain how to find a minimal list of such inequalities.

A representation V is said to be Schur if $\mathrm{End}_Q(V)=\mathbb{C}$. We say that a dimension vector β is a *Schur* root if there exists a Schur representation V of dimension vector β.

Theorem 4.4. [18, Theorem 6.1] *Let Q be a quiver and β a dimension vector. Then the following are equivalent:*

1. β *is a Schur root;*
2. $\sigma_\beta(\beta')<0,\forall\ \beta'\hookrightarrow\beta,\ \beta'\neq 0,\beta,$ *where* $\sigma_\beta=\langle\beta,\cdot\rangle-\langle\cdot,\beta\rangle$.

Let $\mathbb{H}(\beta)=\{\sigma\in\mathbb{R}^{Q_0}\mid\sigma(\beta)=0\}$. Consider the following rational convex polyhedral cone

$$C(Q,\beta)=\{\sigma\in\mathbb{H}(\beta)\mid\sigma(\beta)=0,\sigma(\beta')\leq 0\text{ for all }\beta'\hookrightarrow\beta\}.$$

We call $C(Q,\beta)$ the *cone of effective weights* associated to the quiver setting (Q,β). Note that $C(Q,\beta)\bigcap\mathbb{Z}^{Q_0}=\Sigma(Q,\beta)$ and the dimension of this cone is at most $N-1$, where $N=|Q_0|$ is the number of vertices of Q.

It is a very interesting question to to describe the facets of $C(Q,\beta)$. The answer to this question was given by Derksen and Weyman in [5, Corollary 5.2]:

Proposition 4.5. *Let Q be a quiver with N vertices and β a Schur root. Then:*
(1) $\dim C(Q,\beta) = N-1$.
(2) $\sigma \in C(Q,\beta)$ *if and only if* $\sigma(\beta)=0$ *and* $\sigma(\beta_1)\le 0$ *for every decomposition* $\beta = c_1\beta_1 + c_2\beta_2$ *with* β_1,β_2 *Schur roots,* $\beta_1 \circ \beta_2 = 1$ *and* $c_i = 1$ *whenever* $\langle \beta_i,\beta_i\rangle < 0$.

Remark 4.6. Note that in Proposition 4.5(2), we can replace $\beta_1\circ\beta_2 = 1$ with $\beta_1\circ\beta_2 \neq 0$. Of course, in this case we get a longer list of necessary and sufficient inequalities.

5. The facets of the cone associated to the generalized flag quiver

We use the methods from Section 4 to describe the facets of the cone of effective weights associated to the generalized flag quiver setting.

Throughout this section, we work with the generalized flag quiver setting (Q,β) from Section 3. For the convenience of the reader, we briefly recall this set up. The quiver Q has $m-2$ central vertices with m equioriented $\mathbb{A}_n$ quivers (or flags) $\mathcal{F}(1),\dots,\mathcal{F}(m)$ attached to them. The dimension vector β is defined by $\beta(j,i)=j$ for all $j\in\{1,\dots,n\}$ and $i\in\{1,\dots,m\}$. First, let us prove a simple lemma:

Lemma 5.1. *The dimension vector β is a Schur root.*

Proof. Note that the dimension vector β is indivisible, meaning that the greatest common divisor of its coordinates is one. Next, let us assume that either $n=2, m\ge 4$ or $n\ge 3$. If this is the case then β lies in the so called *fundamental region*, i.e., the support of β is a connected graph and $\langle\varepsilon_i,\beta\rangle + \langle\beta,\varepsilon_i\rangle \le 0$, for all vertices $i\in Q_0$. It follows now from a result of Kac [11, Theorem B(d)] that β is a Schur root. If either $n=2, m=3$ or $n=1$ then β is actually a real Schur root. □

Now, let $\mathcal{D}$ be the set of all dimension vectors β_1 that take one of the following forms:
(1) $\beta_1 = \varepsilon_{(j,2i+1)}$ or $\beta_1 = \beta - \varepsilon_{(j,2i)}$, for $1\le j\le n-1$ (call such a dimension vector *trivial*); or
(2) $\beta_1 \neq \beta$, $\beta_1\circ(\beta-\beta_1)=1$, and β_1 is weakly increasing with jumps of at most one (from the bottom to the top) along the m flags.

Note that if β_1 is in $\mathcal{D}$ then $\beta_1 \hookrightarrow \beta$ and hence $-\beta_1$ is in the dual of the cone $C(Q,\beta)$.

Lemma 5.2. *Keep notation as above. If $\mathcal{F}$ is a facet of $C(Q,\beta)$ then it has to be of the form*
$$\mathcal{F} = \mathbb{H}(\beta_1)\bigcap C(Q,\beta),$$
for some β_1 in $\mathcal{D}$.

Proof. From Proposition 4.5 it follows that there are two Schur roots β_1 and β_2 such that
$$\mathcal{F} = \mathbb{H}(\beta_1)\bigcap C(Q,\beta)$$
with $\beta_1\circ\beta_2 = 1$ and $\beta = c_1\beta_1 + c_2\beta_2$ for some $c_1,c_2\ge 1$.

Now let us assume that β_1 is not trivial. In this case, we will show that β_1 is weakly increasing with jumps of at most one along the flags. Let us denote $c_1\beta_1=\beta', c_2\beta_2=\beta''$. Since $\beta'\circ\beta''\neq 0$ it follows from Theorem 4.2 that any representation of dimension vector β has a subrepresentation of dimension vector β'. Therefore, β' must be weakly increasing along each flag going in and it has jumps of at most one along each flag going out.

