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AN INTERESTING PROBLEM TO "BUG" YOUR STUDENTS WITH

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In this note, we consider some well-known problems relating to "bugs" which have elementary solutions and interesting consequences. First we consider the "four-bug problem" (see [3]).

Four-bug Problem: Four bugs at the corners of a square crawl clockwise at a constant rate, each moving directly towards its neighbor. At any instant, the bugs form a square and, as they crawl towards one another, the square both decreases in size and rotates (see Figure 1). What is the path of motion of the bugs and how far do they travel before they "meet in the center"?

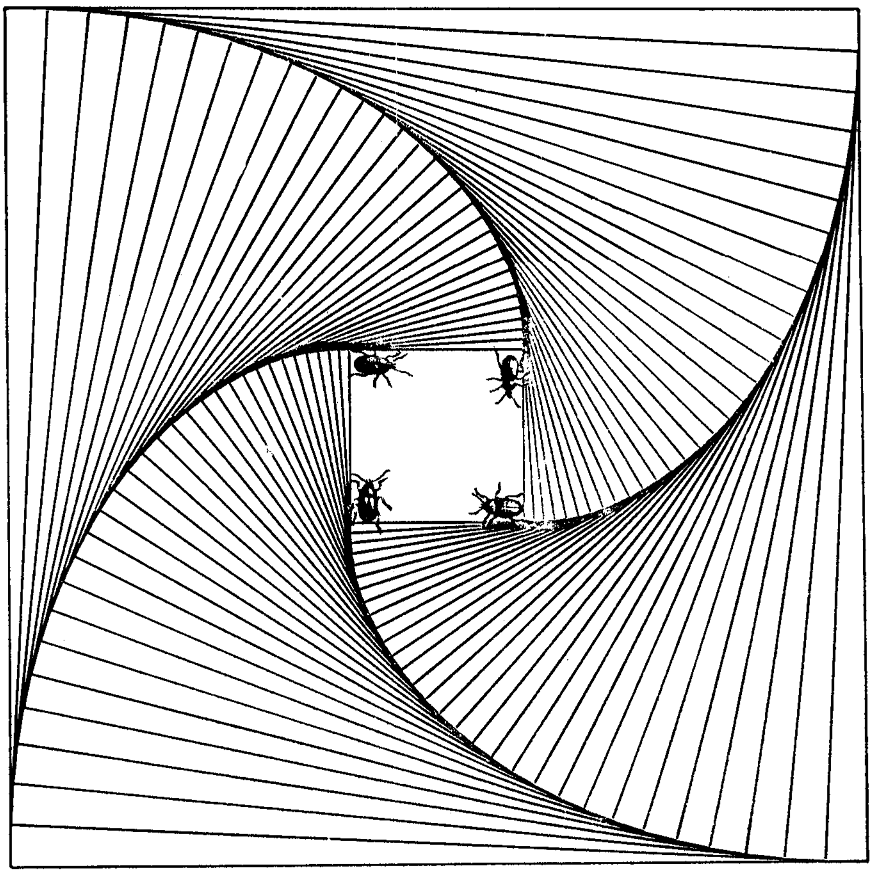

Figure 1

To solve this problem, assume the bugs are at the corners of a square of side $2a$ as in Figure 2. Because of the symmetry in the problem, if bug A has

