Prime divisors in Beatty sequences

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Abstract

We study the values of arithmetic functions taken on the elements of a non-homogeneous Beatty sequence $\lfloor \alpha n + \beta \rfloor$, $n = 1, 2, \ldots$, where $\alpha, \beta \in \mathbb{R}$, and $\alpha > 0$ is irrational. For example, we show that

$$\sum_{n \leq N} \omega(\lfloor \alpha n + \beta \rfloor) \sim N \log \log N \quad \text{and} \quad \sum_{n \leq N} (-1)^{\Omega(\lfloor \alpha n + \beta \rfloor)} = o(N),$$

where $\Omega(k)$ and $\omega(k)$ denote the number of prime divisors of an integer $k \neq 0$ counted with and without multiplicities, respectively.

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1. Introduction

For two fixed real numbers $\alpha$ and $\beta$, the corresponding non-homogeneous Beatty sequence is the sequence of integers defined by

$$B_{\alpha, \beta} = (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty}.$$

Beatty sequences appear in a variety of apparently unrelated mathematical settings, and because of their versatility, the arithmetic properties of these sequences have been extensively explored in the literature; see, for example, [1–4,10,12,13,15,17,23] and the references contained therein.

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In this paper, we show that the methods of [2,3] can be combined with various results about sumsets (see [5–9,18–21]) to obtain new statements about prime divisors of the elements of a Beatty sequence. In particular, for any irrational number $\alpha > 0$, we derive estimates for various sums involving the number of prime divisors (counted with or without multiplicities) of the elements of $B_{\alpha,\beta}$, and we establish an Erdős–Kac type result. We also show that extreme cases occur among the elements of $B_{\alpha,\beta}$; in particular, there are “almost prime” elements as well as elements with almost the maximum possible number of prime divisors.

We remark that the error terms in our main results are all of the form $o(1)$. However, if more information about Diophantine properties of $\alpha$ is available, then (as in [2,3]) one can obtain more explicit bounds for the error terms in our estimates.

2. Preliminaries

2.1. Notation and definitions

In what follows, the letters $k$, $m$, and $n$ (with or without subscripts) always denote non-negative integers.

We use $\Omega(k)$ and $\omega(k)$ to denote the number of prime divisors of an integer $k \geq 1$ counted with and without multiplicities, respectively, and we use $P(k)$ to denote the largest prime divisor of $k$; we also put $\Omega(k) = \omega(k) = P(k) = 0$ if $k \leq 0$.

As usual, $\rho(u)$ denotes the Dickman function; for an account of the basic analytic properties of $\rho(u)$, we refer the reader to [22, Chapter III.5].

For a real number $x$, we denote by $\lfloor x \rfloor$ the greatest integer $\leq x$, and by $\{x\} = x - \lfloor x \rfloor$ the fractional part of $x$.

The discrepancy $D$ of a sequence of (not necessarily distinct) real numbers $a_1, \ldots, a_L \in [0, 1)$ is defined by the relation

$$D = \sup_{\mathcal{I} \subseteq [0, 1]} \left| \frac{V(\mathcal{I}, L)}{L} - |\mathcal{I}| \right|,$$

where the supremum is taken all subintervals $\mathcal{I} = (c, d)$ of the interval $[0, 1)$, $V(\mathcal{I}, L) = \#\{n \leq L: a_n \in \mathcal{I}\}$, and $|\mathcal{I}| = d - c$ is the length of $\mathcal{I}$.

Throughout the paper, any implied constants in the symbols $O$, $\ll$ and $\gg$ may depend (where obvious) on the real numbers $\alpha$ and $\varepsilon$ but are absolute otherwise. We recall that the notations $U = O(V)$, $U \ll V$, and $V \gg U$ are all equivalent to the assertion that the inequality $|U| \leq cV$ holds for some constant $c > 0$. We also use the symbol $o(1)$ to denote a function which tends to 0 and depends only on $\alpha$. It is important to note that our bounds are uniform with respect to all of the other parameters, in particular, with respect to $\beta$.