Next, we will show that β' has jumps of at most one along each flag $\mathcal{F}(i)$ going in a central vertex and β' is weakly increasing along each flag $\mathcal{F}(i)$ going out of a central vertex. For simplicity, let us write

$$\mathcal{F}(i) : 1 \longrightarrow 2 \qquad \cdots \qquad n-1 \longrightarrow n,$$

for a flag going in its central vertex (n,i) (i.e. i is even). Assume to the contrary that there is an $l \in \{1,\dots,n-1\}$ such that $\beta'(l+1) > \beta'(l)+1$. Then $\beta''(l+1) < \beta''(l)$ which implies that $\varepsilon_l \hookrightarrow \beta''$. Since β'' is $\langle \beta', \cdot \rangle$-semi-stable it follows that $\langle \beta', \epsilon_l \rangle \leq 0$. So, $\beta'(l) \leq \beta'(l-1)$ and hence $\beta'(l) = \beta'(l-1)$ or $\beta''(l) = \beta''(l-1)+1$. This shows that $c_2 = 1$ and $\beta'' - \varepsilon_l \hookrightarrow \beta''$. From the fact that $\beta''(=\beta_2)$ is a Schur root and Theorem 4.4 we obtain that β'' is $\sigma_{\beta''}$-stable. Since $\varepsilon_l \hookrightarrow \beta''$,$\beta''-\varepsilon_l \hookrightarrow \beta''$ and $\beta'' \neq \epsilon_l$ it follows $\langle \beta'', \epsilon_l \rangle - \langle \varepsilon_l, \beta'' \rangle < 0$ and $\langle \beta'', \beta''-\varepsilon_l \rangle - \langle \beta''-\varepsilon_l, \beta'' \rangle < 0$. But this is a contradiction. We have just proved that β' has jumps of at most one along each flag going in. Similarly, one can show that β' has to be weakly increasing along each flag going out.

Now, let us show that $c_1 = c_2 = 1$. Since $\beta' = c_1\beta_1$ has jumps of at most one along each flag, we obtain $0 \leq c_1(\beta_1(l+1,i) - \beta_1(l,i)) \leq 1$ for all $l \in \{1,\dots,n-1\}$ and $i \in \{1,\dots,m\}$. If there are l,i such that $\beta_1(l+1,i) - \beta_1(l,i) \neq 0$ then $c_1 = 1$. Otherwise, there is an i such that $\beta'(1,i) = 1$ and so $c_1 = 1$. Similarly, one can show $c_2 = 1$.

In conclusion, $\beta = \beta_1 + \beta_2$ with β_1 weakly increasing with jumps of at most one along the m flags. So, $\beta_1 \in \mathcal{D}$ and this finishes the proof. ☐

Lemma 5.3. *Let $\sigma \in \mathbb{H}(\beta)$. Then $\sigma \in C(Q,\beta)$ if and only if the following are true*

1. (chamber inequalities) $(-1)^i\sigma(\varepsilon_{(j,i)}) \geq 0, \forall 1 \leq j \leq n-1, \forall 1 \leq i \leq m$;
2. (regular inequalities) $\sigma(\beta_1) \leq 0$ *for every $\beta_1 \neq \beta$ weakly increasing with jumps of at most one along the m flags and $\beta_1 \circ (\beta - \beta_1) = 1$.*

Proof. Let us assume that $\sigma \in \mathbb{H}(\beta)$ satisfies the chamber and regular inequalities. Then the description of the facets of $C(Q,\beta)$ given in Lemma 5.2 shows that $\sigma \in C(Q,\beta)$.

Conversely, let $\sigma \in C(Q,\beta)$. We clearly have $\sigma(\beta_1) \leq 0$ for every $\beta_1 \in \mathcal{D}$ by Theorem 4.2. But this is equivalent to (1) and (2). ☐

Remark 5.4. Let σ_λ be the weight defined by the equations (2) – (5) in Section 3. Then by definition we have that

$$\sigma_\lambda(\varepsilon_{(j,i)}) = (-1)^i(\lambda_j(i) - \lambda_{j+1}(i)), \forall 1 \leq j \leq n-1, \forall 1 \leq i \leq m.$$

Consequently, the *chamber inequalities* just tell us that the $\lambda(i)$ are weakly decreasing sequences. This is something that we will always assume.

Example 5.5. For $m = 4$ and $n = 2$, there are exactly 9 dimension vectors β_1 that satisfy the second condition in Lemma 5.3. It turns out that exactly one of the 9 pairs gives us a redundant inequality. Next we find the necessary and sufficient inequalities for σ_λ to be in $C(Q,\beta)$.

For

$$\beta_1 = \beta = \begin{matrix} & 2 & 2 & \\ 1 & 1 & 1 & 1 \end{matrix},$$

we must have the identity $\sigma_\lambda(\beta) = 0$, i.e.,

$$|\lambda(1)| + |\lambda(3)| = |\lambda(2)| + |\lambda(4)|.$$

For $\beta_1 = \begin{matrix} & 1 & 2 & \\ 0 & 0 & 1 & 1 \end{matrix}$ and $\beta_1 = \begin{matrix} & 1 & 2 & \\ 1 & 1 & 1 & 1 \end{matrix}$, we have the inequalities

$$\lambda_2(2) + |\lambda(4)| \leq \lambda_2(1) + |\lambda(3)|,$$

and

$$\lambda_1(2) + |\lambda(4)| \leq \lambda_1(1) + |\lambda(3)|.$$

For $\beta_1 = \begin{matrix} & 0 & 1 & \\ 0 & 0 & 1 & 1 \end{matrix}$ and $\beta_1 = \begin{matrix} & 0 & 1 & \\ 0 & 0 & 0 & 0 \end{matrix}$, we have the inequalities

$$\lambda_1(4) \leq \lambda_1(3), \text{ and } \lambda_2(4) \leq \lambda_2(3).$$

For $\beta_1 = \begin{matrix} & 1 & 1 & \\ 1 & 0 & 1 & 1 \end{matrix}$ and $\beta_1 = \begin{matrix} & 1 & 1 & \\ 1 & 1 & 1 & 0 \end{matrix}$, we have the inequalities

$$\lambda_2(2) + \lambda_1(4) \leq \lambda_1(1) + \lambda_1(3),$$

and

$$\lambda_1(2) + \lambda_2(4) \leq \lambda_1(1) + \lambda_1(3).$$

For $\beta_1 = \begin{matrix} & 1 & 1 & \\ 0 & 0 & 1 & 0 \end{matrix}$ and $\beta_1 = \begin{matrix} & 1 & 1 & \\ 1 & 0 & 0 & 0 \end{matrix}$, we have the inequalities

$$\lambda_2(2) + \lambda_2(4) \leq \lambda_2(1) + \lambda_1(3),$$

and

$$\lambda_2(2) + \lambda_2(4) \leq \lambda_1(1) + \lambda_2(3).$$

For

$$\beta_1 = \begin{matrix} & 0 & 2 & \\ 0 & 0 & 1 & 1 \end{matrix},$$

we obtain the only redundant inequality

$$|\lambda(4)| \leq |\lambda(3)|.$$

6. The Horn type inequalities

Our goal in this section is to give a closed form to the polyhedral inequalities that we obtained in Lemma 5.3.