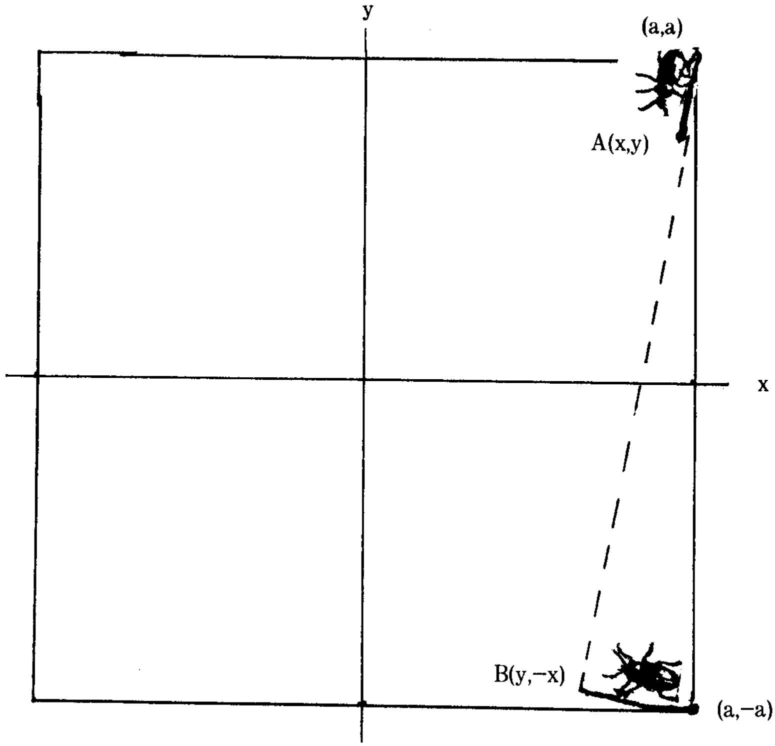

Figure 2

traveled to the point (x,y) in the square, then bug B is at the point (y,−x). So the slope of the line AB is

$$\frac{y+x}{x-y}.$$

Since bug A always travels directly towards bug B, its tangent line lies on top of the line AB and hence has the same slope as that of the line AB. Since the slope of the tangent line to the path of bug A at the point (x,y) is given by $\frac{dy}{dx}$, we have:

$$\frac{dy}{dx} = \frac{x+y}{x-y}.$$

To find the function y describing the path of bug A, we make the change of variables $y = wx$ and see that:

$$\frac{dy}{dx} = \frac{dw}{dx}x + w = \frac{x+wx}{x-wx} = \frac{1+w}{1-w}.$$

Hence

$$\frac{dw}{dx}x = \frac{1+w}{1-w} - w = \frac{1+w^2}{1-w},$$

and so

$$\frac{1-w}{1+w^2}\,dw = \frac{dx}{x}$$

Integrating both sides of this equation gives (for some constant c),

$$\text{Tan}^{-1} w - (1/2) \ln (1 + w^2) = \ln x + c$$

Since

$$w = \frac{y}{x} \text{ we have:}$$

$$\text{Tan}^{-1} \frac{y}{x} = \ln \left(x \sqrt{1 + (y/x)^2}\right) + c$$

$$= \ln \sqrt{x^2 + y^2} + c$$

Since y = a when x = a (the bug starts at the corner (a,a) of the box), we see that

$$\text{Tan}^{-1} 1 = \ln \sqrt{a^2 + a^2} + c,$$

and so

$$c = \frac{\pi}{4} - (\ln a \sqrt{2}).$$

So the equation of motion of the bug is:

$$\text{Tan}^{-1} \frac{y}{x} = \ln \sqrt{x^2 + y^2} + \frac{\pi}{4} - (\ln a \sqrt{2})$$

That this path is a logarithmic spiral is easily seen by changing to polar coordinates with the standard equations,

$$\text{Tan}^{-1} \frac{y}{x} = \Theta, \quad x^2 + y^2 = r^2.$$

Then our equation becomes

$$\Theta - \frac{\pi}{4} + (\ln a \sqrt{2}) = \ln r$$

and by raising e to these powers we have

$$r = a\sqrt{2}\, e^{(\Theta - \pi/4)}$$

This is the equation of motion of the bug in polar coordinates. Because the bugs are traveling in the clockwise direction, they are approaching the center as Θ *decreases* without bound. That is,

$$\lim_{\Theta \to -\infty} r = 0$$

At this point, we could apply the arc length formula to see how far the bugs travel. But this is not necessary since it is quite clear that the bugs travel a distance 2a (equal to the length of the side of the square). Since the path of A is at all times perpendicular to the path of B, there is no component of B's motion which carries B towards or away from A. Therefore, A will reach B in the same time that it would have taken if B had remained stationary. Thus, the length of each spiral path is the same as a side of the square. Actually, however, it is not the "four-bug problem" that we are interested in. What we want to consider here is the "n-bug problem" for n = 2,3,4, ...

N-bug Problem: If n bugs start at the corners of a regular n-sided polygon, and each crawls clockwise at a constant rate directly towards its neighbor, what is the equation of motion of the bugs and how far will they travel until they meet?

We may as well assume that the n-bugs are on a circle of radius 1 centered at the origin and equally spaced around the circle (See Figure 3). Note that $\alpha_n = \frac{\pi}{2} + \frac{\pi}{n}$ is the angle the line from A to B makes with the x-axis. So, in particular, the slope of line AB is tan $-\alpha_n$. In Figure 4, we've let the bugs move to a new position with bug A at the point (x,y) where $x = r \cos \Theta$ and $y = r \sin \Theta$ for some $r>0$. But, the tangent line to the path of A always points at (and hence goes through) B. Hence the slope of the line AB is also given by $\frac{dy}{dx}$.