2.2. Discrepancy of a linear function

It is well known (see, for example, [14, Example 2.1, Chapter 1]) that for every irrational number $\alpha$, the sequence $\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots$, is uniformly distributed modulo 1; this implies the following result:
Lemma 1. Let $\alpha$ be a fixed irrational number. Then, for all $\beta \in \mathbb{R}$, the discrepancy $D_{\alpha, \beta}(L)$ of the sequence $a_1, \ldots, a_L$ defined by

$$a_n = \{\alpha n + \beta\} \quad (n = 1, 2, \ldots, L)$$

satisfies the bound

$$D_{\alpha, \beta}(L) = o(1),$$

where the function implied by $o(1)$ depends only on $\alpha$.

We also need the following statement from [2]:

Lemma 2. Let $\alpha$ be a fixed irrational number. Then, for every positive integer $L$ and every real number $\delta \in (0, 1]$, there exists a real number $\gamma \in [0, 1)$ such that

$$\#\{n \leq L: \{\alpha n + \gamma\} < \delta\} \geq 0.5L\delta.$$ 

2.3. Average number of prime divisors over large sets

The next result can easily be improved and extended in several directions; however, it is perfectly adequate for our purposes.

Lemma 3. Let $\mathcal{M}$ be a set of integers in the interval $[1, M]$ with

$$\#\mathcal{M} \geq \frac{M}{\log M}.$$ 

Then,

$$\sum_{m \in \mathcal{M}} \omega(m) \ll \#\mathcal{M} \log \log M.$$ 

Proof. Let $\tau(m)$ denote the number of positive integer divisors of $m \geq 1$. Combining the trivial bound $\tau(m) \geq 2^{\omega(m)}$ with the asymptotic formula

$$\sum_{m \leq M} \tau(m) = (1 + o(1))M \log M$$

(see [11, Theorem 320]), it follows that

$$\mathcal{E} = \{m \leq M: \omega(m) \geq 5 \log \log M\}$$

is a set of cardinality at most

$$\#\mathcal{E} \leq 2^{-5\log\log M} \sum_{m \leq M} \tau(m) = (1 + o(1))M (\log M)^{1-5\log 2}. \quad (2)$$
Using the crude bound $\omega(m) \ll \log M$ for all $m \in \mathcal{M} \cap \mathcal{E}$, we derive that

$$\sum_{m \in \mathcal{M}} \omega(m) \ll \#\mathcal{E} \log M + \#\mathcal{M} \log \log M.$$ 

Applying (2), we obtain the stated result. □

2.4. Arithmetic functions and sumsets

We begin with the following partial case of [18, Theorem 1], which we have reformulated in a convenient form for our application below:

**Lemma 4.** Let $A$ and $B$ be two sets of integers in the interval $[1, M]$. Then,

$$\sum_{a \in A} \sum_{b \in B} (-1)^{\Omega(a + b)} \ll \frac{M (\#A \#B)^{1/2}}{\log M}.$$ 

We also use the following statement, which combines (and simplifies) some results from [9,19]:

**Lemma 5.** Let $A$ and $B$ be two sets of integers in the interval $[1, M]$ with

$$\min\{\#A, \#B\} \gg M.$$ 

Then there exist integers $a_1, a_2 \in A$ and $b_1, b_2 \in B$ for which

$$\omega(a_1 + b_1) = (1 + o(1)) \frac{\log M}{\log \log M} \quad \text{and} \quad P(a_2 + b_2) \gg M.$$ 

A result of [8] implies that, under certain hypotheses, the number of divisors of a “typical” element of a sumset is about the same as the number of divisors of a “random” integer (see also [7,20,21] for other results in this direction). More precisely:

**Lemma 6.** Let $A$ and $B$ be two sets of integers in the interval $[1, M]$ with

$$\#A \#B = M^2 \exp\left(o\left((\log \log M)^{1/2} \log \log \log M\right)\right).$$ 

Then,

$$\frac{1}{\#A \#B} \sum_{a \in A} \sum_{b \in B} \omega(a + b) = (1 + o(1)) \log \log M.$$ 

The following Erdős–Kac type result is given in [8] (see also [21], which shows that one can take $(\log \log M)^{1/2}$ instead of $(\log \log M)^{1/4}$ in the error term, as well as the related work [7]):
Lemma 7. Let \( A \) and \( B \) be two sets of integers in the interval \([1, M]\), and let \( C \) be a real number. Then the estimate

\[
\# \{(a, b) \in A \times B: \omega(a + b) - \log \log M \leq C(\log \log M)^{1/2}\} = \frac{\#A \#B}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} dt + O\left(\frac{M \#A \#B^{1/2}}{(\log \log M)^{1/4}}\right)
\]

holds uniformly for all \( A, B, M \) and \( C \).