First, let us describe the dimension vectors β_1 that define the regular inequalities from Lemma 5.3(2). Let β_1 be a dimension vector that is weakly increasing with jumps of at most one along the m flags. We define the following jump sets

$$I_i = \{l \mid \beta_1(l,i) > \beta_1(l-1,i), 1 \leq l \leq n\},$$

with the convention that $\beta_1(0,i) = 0$ for all $i \in \{1,\dots,m\}$. We also denote β_1 by β_I.

Note also that $|I_i| = \beta_I(n,i)$ for all $i \in \{1,\dots,m\}$. Therefore, $|I_1| = |I_2| = \beta_I(2)$ and $|I_{m-1}| = |I_m| = \beta_I(m-1)$.

Conversely, it is clear that each m-tuple $I = (I_1,\dots,I_m)$ of subsets of the set $\{1,\dots,n\}$ with $|I_1| = |I_2|$ and $|I_{m-1}| = |I_m|$ uniquely determines the dimension vector β_I. Indeed, if

$$I_i = \{z_1(i) < \cdots < z_r(i)\},$$

we have that

$$\beta_I(k,i) = j-1, \forall\ z_{j-1}(i) \leq k < z_j(i), \forall\ 1 \leq j \leq r+1,$$

with the convention that $z_0(i) = 0$ and $z_{r+1}(i) = n+1$ for all $1 \leq i \leq m$.

Definition 6.1. We define $\mathcal{S}(n,m)$ to be the set consisting of all m-tuples $I = (I_1,\dots,I_m)$ such that $|I_1| = |I_2|$, $|I_{m-1}| = |I_m|$, $\beta_I \neq \beta$ and

$$\beta_I \circ (\beta - \beta_I) = 1.$$

A further description of the set $\mathcal{S}(n,m)$ will be given in Lemma 6.4 and Lemma 6.6.

Proposition 6.2. *Let $\lambda(1),\dots,\lambda(m)$ be weakly decreasing sequences of n reals. Then the following are equivalent:*

(1) $\sigma_\lambda \in C(Q,\beta)$;

(2)
$$\sum_{i\ even} |\lambda(i)| = \sum_{i\ odd} |\lambda(i)|$$
and
$$\sum_{i\ even} \left(\sum_{j\in I_i} \lambda_j(i) \right) \le \sum_{i\ odd} \left(\sum_{j\in I_i} \lambda_j(i) \right)$$
for every m-tuple $(I_1,\dots,I_m)\in\mathcal{S}(n,m)$.

Proof. We have seen that the set of all β_1 occurring in Lemma 5.3(2) are exactly those of the form β_I with $I=(I_1,\dots,I_m)\in\mathcal{S}(n,m)$. Furthermore it is easy to see that
$$\sigma_\lambda(\beta_I) = \sum_{i\ even} \left(\sum_{j\in I_i} \lambda_j(i) \right) - \sum_{i\ odd} \left(\sum_{j\in I_i} \lambda_j(i) \right)$$
and
$$\sigma_\lambda(\beta) = \sum_{i\ even} |\lambda(i)| - \sum_{i\ odd} |\lambda(i)|.$$
The Proposition is now an immediate consequence of Lemma 5.3. □

Example 6.3. In this example we will work out the case when $n=1$. Let $d_1,\dots,d_m$ be $m\ge 3$ positive integers. Then the following are equivalent:

(1) There exists a long exact sequence of the form
$$0\to(\mathbb{Z}/p)^{d_1}\to\cdots\to(\mathbb{Z}/p)^{d_m}\to 0.$$

(2) There exists a long exact sequence of the form
$$0\to\mathbb{Z}/p^{d_1}\to\cdots\to\mathbb{Z}/p^{d_m}\to 0.$$

(3) (*Horn type inequalities*)
$$\sum_{j\ even} d_j = \sum_{j\ odd} d_j$$
and if $m>3$
$$\sum_{j\ even,\ 1\le j\le i} d_j \le \sum_{j\ odd,\ 1\le j\le i} d_j$$
and
$$\sum_{j\ even,\ i\le j\le m} d_j \le \sum_{j\ odd,\ i\le j\le m} d_j,$$
for every i odd with $2\le i\le m-2$, together with $d_m\le d_{m-1}$ if m is even.

Indeed, let $\lambda(i) = (d_i), \forall\, 1 \le i \le m$. The equivalence of (1) and (2) follows from

$$f(\lambda(1), \dots, \lambda(m)) \neq 0 \Longleftrightarrow f(\lambda'(1), \dots, \lambda'(m)) \neq 0.$$

To prove the equivalence (2) $\Longleftrightarrow$ (3), we explicitly describe the facets of the cone $C(Q, \beta)$, where Q is the generalized quiver when $n = 1$. When $m = 3$, the only inequality is $d_2 = d_1 + d_3$. Let us assume that $m \ge 4$. In this case, our quiver Q is an alternating type $\mathbb{A}_{m-2}$ quiver with $m-2$ vertices such that 2 is a source, 3 is a sink and so on. For example if m is odd then our generalized flag quiver becomes:

$$2 \longrightarrow 3 \longleftarrow \cdots \quad m-2 \longleftarrow m-1.$$

First, let β_1, β_2 be two Schur roots (i.e. positive roots of type $\mathbb{A}$) such that $\beta_1 + \beta_2 = \beta = (1, \dots, 1)$ and $\langle \beta_1, \beta_2 \rangle = 0$. Then it is easy to see that

$$\beta_1 = (1, \dots, 1, 0, \dots, 0) \text{ or } \beta_1 = (0, \dots, 0, 1, \dots, 1),$$

with $\mathrm{supp}(\beta_1) = \{2, \dots, i\}$ or $\{i, \dots, m-1\}$ and $2 \le i \le m-1$ odd. To find a minimal list of necessary and sufficient inequalities, we will focus on those m-tuples $I = (I_1, \dots, I_m) \in \mathcal{S}$ for which the corresponding dimension vectors $\beta_I,\ \beta - \beta_I$ are Schur roots. If this the case, we must have that

$$I_j = \begin{cases} \{1\} & \text{if } 1 \le j \le i \\ \emptyset & \text{if } i < j \le m \end{cases}$$

or

$$I_j = \begin{cases} \emptyset & \text{if } 1 \le j < i \\ \{1\} & \text{if } i \le j \le m, \end{cases}$$

where $2 \le i \le m-2$ is odd. If m is even, there is one more possibility, namely $\beta_1 = (0, \dots, 0, 1)$. In this case, $I_1 = \cdots = I_{m-2} = \emptyset$ and $I_{m-1} = Im = \{1\}$. For all such tuples I, we also have that $\beta_I \circ (\beta - \beta_I) = 1$. This way, we obtain the equivalence of (2) and (3). Note that the list of inequalities obtained is minimal.