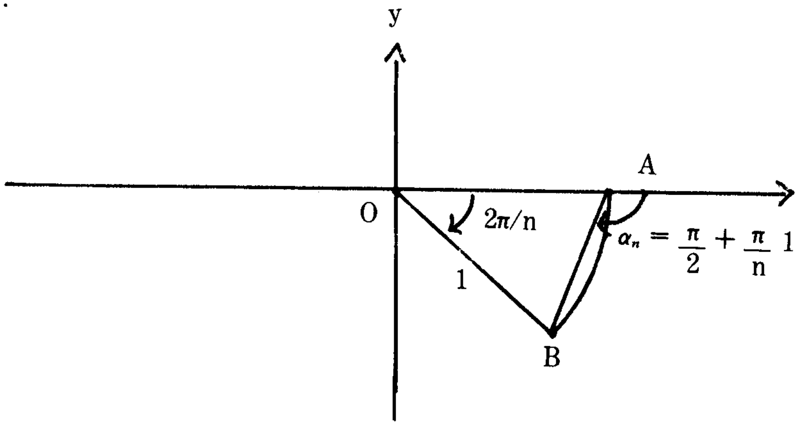

Figure 3

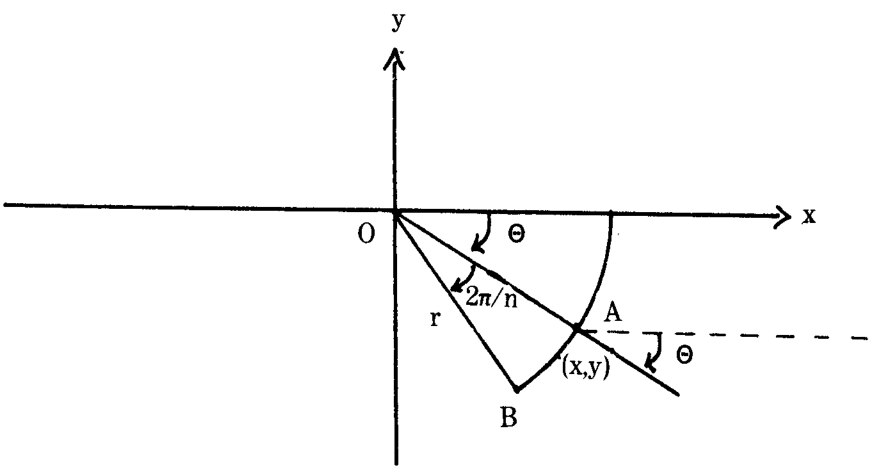

Figure 4

We have then,

$$\text{Tan }(\Theta + \alpha_n) = \frac{dy}{dx} = \frac{dy/d\Theta}{dx/d\Theta}$$

Since $x = r \cos \Theta$ and $y = r \sin \Theta$, it follows that

$$\frac{dx}{d\Theta} = \frac{dr}{d\Theta} \text{ cox } \Theta - r \sin \Theta$$

$$\frac{dy}{d\Theta} = \frac{dr}{d\Theta} \sin \Theta + r \cos \Theta .$$

Hence,

$$\text{Tan }(\Theta + \alpha_n) = \frac{\frac{dr}{d\Theta} \sin \Theta + r \cos \Theta}{\frac{dr}{dr} \cos \Theta - r \sin \Theta}$$

and so

$$(\frac{dr}{d\Theta} \cos \Theta - r \sin \Theta) \sin (\Theta + \alpha_n) = (\frac{dr}{d\Theta} \sin \Theta + r \cos \Theta) \cos (\Theta + \alpha_n).$$

Since

$$\text{Sin }(\Theta + \alpha_n) = \sin \Theta \cos \alpha_n + \cos \Theta \sin \alpha_n$$

and

$$\cos (\Theta + \alpha_n) = \cos \Theta \cos \alpha_n - \sin \Theta \sin \alpha_n$$

we have

$$\frac{dr}{d\Theta} \cos \Theta \sin \Theta \cos \alpha_n + \frac{dr}{d\Theta} \cos^2 \Theta \sin \alpha_n - r \sin^2 \Theta \cos \alpha_n$$
$$- r \sin \Theta \cos \Theta \sin \alpha_n = \frac{dr}{d\Theta} \sin \Theta \cos \Theta \cos \alpha_n$$
$$- \frac{dr}{d\Theta} \sin^2 \Theta \sin \alpha_n + r \cos^2 \Theta \cos \alpha_n - r \cos \Theta \sin \Theta \sin \alpha_n$$

