

NOTES ON GIT-FANS FOR QUIVERS

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1. INTRODUCTION

Given a quiver Q and a dimension vector β , King's various moduli spaces of β -dimensional semi-stable representations depend on the choice of a weight. A central question of Geometric Invariant Theory is to understand how these quotient varieties vary as the weights vary (see [3]). This problem of variation of quotients for quivers was considered by Hille and de la Pena in [6] where they introduced the wall system associated to (Q, β) .

Our goal in this short note is to go through the construction of GIT-fans for quivers. Let $C(Q, \beta)$ be the cone of effective weights associated to (Q, β) . Then, the cones defining the GIT-fan have the property that their relative interiors are precisely the GIT-classes.

Theorem 1.1 (The GIT-fan). *Let Q be a quiver without oriented cycles and β a dimension vector. Define the GIT-cone associated to an effective weight $\sigma \in C(Q, \beta)$ by*

$$C(\sigma) = \{\sigma' \in C(Q, \beta) \mid \text{Rep}(Q, \beta)_{\sigma}^{ss} \subseteq \text{Rep}(Q, \beta)_{\sigma'}^{ss}\}.$$

Then:

- (1) *for every effective weight σ , $C(\sigma)$ is a rational convex polyhedral cone whose relative interior is the GIT-class of σ ;*
- (2) *the cones $C(\sigma)$ form a finite fan covering of $C(Q, \beta)$.*

Consequently, there is a finite number of open subsets of $\text{Rep}(Q, \beta)$ which can be realized as sets of σ -semi-stable representations for some effective weights σ .

The construction of the GIT-fans for normal projective varieties is due to Ressayre [8, Theorem 5.2]. In [4, Theorem 3.2], Halic used Ressayre's theorem to prove the existence of GIT-fans in the affine case. Berchtold and Hausen [2, Theorem 2.11] gave a simple construction of the GIT-fans for torus actions on affine varieties. This case was recently extended to arbitrary reductive groups by Arzhantsev and Hausen in [1].

We prove Theorem 1.1 from scratch following closely the steps outlined in [8] but avoiding Dolgachev and Hu's finiteness theorem [3, Theorem 1.3.9]. Our arguments are based on King [7] semi-stability criterion for quiver representations and Schofield's [9] theory of general representations of quivers. Let us point out that when β is the thin sincere dimension vector (all coordinates equal to one), the GIT-fan in the theorem above coincides with the fan constructed by Hille in [5, Theorem 4.1].

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2. RECOLLECTION FROM QUIVER INVARIANT THEORY

Let $Q = (Q_0, Q_1, t, h)$ be a finite quiver without oriented cycles, where Q_0 is the set of vertices, Q_1 is the set of arrows and $t, h : Q_1 \rightarrow Q_0$ assign to each arrow $a \in Q_1$ its tail ta and head ha , respectively.

For simplicity, we will be working over the field of complex numbers \mathbb{C} . A representation V of Q over \mathbb{C} is a family of finite dimensional \mathbb{C} -vector spaces $\{V(x) \mid x \in Q_0\}$ together with a family $\{V(a) : V(ta) \rightarrow V(ha) \mid a \in Q_1\}$ of \mathbb{C} -linear maps. If V is a representation of Q , we define its dimension vector d_V by $d_V(x) = \dim_{\mathbb{C}} V(x)$ for every $x \in Q_0$. Thus the dimension vectors of representations of Q lie in $\Gamma = \mathbb{Z}^{Q_0}$, the set of all integer-valued functions on Q_0 .

Given two representations V and W of Q , we define a morphism $\phi : V \rightarrow W$ to be a collection of linear maps $\{\phi(x) : V(x) \rightarrow W(x) \mid x \in Q_0\}$ such that for every arrow $a \in Q_1$, we have $\phi(ha)V(a) = W(a)\phi(ta)$. We denote by $\text{Hom}_Q(V, W)$ the \mathbb{C} -vector space of all morphisms from V to W . In this way, we obtain the abelian category $\text{Rep}(Q)$ of all quiver representations of Q . Let W and V be two representations of Q . We say that V is a subrepresentation of W if $V(x)$ is a subspace of $W(x)$ for all vertices $x \in Q_0$ and $V(a)$ is the restriction of $W(a)$ to $V(ta)$ for all arrows $a \in Q_1$.