We also need the following result of [5] (see also [6]):

Lemma 8. Let \( A \) and \( B \) be two sets of integers in the interval \([1, M]\). Then, for any fixed \( \varepsilon > 0 \) and uniformly for \( \exp((\log M)^{2/3+\varepsilon}) < Q \leq M \), we have

\[
\# \{(a, b) \in A \times B: P(a + b) \leq Q\} = \rho(u) \cdot \#A \#B \left(1 + O\left(\frac{M \log(u + 1)}{(\#A \#B)^{1/2} \log Q}\right)\right),
\]

where \( u = (\log M)/\log Q \).

Finally, let \( \sigma_2(m) \) denote the sum of binary digits of \( m \). We need the following estimate, which is a special case of Theorem 1 from [16]:

Lemma 9. Let \( A \) and \( B \) be two sets of integers in the interval \([1, M]\). Then,

\[
\# \{(a, b) \in A \times B: \sigma_2(a + b) \equiv 0 \pmod{2}\} = \frac{\#A \#B}{2} + O\left(2^{\alpha M} (\#A \#B)^{1/2}\right),
\]

where

\[
\alpha = \frac{1}{4} + \frac{\log(\csc(\pi/8))}{2 \log 2} = 0.942888 \ldots
\]

3. Main results

Theorem 1. Let \( \alpha, \beta \in \mathbb{R} \) be fixed, where \( \alpha \) is positive and irrational. Then,

\[
\sum_{n \leq N} (-1)^{\Omega(\lfloor \alpha n + \beta \rfloor)} = o(N).
\]

Proof. First, we consider the case that \( \alpha > 1 \). Let \( K \leq N \) be a positive integer, let \( \Delta \) be a real number in the interval \((0, 1]\), and for every real number \( \gamma \in [0, 1) \), let
\[
\mathcal{N}_\gamma = \{ c \leq n \leq N : \{ \alpha n + \beta - \gamma \} < 1 - \Delta \}, \\
\mathcal{K}_\gamma = \{ c \leq k \leq K : \{ \alpha k + \gamma \} < \Delta \}, \\
\]

where

\[
c = \max \{ \lceil \alpha^{-1} (1 - \beta + \gamma) \rceil, \lceil \alpha^{-1} (1 - \gamma) \rceil \} \asymp 1.
\]

Observe that, by our choice of \( c \), both sets

\[
\mathcal{A}_\gamma = \{ \lfloor \alpha n + \beta - \gamma \rfloor : n \in \mathcal{N}_\gamma \} \quad \text{and} \quad \mathcal{B}_\gamma = \{ \lfloor \alpha k + \gamma \rfloor : k \in \mathcal{K}_\gamma \}
\]

are contained in the interval \([1, M_\gamma]\), where

\[
M_\gamma = \max \{ \lfloor \alpha N + \beta - \gamma \rfloor, \lfloor \alpha K + \gamma \rfloor \} \asymp N.
\]

Also, since \( \alpha > 1 \), the natural maps \( \mathcal{N}_\gamma \to \mathcal{A}_\gamma \) and \( \mathcal{K}_\gamma \to \mathcal{B}_\gamma \) are bijections.

Put \( \mathcal{N}^c_\gamma = \{1, 2, \ldots, N\} \setminus \mathcal{N}_\gamma \). From the definition (1) and Lemma 1, it follows that

\[
\# \mathcal{N}^c_\gamma = N \Delta + o(N). \tag{3}
\]

By Lemma 2, we also have the lower bound

\[
\# \mathcal{K}_\gamma \geq 0.5 K \Delta - c \tag{4}
\]

for some choice of \( \gamma \in [0, 1) \). We now fix \( \gamma \) such that (4) holds and remove the subscript \( \gamma \) from the notation, writing \( \mathcal{N} = \mathcal{N}_\gamma, \mathcal{K} = \mathcal{K}_\gamma \), etc.