Hence,

$$\frac{dr}{d\Theta} (\cos^2 \Theta + \sin^2 \Theta) \sin \alpha_n = r (\cos^2 \Theta + \sin^2 \Theta) \cos \alpha_n,$$

and so

$$\frac{dr}{d\Theta} \sin \alpha_n = r \cos \alpha_n$$

That is,

$$\frac{dr}{r} = (\cot \alpha_n)\, d\,\Theta = -(\tan \frac{\pi}{n})\, d\,\Theta$$

Integrating both sides we obtain,

$$\ln r = -(\tan \frac{\pi}{n})\, \Theta .$$

So the equation of motion of the bugs is the logarithmic spiral whose equation in polar coordinates is,

$$r = e^{-(\tan \frac{\pi}{n})\, \Theta}.$$

Using the formula for arc-length in polar coordinates we obtain for the distance d traveled by the bugs:

$$d = \int_0^{\infty} \sqrt{r^2 + (\frac{dr}{d\Theta})^2}\, d\,\Theta = \int_0^{\infty} \sqrt{1 + \tan^2 \frac{\pi}{n}}\, e^{-(\tan \frac{\pi}{n})} d\Theta$$

$$d = (\sec\frac{\pi}{n}) \lim_{t\to\infty} \int_0^t e^{-(\tan\frac{\pi}{n})\Theta}\, d\Theta$$

$$= (\sec\frac{\pi}{n}) \lim_{t\to\infty} (-\frac{1}{\tan\frac{\pi}{n}}) e^{-(\tan\frac{\pi}{n})\Theta}\Big|_0^t$$

$$= \frac{(\sec\frac{\pi}{n})}{\tan\frac{\pi}{n}} \lim_{t\to\infty} (-e^{-(\tan\frac{\pi}{n})t} + e^0)$$

$$= \frac{\sec\frac{\pi}{n}}{\tan\frac{\pi}{n}} = \csc\frac{\pi}{n}.$$

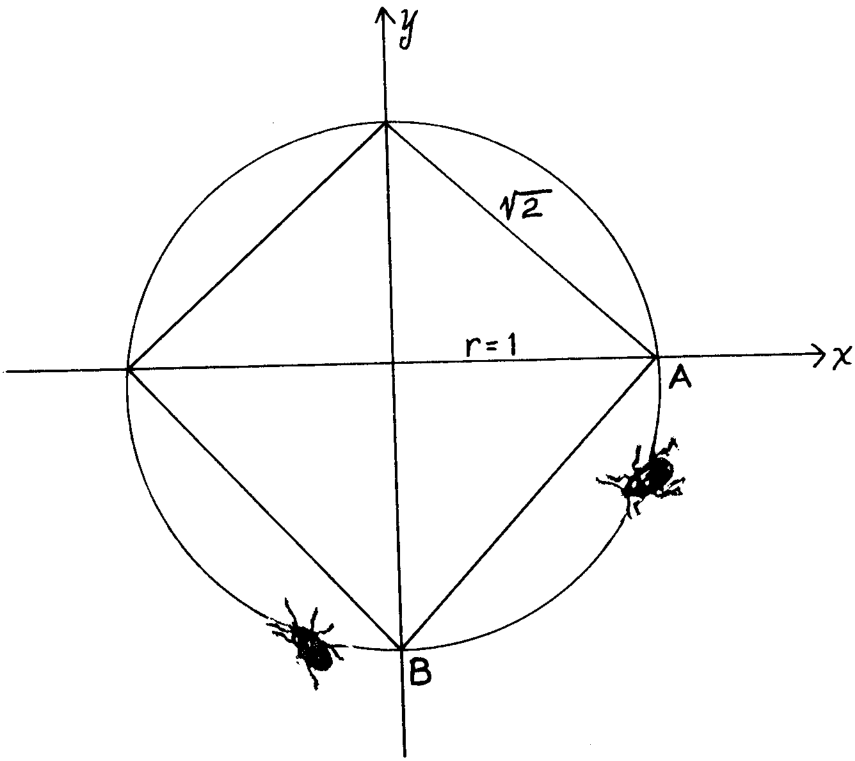

Figure 5

If $n = 4$, the bugs are on a square of side $\sqrt{2}$ (See Figure 5) and $d = \csc \pi/4 = \sqrt{2}$ as predicted earlier. If $n = 2$, so that the bugs are on opposite ends of a straight line of length 2 walking directly towards each other, we get the obvious answer