Now, let us show that $\mathcal{S}(n, m)$ can be described in terms of the generalized Littlewood-Richardson coefficients. For convenience, let us recall some of the notation from Section 1. Let $(I_1, \dots, I_m)$ be an m-tuple of subsets of $\{1, \dots, n\}$ such that at least one of them has cardinality at most $n-1$. We define the following weakly decreasing sequences of integers (using conjugate partitions):

$$\underline{\lambda}(I_1) = \lambda'(I_1), \qquad \underline{\lambda}(I_m) = \begin{cases} \lambda'(I_m) & \text{if } m \text{ is odd} \\ \lambda'(I_m \setminus \{n\}) & \text{if } m \text{ is even,} \end{cases}$$

and for $2 \le i \le m-1$

$$\underline{\lambda}(I_i) = \begin{cases} \lambda'(I_i) & \text{if } i \text{ is even} \\ \lambda'(I_i) - ((|I_i| - |I_{i+1}| - |I_{i-1}|)^{n-|I_i|}) & \text{if } i \le m-2 \text{ is odd} \\ \lambda'(I_i) - ((|I_{m-1}| - |I_{m-2}| - |I_m \setminus \{n\}|)^{n-|I_i|}) & \text{if } i = m-1 \text{ is odd.} \end{cases}$$

Lemma 6.4. *The set $\mathcal{S}(n, m)$ consists of all m-tuples $I = (I_1, \dots, I_m)$ such that:*

(a) $|I_1| = |I_2|$;
(b) $|I_{m-1}| = |I_m|$;
(c) *at least one of the subsets $I_1, \dots, I_m$ has cardinality $< n$;*

(d) $\underline{\lambda}(I_i)$ *is a partition,* $\forall\ 1 \leq i \leq m$*;*
(e) $f(\underline{\lambda}(I_1), \dots, \underline{\lambda}(I_m)) = 1$.

Proof. Let $I = (I_1, \dots, I_m)$ be an m-tuple in $\mathcal{S}(n, m)$. By definition, we know that (a) and (b) are satisfied.

Let us denote $\beta_I = \beta_1$ and $\beta - \beta_I = \beta_2$.

(c) If $\min_{1 \leq i \leq m} |I_i| = n$ then we would have $\beta_1 = \beta$ which is not allowed.

$(d), (e)$ We compute the dimension $\beta_1 \circ \beta_2 = \dim \mathrm{SI}(Q, \beta_2)_{\langle \beta_1, \cdot \rangle}$ using the same arguments as in Lemma 3.1 with β replaced by β_2 and σ by $\sigma_1 = \langle \beta_1, \cdot \rangle$. Since β_1 is weakly increasing and has jumps of at most one along the flags it is easy to see that

$$\sigma_1(l, i) = \begin{cases} 1 & \text{if } l \in I_i \\ 0 & \text{otherwise} \end{cases},$$

for all $l \in \{1, \dots, n-1\}$ and i even and

$$\sigma_1(l, i) = \begin{cases} -1 & \text{if } l+1 \in I_i \\ 0 & \text{otherwise} \end{cases},$$

for all $l \in \{1, \dots, n-1\}$ and i odd. At the central vertices $2, \dots, m-1$, the values of σ_1 are

$$\sigma_1(i) = \begin{cases} 0 & \text{if } i \text{ is even and } n \notin I_i \\ 1 & \text{if } i \text{ is even and } n \in I_i \\ |I_i| - |I_{i+1}| - |I_{i-1}| & \text{if } i \leq m-2 \text{ is odd} \\ |I_{m-1}| - |I_{m-2}| - |I_m \setminus \{n\}| & \text{if } i = m-1 \text{ is odd.} \end{cases}$$

Arguing as in Lemma 3.1, we obtain

$$\begin{aligned} \gamma(1) &= (\beta_2(n-1,1)^{-\sigma_1(n-1,1)}, \dots, \beta_2(1,1)^{-\sigma_1(1,1)})', \\ \gamma(m) &= (\beta_2(n-1,m)^{(-1)^m \cdot \sigma_1(n-1,m)}, \dots, \beta_2(1,m)^{(-1)^m \cdot \sigma_1(1,m)})', \\ \gamma(i) &= (\beta_2(n-1,i)^{(-1)^i \cdot \sigma_1(n-1,i)}, \dots, \beta_2(1,i)^{(-1)^i \cdot \sigma_1(1,i)})' + (((-1)^i \cdot \sigma_1(n,i))^{\beta_2(n,i)}), \end{aligned}$$

must be partitions for all $2 \leq i \leq m-1$ and

$$\dim \mathrm{SI}(Q, \beta_2)_{\sigma_1} = f(\gamma(1), \dots, \gamma(m)).$$

Furthermore, if $I_i = \{z_1(i) < \dots < z_r(i)\}$ then we have

$$\beta_2(z_j(i), i) = z_j(i) - j = \beta_2(z_j(i) - 1, i)$$

for all $j \in \{1, \dots, r\}$.

Therefore, $\gamma(i) = \underline{\lambda}(I_i), 1 \leq i \leq m$ and so

$$f(\underline{\lambda}(I_1), \dots, \underline{\lambda}(I_m)) = 1.$$

We have just proved that if $(I_1, \dots, I_m)$ is in $\mathcal{S}(n, m)$ then $(a) - (e)$ are fulfilled.