Let $\beta \in \mathbb{Z}_{\geq 0}^{Q_0}$ be a dimension vector of Q . We write $\beta_1 \hookrightarrow \beta$ if every β -dimensional representation has a subrepresentation of dimension vector β_1 . If $\sigma \in \mathbb{R}^{Q_0}$ and $\beta \in \mathbb{Z}^{Q_0}$, we define $\sigma(\beta)$ by

$$\sigma(\beta) = \sum_{x \in Q_0} \sigma(x)\beta(x).$$

The *cone of effective weights* associated to (Q, β) is defined by

$$C(Q, \beta) = \{\sigma \in \mathbb{R}^{Q_0} \mid \sigma(\beta) = 0 \text{ and } \sigma(\beta_1) \leq 0 \text{ for all } \beta_1 \hookrightarrow \beta\}.$$

The (saturated, affine) semigroup of lattice points of $C(Q, \beta)$ is

$$\Sigma(Q, \beta) = \{\sigma \in \mathbb{Z}^{Q_0} \mid \sigma(\beta) = 0 \text{ and } \sigma(\beta_1) \leq 0 \text{ for all } \beta_1 \hookrightarrow \beta\}.$$

2.1. Semi-invariants and moduli spaces for quivers. For every vertex x , we denote by ε_x the simple dimension vector corresponding to x , i.e., $\varepsilon_x(y) = \delta_{x,y}, \forall y \in Q_0$, where $\delta_{x,y}$ is the Kronecker symbol. If α, β are two elements of Γ , we define the Euler form by

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

Let β be a dimension vector of Q . The representation space of β -dimensional representations of Q is defined by

$$\text{Rep}(Q, \beta) = \bigoplus_{a \in Q_1} \text{Hom}(\mathbb{C}^{\beta(ta)}, \mathbb{C}^{\beta(ha)}).$$

If $\text{GL}(\beta) = \prod_{x \in Q_0} \text{GL}(\beta(x))$ then $\text{GL}(\beta)$ acts algebraically on $\text{Rep}(Q, \beta)$ by simultaneous conjugation, i.e., for $g = (g(x))_{x \in Q_0} \in \text{GL}(\beta)$ and $V = \{V(a)\}_{a \in Q_1} \in \text{Rep}(Q, \beta)$, we define $g \cdot V$ by

$$(g \cdot V)(a) = g(ha)V(a)g(ta)^{-1} \text{ for each } a \in Q_1.$$

Hence, $\text{Rep}(Q, \beta)$ is a rational representation of the linearly reductive group $\text{GL}(\beta)$ and the $\text{GL}(\beta)$ -orbits in $\text{Rep}(Q, \beta)$ are in one-to-one correspondence with the isomorphism classes of β -dimensional representations of Q . As Q is a quiver without oriented cycles, one can show that there is only one closed $\text{GL}(\beta)$ -orbit in $\text{Rep}(Q, \beta)$ and hence the invariant ring $I(Q, \beta) = \mathbb{C}[\text{Rep}(Q, \beta)]^{\text{GL}(\beta)}$ is exactly the base field \mathbb{C} .

Now, consider the subgroup $\mathrm{SL}(\beta) \subseteq \mathrm{GL}(\beta)$ defined by

$$\mathrm{SL}(\beta) = \prod_{x \in Q_0} \mathrm{SL}(\beta(x)).$$

Although there are only constant $\mathrm{GL}(\beta)$ -invariant polynomial functions on $\mathrm{Rep}(Q, \beta)$, the action of $\mathrm{SL}(\beta)$ on $\mathrm{Rep}(Q, \beta)$ provides us with a highly non-trivial ring of semi-invariants. Note that any $\sigma \in \mathbb{Z}^{Q_0}$ defines a rational character of $\mathrm{GL}(\beta)$ by

$$\{g(x) \mid x \in Q_0\} \in \mathrm{GL}(\beta) \mapsto \prod_{x \in Q_0} (\det g(x))^{\sigma(x)}.$$

In this way, we can identify $\Gamma = \mathbb{Z}^{Q_0}$ with the group $X^*(\mathrm{GL}(\beta))$ of rational characters of $\mathrm{GL}(\beta)$, assuming that β is a sincere dimension vector (i.e., $\beta(x) > 0$ for all vertices $x \in Q_0$). We also refer to the rational characters of $\mathrm{GL}(\beta)$ as (integral) weights.