$$d = \csc\frac{\pi}{2} = 1.$$

If $n = 3$, the bugs travel $\frac{2}{3}$ the length of the side of the triangle since,

$$d = \csc\frac{\pi}{3} = \frac{2}{\sqrt{3}}$$

while they sit on the corners of a triangle of side $\frac{3}{\sqrt{3}}$. That is, for $n<4$, the bugs travel a distance less than the original distance between them. If $n>4$, they travel a distance greater than the distance between them. If we think of the circle as an "infinite sided polygon" and we put bugs at an "infinite number of corners", the bugs march forever in a circle (i.e. they travel an infinite distance). This follows from our formula for the distance since $\csc\frac{\pi}{n}$ is monotone increasing as a function of n for $n\geqslant 2$, and

$$\lim_{n\to\infty} \csc\frac{\pi}{n} = +\infty.$$

Another method for finding the distance traveled by the bugs was given by J. Charles Clapham [5], in which he shows that the distance traveled by a bug can be found trigonometricly by extending a side AB of the polygon (See Figure 6 for the case $n = 6$) and locating it at a point X such that angle A0X is 90 degrees. The length of AX is then the distance traveled by the bugs. This follows from our formula since the length of AX is merely sec Θ and for a polygon with n sides, $\Theta = \frac{1}{2}\frac{(n-2)\pi}{n}$ and so

$$\csc\frac{\pi}{n} = \sec\Theta = \text{length of AX}.$$

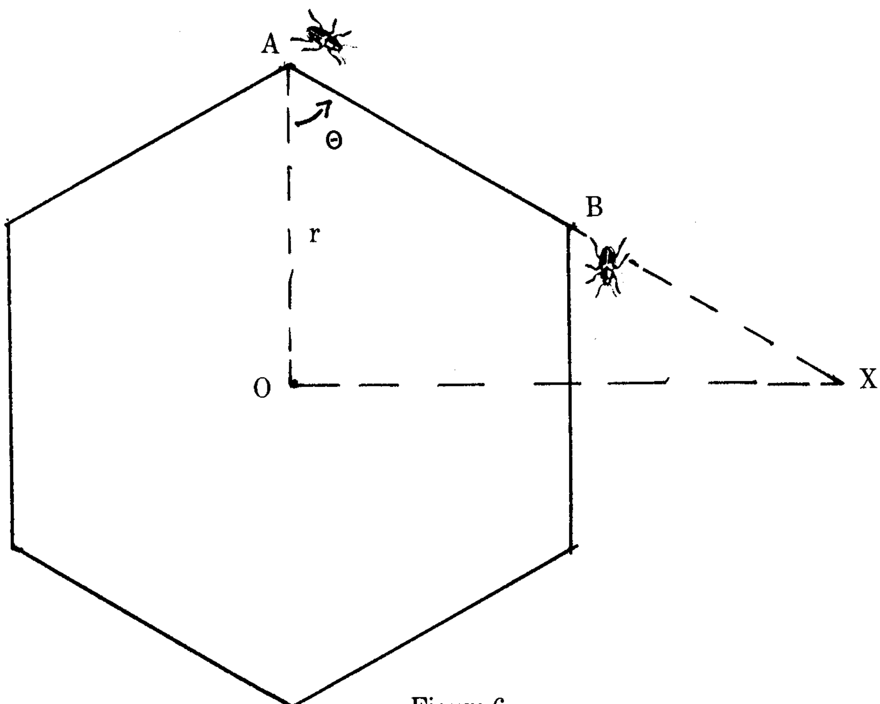

Figure 6

Another thing you can do is have your students make pictures using the paths of the bugs. Figure 7 is a design made by Rutherford Boyd using the paths from the "three-bug problem". The picture is made entirely of triangles although your eye will have trouble following these lines. (See [5] for some interesting pictures made with mathematical designs.)