Conversely, let $I = (I_1, \dots, I_m)$ be an m-tuple of subsets of $\{1, \dots, n\}$ satisfying $(a) - (e)$. Then we can define β_I such that $\beta_I \neq \beta$ and

$$\beta_I \circ (\beta - \beta_I) = f(\underline{\lambda}(I_1), \dots, \underline{\lambda}(I_m)) = 1.$$

Thus, $I = (I_1, \dots, I_m) \in \mathcal{S}(n, m)$ and so we are done. $\square$

Proposition 6.5. *Let $\lambda(i) = (\lambda_1(i), \dots, \lambda_n(i)),\ i \in \{1, \dots, m\}$ be m weakly decreasing sequences of n reals. Then the following are equivalent:*

(1) $\sigma_\lambda \in C(Q, \beta)$;

(2) *the numbers $\lambda_j(i)$ satisfy*

$$\sum_{i \text{ even}} |\lambda(i)| = \sum_{i \text{ odd}} |\lambda(i)|$$

together with

$$(*) \qquad \sum_{i \text{ even}} \left(\sum_{j \in I_i} \lambda_j(i) \right) \leq \sum_{i \text{ odd}} \left(\sum_{j \in I_i} \lambda_j(i) \right)$$

for every m-tuple $(I_1, \dots, I_m)$ for which $|I_1| = |I_2|$, $|I_{m-1}| = |I_m|$, $\underline{\lambda}(I_i),\ 1 \leq i \leq m$ are partitions and

$$f(\underline{\lambda}(I_1), \dots, \underline{\lambda}(I_m)) \neq 0;$$

(3) *the numbers $\lambda_j(i)$ satisfy*

$$\sum_{i \text{ even}} |\lambda(i)| = \sum_{i \text{ odd}} |\lambda(i)|$$

and $(*)$ for every m-tuple $(I_1, \dots, I_m)$ for which $|I_1| = |I_2|$, $|I_{m-1}| = |I_m|$, $\underline{\lambda}(I_i),\ 1 \leq i \leq m$ are partitions and

$$f(\underline{\lambda}(I_1), \dots, \underline{\lambda}(I_m)) = 1.$$

Proof. The proof follows from Proposition 6.2, Lemma 6.4 and Remark 4.6. □

We end this section with some further remarks on the set $\mathcal{S}(n, m)$. The next Lemma gives us constraints on the possible m-tuples $I = (I_1, \dots, I_m)$ of the set $\mathcal{S}(n, m)$.

Lemma 6.6. *Let $I = (I_1, \dots, I_m)$ be in $\mathcal{S}(n, m)$. Then the subsets $I_1, \dots, I_m$ satisfy:*

(a) *(if $m > 3$) for each i odd, $2 \leq i \leq m - 2$*

$$\max\{|I_{i-1}|, |I_{i+1}|\} \leq |I_i| \leq |I_{i-1}| + |I_{i+1}| + s_i,$$

where s_i is the smallest $k \in \{0, \dots, |I_i|\}$ such that $n - k \notin |I_i|$;

(b) *if $i = m - 1$ is odd we have $|I_{m-2}| \leq |I_{m-1}|$ and if $n \in I_m$ then either $n \in I_{m-1}$ or $I_{m-2} \neq \emptyset$.*

Proof. (a) Let us denote $\beta_I = \beta_1$ and $\beta - \beta_I = \beta_2$. Since $\beta_1 \circ \beta_2 \neq 0$ we have from Theorem 4.2 that any representation V of dimension vector $\beta = \beta_1 + \beta_2$ has a subrepresentation of dimension vector β_1. Choose V such that $V(a)$ is invertible for every main arrow a. Then for each i odd, $2 \leq i \leq m - 1$, we clearly have

$$\max\{|I_{i-1}|, |I_{i+1}|\} \leq |I_i|.$$

Let us denote $\langle \beta_1, \cdot \rangle$ by σ_1. A necessary condition for $\dim \mathrm{SI}(Q, \beta_2)_{\langle \beta_1, \cdot \rangle}$ not to be zero is that $\underline{\lambda}(I_i)$, $\forall\ 1 \leq i \leq m$ be partitions, i.e. they must have non-negative parts.

Suppose that $2 \leq i \leq m - 1$ is odd and let s_i be the smallest $k \in \{0, \dots, |I_i|\}$ such that $n - k \notin I_i$. Then the smallest part of $\lambda'(I_i)$ is exactly s_i.

For $2 \leq i \leq m - 2$ odd, we have seen that $\underline{\lambda}(I_i) = \lambda'(I_i) - (\sigma_1(i)^{n - |I_i|})$. On the other hand, we know that $\sigma_1(i) = |I_i| - |I_{i-1}| - |I_{i+1}|$ and the smallest part of $\lambda'(I_i)$ is precisely s_i. Thus, $\underline{\lambda}(I_i)$ is a partition if and only if

$$0 \leq |I_{i-1}| + |I_{i+1}| - |I_i| + s_i.$$

(b) If $i = m-1$ is odd and $n \notin I_m$ then

$$\sigma_1(m-1) = |I_{m-1}| - |I_m| - |I_{m-2}| = -|I_{m-2}| \leq 0$$

in which case $\underline{\lambda}(I_{m-1})$ is clearly a partition.

Now let assume that $i = m-1$ is odd and $n \in I_m$. Then

$$\sigma_1(m-1) = |I_{m-1}| - |I_m| + 1 - |I_{m-2}| = 1 - |I_{m-2}|$$

and hence $\underline{\lambda}(I_{m-1})$ is a partition when

$$s_{m-1} + |I_{m-2}| \geq 1.$$

So, in this case we must have that either $n \in I_{m-1}$ or $I_{m-2} \neq \emptyset$. □

Remark 6.7. When $m = 3$, the set $\mathcal{S}(n,3)$ is just the set of all triples (I_1, I_2, I_3) of subsets of $\{1, \ldots, n\}$ of the same cardinality r with $r < n$ and $c^{\lambda(I_2)}_{\lambda(I_1),\lambda(I_3)} = 1$. So, $\mathcal{K}(n,3)$ is indeed the Klyachko's cone. Therefore, in this case we recover the Horn type inequalities that solve the non-vanishing of the Littlewood-Richardson coefficients problem and Horn's conjecture.

7. Proof of Theorem 1.6 and Proposition 1.7

Before we prove our main theorem, we briefly recall the following moment map description of the cone of effective weights.

Proposition 7.1. [1, Proposition 1.3] *Let Q be a quiver without oriented cycles, β be a dimension vector and $\sigma \in \mathbb{R}^{Q_0}$. Then the following statements are equivalent:*

(1) $\sigma \in C(Q, \beta)$;
(2) *there exists $W = \{W(a)\}_{a \in Q_1} \in \mathrm{Rep}(Q, \beta)$ satisfying*

$$(\dagger) \quad \sum_{\substack{a \in Q_1 \\ ta = x}} W(a)^* W(a) - \sum_{\substack{a \in Q_1 \\ ha = x}} W(a) W(a)^* = \sigma(x) Id_{\beta(x)},$$

for all $x \in Q_0$, where $W(a)^*$ is the adjoint of $W(a)$ with respect to the standard Hermitian inner product on $\mathbb{C}^n$.