From now on, we assume that \( K \Delta \geq 10c \); in particular, (4) implies

\[
\# \mathcal{K} \geq 0.4 K \Delta. \tag{5}
\]

For every \( k \in \mathcal{K} \), we have

\[
\sum_{n \leq N} (-1)^{\Omega(\lfloor \alpha n + \beta \rfloor)} = \sum_{n \leq N} (-1)^{\Omega(\lfloor \alpha (n + k) + \beta \rfloor)} + O(K) \\
= \sum_{n \in \mathcal{N}} (-1)^{\Omega(\lfloor \alpha (n + k) + \beta \rfloor)} + O(K + \# \mathcal{N}^c).
\]

Consequently,

\[
\sum_{n \leq N} (-1)^{\Omega(\lfloor \alpha n + \beta \rfloor)} = \frac{W}{\# \mathcal{K}} + O(K + \# \mathcal{N}^c), \tag{6}
\]

where

\[
W = \sum_{n \in \mathcal{N}} \sum_{k \in \mathcal{K}} (-1)^{\Omega(\lfloor \alpha (n + k) + \beta \rfloor)}.
\]
For all \( n \in \mathcal{N} \) and \( k \in \mathcal{K} \), we have
\[
\left\lfloor \alpha(n + k) + \beta \right\rfloor = \alpha(n + k) + \beta - \{ \alpha(n + k) + \beta \}
= (\alpha n + \beta - \gamma) + (\alpha k + \gamma) - \{ \alpha n + \beta - \gamma \} - \{ \alpha k + \gamma \}
= [\alpha n + \beta - \gamma] + [\alpha k + \gamma],
\]
and therefore,
\[
W = \sum_{n \in \mathcal{N}} \sum_{k \in \mathcal{K}} (-1)^{\Omega([\alpha n + \beta - \gamma] + [\alpha k + \gamma])} = \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} (-1)^{\Omega(a + b)}.
\]
By Lemma 4, we derive that
\[
W \ll M(\#A\#B)^{1/2} \frac{\log M}{\log N} \ll N(\#\mathcal{N}\#\mathcal{K})^{1/2} \frac{\log N}{\log N}.
\]
Substituting this estimate into (6), and using (3), (4) and the trivial bound \( \#\mathcal{N} \leq N \), it follows that
\[
\sum_{n \leq N} (-1)^{\Omega([\alpha n + \beta])} \ll \frac{N^{3/2}}{(K\Delta)^{1/2} \log N} + K + N\Delta + o(N).
\]
Choosing \( K = \lceil N(\log N)^{-1/2} \rceil \) and \( \Delta = (\log N)^{-1/2} \), we have \( K\Delta \geq 10c \) once \( N \) is sufficiently large, and the result follows for the case that \( \alpha > 1 \).

If \( \alpha < 1 \), we put \( t = \lfloor \alpha^{-1} \rfloor \) and write
\[
\sum_{n \leq N} (-1)^{\Omega([\alpha n + \beta])} = \sum_{j=0}^{t-1} \sum_{m \leq (N-j)/t} (-1)^{\Omega([\alpha m + \alpha j + \beta])}.
\]
Applying the preceding argument with the irrational number \( \alpha t > 1 \), we conclude the proof. \( \square \)

**Theorem 2.** Let \( \alpha, \beta \in \mathbb{R} \) be fixed, where \( \alpha \) is positive and irrational. Then there exist positive integers \( m, n \leq N \) such that
\[
\omega([\alpha m + \beta]) = (1 + o(1)) \frac{\log N}{\log \log N} \quad \text{and} \quad P([\alpha n + \beta]) \gg N.
\]

**Proof.** Put \( t = \lfloor \alpha^{-1} \rfloor \) and define the sets
\[
\mathcal{N}_1 = \{ c \leq n_1 \leq N/(2t): \{ \alpha t n_1 + \beta \} < 1/2 \},
\]
\[
\mathcal{N}_2 = \{ c \leq n_2 \leq N/(2t): \{ \alpha t n_2 \} < 1/2 \},
\]
where
\[
c = \max\left\{ \lceil (\alpha t)^{-1} (1 - \beta) \rceil, \lfloor (\alpha t)^{-1} \rfloor \right\}.
\]
From the definition (1) and Lemma 1, it follows that
\[ \#N_j = \frac{N}{4t} + o(N) \quad (j = 1, 2). \]