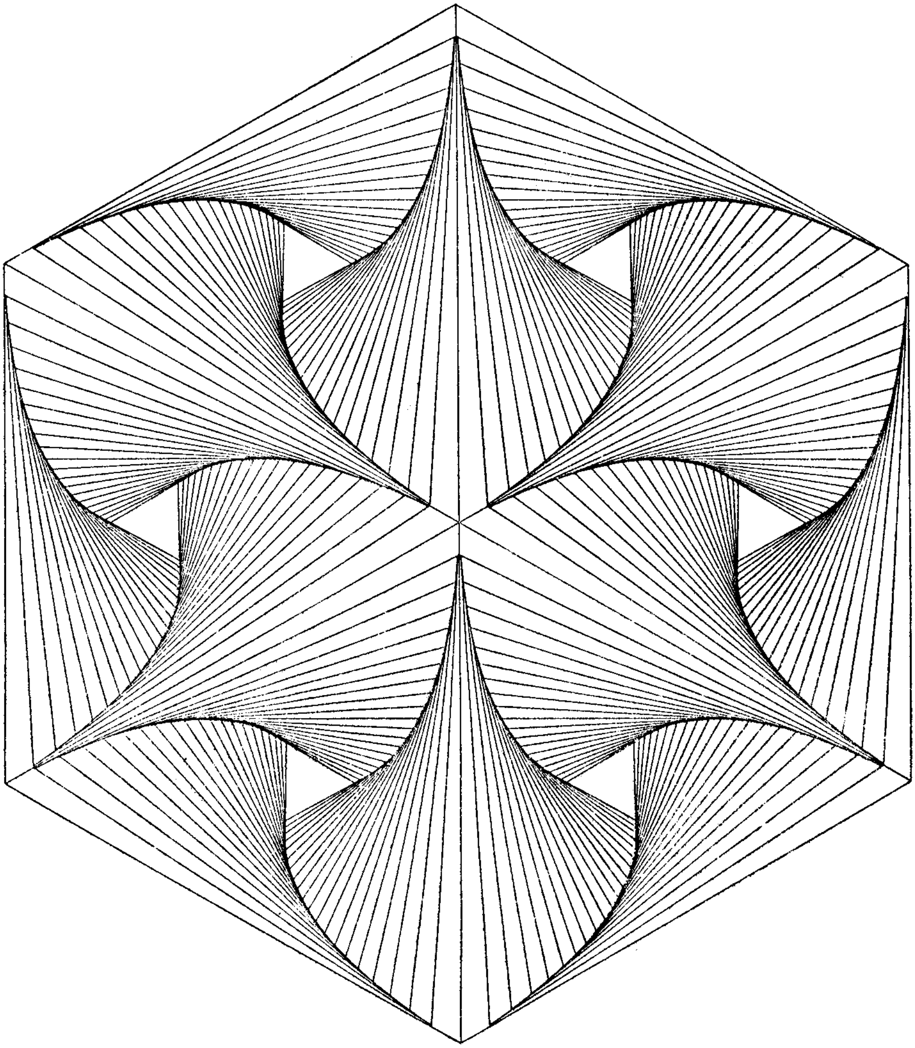

Figure 7

If you haven't "bugged" your students enough at this point, you can give them a homework problem proposed by Richard Hess [2]:

Problem: A bug starts at Monday noon at the upper left corner (X) of a p by q rectangle and crawls within the rectangle to the diagonally opposite corner

(Y), arriving at 6:00 p.m.. (See Figure 8). Exhausted, he sleeps till noon Tuesday. At that time he embarks for X, crawling along another path in the rectangle and arriving at X at 6:00 p.m. on Tuesday. Prove that at some time Tuesday the bug was at a point no further than p from where he was at the same time on Monday.

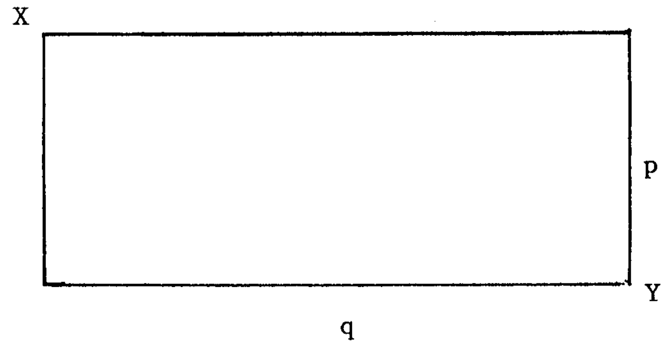

Figure 8

The solution of this problem is similar to that of the "up and down the mountain problem". (See [1], p-68).

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