In what follows, we work with the generalized flag quiver setting from Section 3. To apply Proposition 7.1, we need the following useful fact from linear algebra:

Lemma 7.2. *Let $\sigma(1), \ldots, \sigma(n-1)$ be non-positive real numbers. Then the following are equivalent:*

(1) *there exist $W_i \in \mathrm{Mat}_{i \times (i+1)}(\mathbb{C}), 1 \leq i \leq n-1$ such that*

$$W_i \cdot W_i^\star - W_{i-1}^\star \cdot W_{i-1} = -\sigma(i) \,\mathrm{Id}_{\mathbb{C}^i} \text{ for } 2 \leq i \leq n-1,$$
$$W_1 \cdot W_1^\star = -\sigma(1);$$

(2) *there exists a $n \times n$ Hermitian matrix $H (= W_{n-1}^\star \cdot W_{n-1})$ with eigenvalues*

$$\nu(i) = -\sum_{j=i}^{n-1} \sigma(j), \forall\, 1 \leq i \leq n-1 \text{ and } \nu(n) = 0.$$

Proof. See [3, Section 3.4]. □

Proposition 7.3. *Let $\lambda(i) = (\lambda_1(i), \dots, \lambda_n(i)),\ 1 \leq i \leq m$ be m weakly decreasing sequences of n reals. Then*

$$\sigma_\lambda \in C(Q, \beta) \Longleftrightarrow (\lambda(1), \dots, \lambda(m) \in \mathcal{K}(n, m).$$

Proof. From Proposition 7.1, we know that $\sigma_\lambda \in C(Q, \beta)$ if and only if there exists $W \in \mathrm{Rep}(Q, \beta)$ satisfying the quiver matrix equations (†).

The matrix equations coming from the first $n-1$ vertices of the flag $\mathcal{F}(i)$ are essentially those from Lemma 7.2. So, they are equivalent to the existence of Hermitian matrices $H(i)$ with eigenvalues

$$(\lambda_1(i) - \lambda_n(i), \dots, \lambda_{n-1}(i) - \lambda_n(i), 0).$$

Let $a_1, \dots, a_{m-3}$ denote the main arrows, i.e., those connecting the central vertices. Taking into account the matrix equations coming from the main vertices, we see that $\sigma_\lambda \in C(Q, \beta)$ if and only if there exist Hermitian matrices $H'(i)$ with spectrum $\lambda(i), 1 \leq i \leq m$ and $n \times n$ complex matrices $W(a_i)$ such that:

$$\begin{aligned}
&H'(1) + W(a_1)^* \cdot W(a_1) = H'(2),\\
&W(a_1) \cdot W(a_1)^* + W(a_2) \cdot W(a_2)^* = H'(3),\\
&\dots\\
&H'(m) + W(a_{m-3})^* \cdot W(a_{m-3}) = H'(m-1)
\end{aligned}$$

When writing the last equation of the system above, we assumed that m is odd. Of course, if m is even, the last equation looks like

$$H'(m) + W(a_{m-3}) \cdot W(a_{m-3})^* = H'(m-1).$$

To bring the matrix equations above in a for us convenient form, we can conjugate (if necessary) the equations by unitary matrices. Also, note that for any $n \times n$ matrix, say A, we have that $A \cdot A^*$ and $A^* \cdot A$ are both positive semi-definite and have the same spectrum. Moreover, any positive semi-definite Hermitian matrix B can be written as $W \cdot W^*$ or $W^* \cdot W$.

Thus, we obtain that $\sigma_\lambda \in C(Q, \beta)$ if and only if there exist Hermitian matrices $H(i)$ with spectrum $\lambda(i), 1 \leq i \leq m$ and positive semi-definite $n \times n$ matrices $B(i)$ such that:

$$\begin{aligned}
&H(1) + B(1) = H(2),\\
&B(1) + B(2) = H(3),\\
&\dots\\
&H(m) + B(m-3) = H(m-1).
\end{aligned}$$

Solving this system of matrix equations for $B(i)$, we deduce that

$$B(i-1) = \sum_{j=1}^{i} (-1)^{j+i} H(j), \forall\ 2 \leq i \leq m-2$$

and

$$B(m-3) = H(m-1) - H(m).$$

Now, the proof follows. □

Proof of Theorem 1.6. (1) $\Longleftrightarrow$ (2) This equivalence follows from Proposition 6.5 and Proposition 7.3.

(1) $\Longleftrightarrow$ (3) Using Lemma 3.2 and Proposition 7.3 the equivalence follows.

(3) $\iff$ (4) Note that any long exact sequence breaks into short exact sequences by taking cokernels. Thus, (3) is equivalent to the existence of short exact sequences

$$0 \to M_1 \to M_2 \to N_1 \to 0,$$
$$0 \to N_1 \to M_3 \to N_2 \to 0,$$
$$\dots$$
$$0 \to N_{m-3} \to M_{m-1} \to M_m \to 0,$$

where $\mu(1), \cdots, \mu(m-3)$ are some partitions of length at most n and $N_1, \dots, N_{m-3}$ are finite abelian p-groups of types $\mu(1), \dots, \mu(m-3)$. This is equivalent to (4) by Klein's Theorem (see [12]). □

Remark 7.4. By definition, we know that $(\lambda(1), \dots, \lambda(m)) \in \mathcal{K}(n,m)$ if and only if there exist Hermitian matrices with prescribed eigenvalues and such that they satisfy a system of matrix (in)equalities. In principle, one can use the eigenvalue and the majorization problems (see [1] or [8]) to find necessary and sufficient Horn type inequalities for each of the matrix (in)equality defining the cone $\mathcal{K}(n,m)$. When we put together these inequalities we obtain a list of necessary but not sufficient Horn type inequalities. Indeed, let us look at these inequalities when $m=4$ and $n=2$. In this case, we want to find inequalities in the parts of $\lambda(1)$, $\lambda(2)$, $\lambda(3)$, $\lambda(4)$ such that there exist 2×2 Hermitian matrices $H(1)$, $H(2)$, $H(3)$, $H(4)$ with eigenvalues $\lambda(1)$, $\lambda(2)$, $\lambda(3)$, $\lambda(4)$ and

$$H(2) + H(4) = H(1) + H(3)$$

and

$$H(1) \le H(2).$$

The two conditions above imply the following list of necessary Horn type inequalities:

$$|\lambda(2)| + |\lambda(4)| = |\lambda(1)| + |\lambda(3)|,$$
$$\lambda_2(2) + \lambda_1(4) \le \lambda_1(1) + \lambda_1(3),$$
$$\lambda_1(2) + \lambda_2(4) \le \lambda_1(1) + \lambda_1(3),$$

and

$$\lambda_2(2) + \lambda_2(4) \le \lambda_2(1) + \lambda_1(3),$$
$$\lambda_2(2) + \lambda_2(4) \le \lambda_1(1) + \lambda_2(3),$$

and

$$\lambda_1(1) \le \lambda_1(2),\ \ \lambda_2(1) \le \lambda_2(2).$$

Comparing this list with the one worked out in Example 5.5, we see that the eigenvalue and the majorization problems give necessary Horn type inequalities which are not sufficient. For example, one can take $\lambda(1) = (2,1)$, $\lambda(2) = (3,1)$, $\lambda(3) = (4,1)$, and $\lambda(4) = (2,2)$.

Proof of Proposition 1.7. (1) The chamber inequalities of Lemma 5.3(1) and Proposition 7.3 show that

$$\mathcal{K}(n,m) \longrightarrow C(Q,\beta) \times \mathbb{R}^2$$
$$\lambda = (\lambda(1), \dots, \lambda(m)) \longrightarrow (\sigma_\lambda, \lambda_n(1), \lambda_n(m))$$

is an isomorphism of cones. Since β is a Schur root, the dimension of the cone $C(Q,\beta)$ is the number of the vertices of the generalized flag quiver minus one and so (1) follows.

(2) This is a consequence of Proposition 6.5.

□

8. Representation theoretic interpretations

In this section, we give two representation theoretic interpretations of the generalized Littlewood-Richardson coefficients.

8.1. **Parabolic Kazhdan-Lusztig polynomials.** In [16], Leclerc and Miyachi obtained some remarkable closed formulas for certain vectors of the canonical bases of the Fock space representation of the quantum affine algebra $U_q(\widehat{\mathrm{sl}}_n)$. As a direct consequence, they derived a combinatorial description of certain parabolic affine Kazhdan-Lusztig polynomials. To state some of their results, we need to review some definitions from [16, Section 5]. Let v be an indeterminate. We denote by $K = \mathbb{C}(v)$ the field of rational functions in v and let Sym be the algebra over K of symmetric functions in a countable set X of variables. Let $\mathcal{P}$ be the set of all partitions and S_λ be the Schur function labelled by $\lambda \in \mathcal{P}$. It is well known that the functions S_λ form a linear basis for Sym. We denote by $\langle \cdot, \cdot \rangle$ the scalar product for which this basis is orthonormal.

Now, let $N \geq 1$ be an integer and let $A_0, \dots, A_{N-1}$ be N countable sets of indeterminates. Let

$$\mathcal{S} = Sym(A_0, \dots, A_{N-1})$$

be the algebra over K of functions symmetric in each set $A_0, \dots, A_{N-1}$ separately. If $\underline{\lambda} = (\lambda^0, \dots, \lambda^{N-1}) \in \mathcal{P}^N$, consider

$$S_{\underline{\lambda}} = S_{\lambda^0}(A_0) \cdots S_{\lambda^{N-1}}(A_{N-1}).$$

Then $\{S_{\underline{\lambda}} \mid \underline{\lambda} \in \mathcal{P}^N\}$ forms a linear basis which is orthonormal with respect with the induced scalar product. In [16, Section 5.6], the authors introduced a canonical basis $\{\eta_{\underline{\lambda}}(v) \mid \underline{\lambda} \in \mathcal{P}^N\}$ and showed that:

Lemma 8.1. [16, Lemma 4] *For $\underline{\lambda}, \underline{\mu} \in \mathcal{P}^N$, we have*

$$\langle S_{\underline{\lambda}}, \eta_{\underline{\mu}}(v) \rangle = (-v)^{\delta(\underline{\lambda}, \underline{\mu})} \sum \prod_{0 \leq j \leq N-1} c^{\mu^j}_{\alpha^j, \beta^j} \cdot c^{\lambda^j}_{\beta^j, (\alpha^{j+1})'}$$

where the sum runs through all $\alpha^0, \dots, \alpha^N$, $\beta^0, \dots, \beta^{N-1}$ in $\mathcal{P}$ subject to:

$$|\alpha^i| = \sum_{0 \leq j \leq i-1} |\lambda^j| - |\mu^j|, \qquad |\beta^i| = |\mu^i| + \sum_{0 \leq j \leq i-1} |\mu^j| - |\lambda^j|,$$

and

$$\delta(\underline{\lambda}, \underline{\mu}) = \sum_{0 \leq j \leq N-2} (N-1-j)(|\lambda^j| - |\mu^j|).$$

Here the convention is that an empty sum is equal to zero. Hence, α^0 is the empty partition, $|\beta^0| = |\mu^0|$ and so

$$c^{\mu^0}_{\alpha^0, \beta^0} \cdot c^{\lambda^0}_{\beta^0, (\alpha^1)'} = c^{\lambda^0}_{\mu^0, (\alpha^1)'}.$$

By convention, α^N is the empty partition and hence

$$c^{\mu^{N-1}}_{\alpha^{N-1}, \beta^{N-1}} \cdot c^{\lambda^{N-1}}_{\beta^{N-1}, (\alpha^N)'} = c^{\mu^{N-1}}_{\alpha^{N-1}, \lambda^{N-1}}.$$

Now, let us rewrite the above scalar product using our generalized Littlewood-Richardson coefficients. It is easy to see that for $\underline{\lambda}, \underline{\mu} \in \mathcal{P}^N$ we have

$$\langle S_{\underline{\lambda}}, \eta_{\underline{\mu}}(v) \rangle = (-v)^{\delta(\underline{\lambda}, \underline{\mu})} \cdot f(\mu^0, \lambda^0, (\mu^1)', (\lambda^1)', \dots, \mu^{N-1}, \lambda^{N-1}).$$

Note that in the above formula we assumed that N is odd. For N even, just replace μ^{N-1} and λ^{N-1} in f with $(\mu^{N-1})'$ and $(\lambda^{N-1})'$ respectively.