Let $\mathrm{SI}(Q, \beta) = \mathbb{C}[\mathrm{Rep}(Q, \beta)]^{\mathrm{SL}(\beta)}$ be the ring of semi-invariants. As $\mathrm{SL}(\beta)$ is the commutator subgroup of $\mathrm{GL}(\beta)$ and $\mathrm{GL}(\beta)$ is linearly reductive, we have

$$\mathrm{SI}(Q, \beta) = \bigoplus_{\sigma \in X^*(\mathrm{GL}(\beta))} \mathrm{SI}(Q, \beta)_\sigma,$$

where

$$\mathrm{SI}(Q, \beta)_\sigma = \{f \in \mathbb{C}[\mathrm{Rep}(Q, \beta)] \mid g \cdot f = \sigma(g)f \text{ for all } g \in \mathrm{GL}(\beta)\}$$

is the space of semi-invariants of weight σ .

Definition 2.1. [7, Proposition 3.1] Let β be a dimension vector and σ an integral weight such that $\sigma(\beta) = 0$. A β -dimensional representation W is said to be:

- (1) σ -semi-stable if $\sigma(\underline{d}_{W'}) \leq 0$ for every subrepresentation W' of W ;
- (2) σ -stable if $\sigma(\underline{d}_{W'}) < 0$ for every proper subrepresentation $0 \neq W' \subsetneq W$.

We say that a dimension vector β is σ -(-semi)-stable if there exists a β -dimensional representation which is σ -(-semi)-stable.

Let β be a σ -semi-stable dimension vector. The set of σ -semi-stable representations in $\mathrm{Rep}(Q, \beta)$ is denoted by $\mathrm{Rep}(Q, \beta)_\sigma^{ss}$ while the set of σ -stable representations in $\mathrm{Rep}(Q, \beta)$ is denoted by $\mathrm{Rep}(Q, \beta)_\sigma^s$. The one dimensional torus

$$T = \{(t \mathrm{Id}_{\beta(x)})_{x \in Q_0} \mid t \in \mathbb{C}^*\} \subseteq \mathrm{GL}(\beta)$$

acts trivially on $\mathrm{Rep}(Q, \beta)$ and so there is a well-defined action of $\mathrm{PGL}(\beta) = \mathrm{GL}(\beta)/T$ on $\mathrm{Rep}(Q, \beta)$. Using methods from geometric invariant theory, one can construct the following GIT-quotient of $\mathrm{Rep}(Q, \beta)$:

$$\mathcal{M}(Q, \beta)_\sigma^{ss} = \mathrm{Proj}(\bigoplus_{n \geq 0} \mathrm{SI}(Q, \beta)_{n\sigma}).$$

It was proved by King [7] that $\mathcal{M}(Q, \beta)_\sigma^{ss}$ is a categorical quotient of $\mathrm{Rep}(Q, \beta)_\sigma^{ss}$ by $\mathrm{PGL}(\beta)$. Note that $\mathcal{M}(Q, \beta)_\sigma^{ss}$ is an irreducible projective variety, called the moduli space of β -dimensional σ -semi-stable representations (for more details, see [7]).

2.2. More on semi-stability and GIT-classes. If $\sigma \in \mathbb{R}^{Q_0}$ is a weight, define the set of β -dimensional σ -semi-stable representations by

$$\mathrm{Rep}(Q, \beta)_\sigma^{ss} = \{W \in \mathrm{Rep}(Q, \beta) \mid \sigma(\underline{d}_W) = 0 \text{ and } \sigma(\underline{d}_{W'}) \leq 0, \forall W' \leq W\}.$$

We know that when σ is an integral weight, this set is a (possibly empty) open subset of $\mathrm{Rep}(Q, \beta)$. In fact, this is going to be true (see Corollary 2.3) for arbitrary weights.

Given a weight $\sigma \in \mathbb{R}^{Q_0}$, we can construct the full subcategory $\text{Rep}(Q)_\sigma^{ss}$ of $\text{Rep}(Q)$ consisting of all σ -semi-stable representations of Q . It turns out that $\text{Rep}(Q)_\sigma^{ss}$ is an abelian category whose simple objects are precisely the σ -stable representations. Furthermore, every object of $\text{Rep}(Q)_\sigma^{ss}$ has a Jordan-Hölder filtration whose factors are σ -stable representations.

Following Dolgachev and Hu [3], we say that two (effective) weights σ_1 and σ_2 are *GIT-equivalent*, and write $\sigma_1 \sim \sigma_2$, if

$$\text{Rep}(Q, \beta)_{\sigma_1}^{ss} = \text{Rep}(Q, \beta)_{\sigma_2}^{ss}.$$

For example, $\sigma \sim m\sigma$ for any positive integer m . The GIT-equivalence class of a weight $\sigma \in C(Q, \beta)$ is denoted by $\langle \sigma \rangle$.

The *GIT-cone* associated to a weight $\sigma \in \mathbb{R}^{Q_0}$ is defined by:

$$C(\sigma) = \{\sigma' \in \mathbb{R}^{Q_0} \mid \text{Rep}(Q, \beta)_\sigma^{ss} \subseteq \text{Rep}(Q, \beta)_{\sigma'}^{ss}\}.$$

Note that two (effective) weights σ_1 and σ_2 are GIT-equivalent if and only if $\sigma_1 \in C(\sigma_2)$ and $\sigma_2 \in C(\sigma_1)$.

Let us record the following useful finiteness result:

Lemma 2.2. *The GIT-cones are rational convex polyhedral cones and there only finitely many of them.*

Proof. For every $\sigma \in C(Q, \beta)$, let D_σ be the set of all (sub)dimension vectors of the form $\beta' = \underline{d}_{W'}$, where W' is a subrepresentation of a σ -semi-stable representation $W \in \text{Rep}(Q, \beta)$. Then:

$$C(\sigma) = \{\sigma' \in \mathbb{R}^{Q_0} \mid \sigma'(\beta) = 0 \text{ and } \sigma'(\beta') \leq 0 \text{ for all } \beta' \in D_\sigma\}.$$

Since there are finitely many subdimension vectors $\beta' \leq \beta$, we clearly have only finitely many GIT-cones. \square

Corollary 2.3. *Any effective weight is GIT-equivalent to some integral effective weight.*

Proof. Let $\sigma \in C(Q, \beta)$. As $C(\sigma)$ is a rational convex polyhedral cone, we can find a sequence $\{\sigma_n\}_{n \geq 1} \subseteq C(\sigma) \cap \mathbb{Q}^{Q_0}$ such that $\sigma_n \rightarrow \sigma$. Since there are finitely many GIT-cones, we can assume that $C(\sigma_n) = C(\sigma_1)$, $\forall n \geq 1$. This implies that $\sigma \in C(\sigma_1)$ as $C(\sigma_1)$ is a closed subset of \mathbb{R}^{Q_0} . Hence, $\sigma \sim \sigma_1$ and we are done. \square

Definition 2.4. Let $\sigma \in C(Q, \beta)$ and $W \in \text{Rep}(Q, \beta)_\sigma^{ss}$. We say that W is σ -*polystable* if $\text{GL}(\beta)W$ is closed in $\text{Rep}(Q, \beta)_\sigma^{ss}$.

Remark 2.5. Note that the σ -polystable points are said to be pivotal for σ in the terminology of [8]. Our terminology is motivated by the following result of King:

Proposition 2.6. [7, Proposition 3.2] *Let $\sigma \in \Sigma(Q, \beta)$ and $W \in \text{Rep}(Q, \beta)_\sigma^{ss}$. Then the following are equivalent:*

- (1) W is a direct sum of σ -stable representations;
- (2) W is σ -polystable.

Remark 2.7. Let $\sigma \in \Sigma(Q, \beta)$ and consider the action of $\text{GL}(\beta)$ on the variety $\text{Rep}(Q, \beta)_\sigma^{ss}$. Then for every $W \in \text{Rep}(Q, \beta)_\sigma^{ss}$ there exists a unique (up to isomorphism) $\widetilde{W} \in \overline{\text{GL}(\beta)W} \cap \text{Rep}(Q, \beta)_\sigma^{ss}$ such that \widetilde{W} is σ -polystable. More precisely, consider a Jordan-Holder filtration of W (in the category $\text{Rep}(Q)_\sigma^{ss}$):

$$F_\bullet(W) : 0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{l-1} \subsetneq W_l = W,$$

with W_i/W_{i-1} σ -stable representations. Viewing the associated graded representation $gr(F_\bullet(W))$ as the limit of a 1-parameter subgroup of $\mathrm{GL}(\beta)$ acting on W , we deduce that

$$gr(F_\bullet(W)) := \bigoplus_{i=1}^l W_i/W_{i-1} \in \overline{\mathrm{GL}(\beta)W} \cap \mathrm{Rep}(Q, \beta)_{\sigma}^{ss}.$$

(Technically speaking, $gr(F_\bullet(W))$ is isomorphic to a point in the closure of the orbit of W .) In any case, Proposition 2.6 implies that

$$gr(F_\bullet(W)) \cong \widetilde{W}.$$

The following lemma will come in handy when finding the faces of GIT-cones (Proposition 4.2):

Lemma 2.8. *Let $\sigma_1, \sigma_2 \in \Sigma(Q, \beta)$.*

(1) *Assume that every σ_1 -polystable representation belongs to $\mathrm{Rep}(Q, \beta)_{\sigma_2}^{ss}$. Then:*

$$\mathrm{Rep}(Q, \beta)_{\sigma_1}^{ss} \subseteq \mathrm{Rep}(Q, \beta)_{\sigma_2}^{ss}.$$