Next, we explain how these formulas are related to some parabolic Kazhdan-Lusztig polynomials. Let $w \geq 1$ be an integer and let $\rho = (\rho_1, \ldots, \rho_l)$ be the large N-core associated with w. By $\mathcal{P}(\rho)$, we denote the set of partitions with N-core ρ. Let $\mathcal{P}(\rho, w) \subseteq \mathcal{P}(\rho)$ be the subset of partitions with N-weight $\leq w$. To each $\lambda \in \mathcal{P}(\rho)$, one can associate its N-quotient denoted by $\underline{\lambda} = (\lambda^0, \ldots, \lambda^{N-1})$. For all these definitions, we refer to [16, Section 6].

Corollary 8.2. [16, Corollary 10] *Let $\lambda, \mu \in \mathcal{P}(\rho, w)$. Then*

$$d_{\lambda,\mu}(v) = (-1)^{\delta(\underline{\lambda},\underline{\mu})} \langle S_{\underline{\lambda}}, \eta_{\underline{\mu}}(v) \rangle \in \mathbb{N}[v]$$

is a parabolic Kazhdan-Lusztig polynomial.

Furthermore, one has that $d_{\lambda,\mu}(1)$ is a decomposition number of a q-Schur algebra at a primitive N^{th} root q of unity (see also [10, Theorem 2]). Note that in this case, $d_{\lambda,\mu}(1)$ is a generalized Littlewood-Richardson coefficient.

8.2. **Multiplicities in representation spaces.** We show that the generalized Littlewood-Richardson coefficients can be viewed as multiplicities of some irreducible representations of a product of general linear groups in the affine coordinate ring of some representation space. For this, let us consider the alternating type $\mathbb{A}_m$ quiver with vertices $1, 2, \ldots, m$ such that 1 is a source, 2 is a sink, and so on. For example, if m is odd the alternating quiver looks like:

$$1 \longrightarrow 2 \qquad \cdots \qquad m-1 \longleftarrow m.$$

Now, let α be the dimension vector $\alpha = (n, \ldots, n)$. For simplicity, let us write $V(i) = \mathbb{C}^n$. Without loss of generality, let us assume that m is odd. Using the Littlewood-Richardson rule, we can decompose $\mathbb{C}[\mathrm{Rep}(Q, \alpha)]$ as follows:

$$\bigoplus f(\lambda(1), \ldots, \lambda(m)) \left(S^{\lambda(1)}V(1) \otimes S^{\lambda(2)}V^*(2) \otimes \cdots \otimes S^{\lambda(m)}V(m) \right),$$

where the sum is taken over all partitions $\lambda(i),\ 1 \leq i \leq m$ of length at most n. Thus, $f(\lambda(1), \ldots, \lambda(m))$ is equal to the multiplicity:

$$\mathrm{mult}_{\mathrm{GL}(\alpha)} \left(S^{\lambda(1)}V(1) \otimes S^{\lambda(2)}V^*(2) \otimes \cdots \otimes S^{\lambda(m)}V(m), \mathbb{C}[\mathrm{Rep}(Q, \alpha)] \right).$$

If m is even then $f(\lambda(1), \ldots, \lambda(m))$ is equal to the multiplicity:

$$\mathrm{mult}_{\mathrm{GL}(\alpha)} \left(S^{\lambda(1)}V(1) \otimes S^{\lambda(2)}V^*(2) \otimes \cdots \otimes S^{\lambda(m)}V^*(m), \mathbb{C}[\mathrm{Rep}(Q, \alpha)] \right).$$

References

[1] C. Chindris. Eigenvalues of Hermitian matrices and cones arising from quivers. To appear in IMRN, Volume 2006 (Preprint avilable at www.math.umn.edu/˜chindris/papers.html).

[2] C. Chindris. The cone of effective weights for quivers and Horn type problems. Ph.D Thesis, University of Michigan, 2005.

[3] W. Crawley-Boevey and Ch. Geiss. Horn's problem and semi-stability for quiver representations. In *Representations of algebra. Vol. I, II*, pages 40–48. Beijing Norm. Univ. Press, Beijing, 2002.

[4] H. Derksen and J. Weyman. Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients. *J. Amer. Math. Soc.*, 13(3):467–479, 2000.

[5] H. Derksen and J. Weyman. The combinatorics of quiver representations. Preprint, arXiv.math.RT/0608288, 2006.

[6] W. Fulton. *Young tableaux. With applications to representation theory and geometry*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.

[7] W. Fulton. Eigenvalues, invariant factors, highest weights, and Schubert calculus. *Bull. Amer. Math. Soc. (N. S.)*, 37(3):209–249, 2000.

[8] W. Fulton. Eigenvalues of majorized Hermitian matrices and Littlewood-Richardson coefficients. Special Issue: Workshop on Geometric and Combinatorial Methods in the Hermitian Sum Spectral Problem. *Linear Algebra Appl.*, 319(1-3):23–36, 2000.

[9] A. Horn. Eigenvalues of sums of Hermitian matrices. *Pacific J. Math.*, 12:225–241, 1962.

[10] G. James, S. Lyle, and A. Mathas. Rouquier blocks. *Math. Z.*, 252(3):511–531, 2006.

[11] V.G. Kac. Infinite root systems, representations of graphs and invariant theory II. *J. Algebra*, 78(1):141–162, 1982.

[12] T. Klein. The multiplication of Schur-functions and extensions of p-modules. *J. London Math. Soc.*, 43:280–284, 1968.

[13] A. Klyachko. Stable bundles, representation theory and Hermitian operators. *Selecta Math. (N.S.)*, 4(3):419–445, 1998.

[14] A. Knutson and T. Tao. The honeycomb model of $\mathrm{gl}_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.*, 12(4):1055–1090, 1999.

[15] A. Knutson, T. Tao, and Ch. Woodward. The honeycomb model of $\mathrm{gl}_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone. *J. Amer. Math. Soc.*, 17(1):19–48, 2004.

[16] B. Leclerc and H. Miyachi. Some closed formulas for canonical bases of Fock spaces. *Represent. Theory*, (6):290–312, 2002.

[17] A. Schofield. Semi-invariants of quivers. *J. London Math. Soc. (2)*, 43(3):385–395, 1991.

[18] A. Schofield. General representations of quivers. *Proc. London Math. Soc. (3)*, 65(1):46–64, 1992.

[19] A. Schofield and M. van den Bergh. Semi-invariants of quivers for arbitrary dimension vectors. *Indag. Math. (N.S.)*, 12(1):125–138, 2001.

University of Minnesota, School of Mathematics, Minneapolis, MN, USA
E-mail address: chindris@math.umn.edu