(2) [8, Lemma 4.1] *Assume that*

$$\mathrm{Rep}(Q, \beta)_{\sigma_1}^{ss} \subseteq \mathrm{Rep}(Q, \beta)_{\sigma_2}^{ss}.$$

Let W_2 be σ_2 -polystable. Then there exists a σ_1 -polystable representation W_1 such that $W_2 \in \overline{\mathrm{GL}(\beta)W_1}$. Moreover, $\mathrm{Rep}(Q, \beta)_{\sigma_2}^s \subseteq \mathrm{Rep}_{\sigma_1}^s$.

Proof. (1) Let $W \in \mathrm{Rep}(Q, \beta)_{\sigma_1}^{ss}$ and let \widetilde{W} be a σ_1 -polystable representation so that

$$\widetilde{W} \in \overline{\mathrm{GL}(\beta)W} \cap \mathrm{Rep}(Q, \beta)_{\sigma_1}^{ss}.$$

Then, by assumption, \widetilde{W} is σ_2 -semi-stable. But this clearly implies that W is σ_2 -semi-stable, too. Indeed, one can use the description [7, Proposition 3.1] of semi-stable representations in terms of semi-invariants. So, $\mathrm{Rep}(Q, \beta)_{\sigma_1}^{ss} \subseteq \mathrm{Rep}(Q, \beta)_{\sigma_2}^{ss}$.

(2) See [8, Lemma 4.1]. □

3. ORBIT CONES OF QUIVER REPRESENTATIONS

Fix a quiver Q without oriented cycles and a dimension vector $\beta \in \mathbb{Z}_{\geq 0}^{Q_0}$.

Definition 3.1. Let $W \in \mathrm{Rep}(Q, \beta)$. The *orbit cone* of W is defined by

$$\Omega(W) = \{\sigma \in \mathbb{R}^{Q_0} \mid W \in \mathrm{Rep}(Q, \beta)_{\sigma}^{ss}\}.$$

Remark 3.2. Note that $\Omega(W)$ is a rational convex polyhedral cone:

$$\Omega(W) = \{\sigma \in \mathbb{R}^{Q_0} \mid \sigma(\beta) = 0 \text{ and } \sigma(\underline{d}_{W'}) \leq 0, \text{ for all subrepresentations } W' \subseteq W\}.$$

Furthermore, since there are finitely many dimension vectors β' with $\beta' \leq \beta$, it follows that there are finitely many orbit cones.

First, a simple lemma:

Lemma 3.3. *Let $\sigma \in C(Q, \beta)$ and $W \in \mathrm{Rep}(Q, \beta)_{\sigma}^{ss}$ be a σ -polystable representation. Then*

$$\sigma \in \mathrm{relint}(\Omega(W)).$$

Proof. Let $W' \subseteq W$ be a subrepresentation such that $\sigma(\underline{d}_{W'}) = 0$. This implies that $W', W/W'$, and hence $W' \oplus W/W'$ are σ -semi-stable. Now, it is well-known that $W' \oplus W/W' \in \overline{\mathrm{GL}(\beta)W}$. As the orbit of W is closed in $\mathrm{Rep}(Q, \beta)_{\sigma}^{ss}$, it follows

$$W' \oplus W/W' \cong W.$$

From this we obtain that $\Omega(W) = \Omega(W) \cap \mathbb{H}(\underline{d}_{W'})$. In other words, σ can not lie on any proper face of $\Omega(W)$ and this finishes the proof. \square

Now, let us identify some of the faces of $\Omega(W)$ (compare to [8, Lemma 3.5]):

Lemma 3.4. *Let $\sigma_0 \in \Sigma(Q, \beta)$ and $W \in \mathrm{Rep}(Q, \beta)_{\sigma_0}^{ss}$. Consider a Jordan-Hölder filtration of W (in the category $\mathrm{Rep}(Q)_{\sigma_0}^{ss}$):*

$$F_{\bullet}(W) : 0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{l-1} \subsetneq W_l = W,$$

with W_i/W_{i-1} σ_0 -stable representations. Let $\widetilde{W} \in \overline{\mathrm{GL}(\beta)W} \cap \mathrm{Rep}(Q, \beta)_{\sigma_0}^{ss}$ be so that \widetilde{W} is σ_0 -polystable. Then

$$\Omega(\widetilde{W}) = \Omega(W) \cap \{\sigma \in \mathbb{R}^{Q_0} \mid \sigma(\beta_i) = 0, 1 \leq i \leq l\},$$

where $\beta_i = \underline{d}_{W_i}, 1 \leq i \leq l$. In particular, $\Omega(\widetilde{W})$ is a face of $\Omega(W)$.

Proof. We have seen in Remark 2.7 that

$$(1) \quad \mathrm{gr}(F_{\bullet}(W)) := \bigoplus_{i=1}^l W_i/W_{i-1} \cong \widetilde{W}.$$

Now, if $\sigma \in \Omega(W) \cap \{\sigma \in \mathbb{R}^{Q_0} \mid \sigma(\beta_i) = 0, 1 \leq i \leq l\}$ then W_i are clearly σ -semi-stable. This implies that W_i/W_{i-1} are σ -semi-stable and hence \widetilde{W} is σ -semi-stable.

Conversely, let $\sigma \in \Omega(\widetilde{W})$. Then (1) implies that the W_i/W_{i-1} are σ -semi-stable. Hence, $W = W_l$ is σ -semi-stable and $\sigma(\beta_i) = 0$ for all $1 \leq i \leq l$. This finishes the proof. \square

The next proposition is essential for the description of the GIT-classes and for the construction of the GIT-fan associated to (Q, β) . It was proved for points of normal projective varieties in [8, Proposition 3.6]. The proof in that case makes use of a fundamental finiteness result of Dolgachev and Hu [3, Theorem 1.5]. Our arguments are conceptually much simpler and do not use Dolgachev and Hu's finiteness result.

Proposition 3.5 (Faces of orbit cones). *Let $W \in \mathrm{Rep}(Q, \beta)$. Then the following are true.*

- (1) *The faces of $\Omega(W)$ are exactly those of the form $\Omega(W')$ with $W' \in \overline{\mathrm{GL}(\beta)W}$.*
- (2) *There exists $W_0 \in \overline{\mathrm{GL}(\beta)W}$ such that $\Omega(W) = \Omega(W_0)$, and*

$$\sigma \in \mathrm{relint}(\Omega(W)) \iff W_0 \text{ is } \sigma \text{-polystable.}$$

Proof. (1) First, let us show that if $W' \in \overline{\mathrm{GL}(\beta)W}$ then $\Omega(W')$ is a face of $\Omega(W)$. Assume $\Omega(W')$ is not the trivial cone (i.e., it has positive dimension) and pick $\sigma' \in \mathrm{relint}(\Omega(W')) \cap \mathbb{Z}^{Q_0}$. Let $W'' \in \overline{\mathrm{GL}(\beta)W'} \cap \mathrm{Rep}(Q, \beta)_{\sigma'}^{ss}$ be so that W'' is σ' -polystable. By Lemma 3.4, we know that $\Omega(W'')$ is a face of $\Omega(W')$. But $\Omega(W'')$ contains a relative interior point of $\Omega(W')$ and therefore, we must have $\Omega(W'') = \Omega(W')$.

We clearly have that $W'' \in \overline{\mathrm{GL}(\beta)W} \cap \mathrm{Rep}(Q, \beta)_{\sigma'}^{ss}$. Applying Lemma 3.4 again, we obtain that $\Omega(W'')$ is a face of $\Omega(W)$ and so $\Omega(W')$ is indeed a face of $\Omega(W)$.

Now, let \mathcal{F} be a proper face of $\Omega(W)$. If \mathcal{F} is the trivial face then $\mathcal{F} = \Omega(S)$, where S is the zero element of the vector space $\text{Rep}(Q, \beta)$ (that is to say, S is the unique semi-simple representation belonging to $\text{Rep}(Q, \beta)$). Now, let us assume that \mathcal{F} is not the trivial face and choose $\tilde{\sigma} \in \text{relint } \mathcal{F} \cap \mathbb{Z}^{Q_0}$. Then, \mathcal{F} is the intersection of all facets of $\Omega(W)$ containing $\tilde{\sigma}$ and so we can write:

$$\mathcal{F} = \Omega(W) \bigcap \bigcap_{W'} \{\sigma \in \mathbb{R}^{Q_0} \mid \sigma(\underline{d}_{W'}) = 0\},$$

where the intersection on the right is over all subrepresentations $W' \subseteq W$ with $\tilde{\sigma}(\underline{d}_{W'}) = 0$.

Consider a Jordan-Holder filtration of W (in the category $\text{Rep}(Q)_{\tilde{\sigma}}^{ss}$):

$$F_{\bullet}(W) : 0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{l-1} \subsetneq W_l = W,$$

with W_i/W_{i-1} $\tilde{\sigma}$ -stable representations. Let $\widetilde{W} \in \overline{\text{GL}(\beta)W} \cap \text{Rep}(Q, \beta)_{\tilde{\sigma}}^{ss}$ be so that \widetilde{W} is $\tilde{\sigma}$ -polystable. In what follows, we show that

$$\mathcal{F} = \Omega(\widetilde{W}).$$

We have seen in Lemma 3.4 that

$$\Omega(\widetilde{W}) = \Omega(W) \bigcap \bigcap_{i=1}^l \mathbb{H}(\underline{d}_{W_i})$$

and $\Omega(\widetilde{W})$ is a face of $\Omega(W)$. So, $\mathcal{F} \subseteq \Omega(\widetilde{W})$ as $\Omega(\widetilde{W})$ is a face containing a relative interior point of \mathcal{F} .

Now, if $W' \subseteq W$ is a subrepresentation such that $\tilde{\sigma}(\underline{d}_{W'}) = 0$ then W' becomes $\tilde{\sigma}$ -semi-stable. The uniqueness of the factors of the Jordan-Holder filtration $F_{\bullet}(W)$ implies that $\underline{d}_{W'}$ is a non-negative integer linear combination of vectors of the form $\underline{d}_{W_i} - \underline{d}_{W_{i-1}}$. So, $\sigma(\underline{d}_{W'}) = 0$ for all $\sigma \in \Omega(\widetilde{W})$ and hence $\Omega(\widetilde{W}) \subseteq \mathcal{F}$. We have just proved that $\mathcal{F} = \Omega(\widetilde{W})$ with $\widetilde{W} \in \overline{\text{GL}(\beta)W}$.

(2) Let $\sigma_0 \in \text{relint}(\Omega(W)) \cap \mathbb{Z}^{Q_0}$ and let $W_0 \in \overline{\text{GL}(\beta)W} \cap \text{Rep}(Q, \beta)_{\sigma_0}^{ss}$ be so that W_0 is σ_0 -polystable. We are going to show that W_0 satisfies the required properties. Using (1), it is clear that

$$\Omega(W_0) = \Omega(W).$$

The implication " \Leftarrow " follows immediately from Lemma 3.3. For the implication " \Rightarrow ", let $\sigma \in \text{relint}(\Omega(W))$ and assume that $\text{GL}(\beta)W_0$ is not closed in $\text{Rep}(Q, \beta)_{\sigma}^{ss}$. Then there exists $W'' \in (\overline{\text{GL}(\beta)W_0} - \text{GL}(\beta)W_0) \cap \text{Rep}(Q, \beta)_{\sigma}^{ss}$. Note that $\sigma_0 \notin \Omega(W'')$ as $\text{GL}(\beta)W_0$ is closed in $\text{Rep}(Q, \beta)_{\sigma_0}^{ss}$. From this and part (1), we get that $\Omega(W'')$ is a proper face of $\Omega(W_0)$. But this contradicts the fact that $\Omega(W'')$ contains the relative interior point σ of $\Omega(W)$. \square

4. THE GIT-FAN

First a simple lemma:

Lemma 4.1. *If $\sigma \in C(Q, \beta)$ then*

$$C(\sigma) = \bigcap_W \Omega(W),$$

where the intersection is over all σ -polystable representations $W \in \text{Rep}(Q, \beta)$.

Proof. By Corollary 2.3, we can assume that $\sigma \in \Sigma(Q, \beta)$. The inclusion $C(\sigma) \subseteq \bigcap_W \Omega(W)$ is obvious while the other inclusion follows immediately from Lemma 2.8(1). This finishes the proof. \square

It is clear that in order to prove Theorem 1.1, we need to find the faces and relative interiors of GIT-cones.

Proposition 4.2 (Faces of GIT-cones). *Let $\sigma \in C(Q, \beta) \cap \mathbb{Z}^{Q_0}$ and let \mathcal{F} be a face of $C(\sigma)$. If $\sigma_0 \in \text{relint}(\mathcal{F}) \cap \mathbb{Z}^{Q_0}$ then*

$$\mathcal{F} = C(\sigma_0).$$

In particular, any $\sigma_0 \in \text{relint}(C(\sigma)) \cap \mathbb{Z}^{Q_0}$ is GIT-equivalent to σ .

Proof. By Lemma 4.1, we know that

$$C(\sigma) = \bigcap_W \Omega(W),$$

where the intersection is over all σ -polystable representations $W \in \text{Rep}(Q, \beta)$. Moreover, this is a finite intersection and so we have

$$\mathcal{F} = \bigcap_W \mathcal{F}_{W, \sigma_0},$$

where $\mathcal{F}_{W, \sigma_0}$ is the unique face of $\Omega(W)$ such that $\sigma_0 \in \text{relint}(\mathcal{F}_{W, \sigma_0})$.

Now, for every σ -polystable representation $W \in \text{Rep}(Q, \beta)$, let $\widetilde{W} \in \text{Rep}(Q, \beta)$ be the unique (up to isomorphism) σ_0 -polystable representation such that $\widetilde{W} \in \overline{\text{GL}(\beta)W}$. From the proof of Proposition 3.5, we know that each $\mathcal{F}_{W, \sigma_0} = \Omega(\widetilde{W})$ and hence

$$\mathcal{F} = \bigcap_W \Omega(\widetilde{W}),$$

where the intersection is over all σ -polystable representations $W \in \text{Rep}(Q, \beta)$. Using Lemma 2.8(2) and Lemma 4.1, this intersection is exactly $C(\sigma_0)$ and so $\mathcal{F} = C(\sigma_0)$. \square

The next result gives a polyhedral description of a GIT-class (compare to [8, Lemma 4.2 ¹]):

Proposition 4.3 (GIT-classes). *Let $\sigma \in C(Q, \beta)$ and $F = \langle \sigma \rangle$. Then*

$$F = \text{relint}(C(\sigma)).$$

Proof. First of all, we can assume that $\sigma \in \Sigma(Q, \beta)$ by Corollary 2.3. Now, let $\sigma' \in \text{relint}(C(\sigma))$.

As $\text{relint}(C(\sigma')) \cap \text{relint}(C(\sigma)) \neq \emptyset$, we know that $C(\sigma') \cap C(\sigma)$ is a rational convex polyhedral cone whose relative interior is non-empty. Hence, we can choose

$$\sigma_0 \in \text{relint}(C(\sigma')) \cap \text{relint}(C(\sigma)) \cap \mathbb{Z}^{Q_0}.$$

This implies that $\sigma_0 \sim \sigma$ by Proposition 4.2. We also have that $\text{Rep}(Q, \beta)_{\sigma'}^{ss} \subseteq \text{Rep}(Q, \beta)_{\sigma}^{ss} \subseteq \text{Rep}(Q, \beta)_{\sigma_0}^{ss}$ and so $\sigma' \sim \sigma$. We have just proved

$$\text{relint}(C(\sigma)) \subseteq F.$$

Now, let us prove the other inclusion. From Lemma 3.3, we know that

$$\sigma \in \bigcap_W \text{relint}(\Omega(W)),$$

where the intersection is over all σ -polystable representations $W \in \text{Rep}(Q, \beta)$. Since this is a non-empty intersection and we work with a finite intersection of rational convex polyhedral cones, we have:

$$\bigcap_W \text{relint}(\Omega(W)) = \text{relint}\left(\bigcap_W \Omega(W)\right).$$

¹Note that Rassayre's proof works only for rational points of the ample cone.

Using Lemma 4.1 and applying Lemma 3.3 for every point of F , we obtain $F \subseteq \text{relint}(C(\sigma))$. \square

Proof of Theorem 1.1. (1) This part follows now from Lemma 2.2 and Proposition 4.3.

(2) It is clear that $\mathcal{F}(Q, \beta) = \{C(\sigma) \mid \sigma \in C(Q, \beta)\}$ is a finite cover of $C(Q, \beta)$. Moreover, Proposition 4.2 shows that every face of a GIT-cone is again a GIT-cone. It remains to prove that the intersection of two GIT-cones is also a GIT-cone. Let $\sigma_1, \sigma_2 \in C(Q, \beta)$ be so that $C(\sigma_1) \cap C(\sigma_2) \not\supseteq \{0\}$. Now, pick a $\sigma_0 \in \text{relint}(C(\sigma_1) \cap C(\sigma_2)) \cap \mathbb{Z}^{Q_0}$. Let F_i be the unique (not necessarily proper) face of $C(\sigma_i)$ so that $\sigma_0 \in \text{relint}(F_i)$, $i \in \{1, 2\}$. From Proposition 4.2 follows $F_1 = F_2 = C(\sigma_0)$. So, $C(\sigma_0)$ is a face of $C(\sigma_1) \cap C(\sigma_2)$ containing a relative interior point this intersection. Therefore,

$$C(\sigma_1) \cap C(\sigma_2) = C(\sigma_0)$$

and this finishes the proof. \square

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