By our choice of \( c \), we see that the sets
\[ A = \{ \lfloor \alpha t n_1 + \beta \rfloor : n_1 \in \mathcal{N}_1 \} \quad \text{and} \quad B = \{ \lfloor \alpha t n_2 \rfloor : n_2 \in \mathcal{N}_2 \} \]
are both contained in the interval \([1, M]\), where
\[ M = \max\{ \lfloor \alpha t N + \beta \rfloor, \lfloor \alpha t N \rfloor \}. \]
Since \( \alpha t > 1 \), the natural maps \( \mathcal{N}_1 \to A \) and \( \mathcal{N}_2 \to B \) are bijections; thus,
\[ \max\{ \#A, \#B \} = \max\{ \#N_1, \#N_2 \} \asymp N \asymp M. \]

Finally, as in the proof of Theorem 1 we have
\[ \lfloor \alpha (tn_1 + tn_2) + \beta \rfloor = \lfloor \alpha tn_1 + \beta \rfloor + \lfloor \alpha tn_2 \rfloor \quad (n_1 \in \mathcal{N}_1, \ n_2 \in \mathcal{N}_2). \]

Applying Lemma 5, the result follows immediately. \( \square \)

**Theorem 3.** Let \( \alpha, \beta \in \mathbb{R} \) be fixed, where \( \alpha \) is positive and irrational, and let
\[ F(N, C) = \#\{ n \leq N : \omega(\lfloor \alpha n + \beta \rfloor) - \log \log N \leq C(\log \log N)^{1/2} \}. \]

Then the following estimate holds:
\[ F(N, C) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{C} e^{-t^2/2} dt + o(N), \]
where the function implied by \( o(N) \) depends only on \( \alpha, \beta \) and \( C \).

**Proof.** First, we consider the case that \( \alpha > 1 \). Put
\[ K = \left\lceil N(\log \log \log N)^{-1/2} \right\rceil \quad \text{and} \quad \Delta = (\log \log \log N)^{-1/2}, \]
and let \( \gamma, c, \mathcal{N}, \mathcal{N}^c, \mathcal{K}, \mathcal{A}, \mathcal{B} \) and \( M \) be defined as in the proof of Theorem 1. In particular, from (3) and (5) we derive that
\begin{align*}
\#\mathcal{N}^c &= o(N), \\
\#\mathcal{A} &= \#\mathcal{N} = N + o(N), \\
\#\mathcal{B} &= \#\mathcal{K} \gg \frac{N}{\log \log \log N},
\end{align*}
(7) (8) (9)
provided that $N$ is sufficiently large. Let $C'$ be chosen to satisfy the relation

$$\log \log N + C (\log \log N)^{1/2} = \log \log M + C' (\log \log M)^{1/2}. \quad (10)$$

Since $M \asymp N$, we have $C' = C(1 + o(1))$, hence it follows that

$$\int_{-\infty}^{C'} e^{-t^2/2} dt = \int_{-\infty}^{C} e^{-t^2/2} dt + o(1). \quad (11)$$

Finally, let

$$f(m) = \begin{cases} 1 & \text{if } \omega(m) - \log \log M \leq C' (\log \log M)^{1/2}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, using (10) we have

$$F(N, C) = \sum_{n \leq N} f(\lfloor \alpha n + \beta \rfloor).$$

For every $k \in K$,

$$F(N, C) = \sum_{n \leq N} f(\lfloor \alpha(n + k) + \beta \rfloor) + O(K) = \sum_{n \in \mathcal{N}} f(\lfloor \alpha(n + k) + \beta \rfloor) + O(K + \#N^c).$$

By our choice of $K$ and the estimate (7), we have $K + \#N^c = o(N)$; also,

$$\lfloor \alpha(n + k) + \beta \rfloor = \lfloor \alpha n + \beta - \gamma \rfloor + \lfloor \alpha k + \gamma \rfloor \quad (n \in \mathcal{N}, \ k \in K)$$

as before. Therefore,

$$F(N, C) = \frac{1}{\#K} \sum_{n \in \mathcal{N}} \sum_{k \in K} f(\lfloor \alpha n + \beta - \gamma \rfloor + \lfloor \alpha k + \gamma \rfloor) + o(N) = \frac{1}{\#B} \sum_{a \in A} \sum_{b \in B} f(a + b) + o(N).$$

Applying Lemma 7, we derive that

$$F(N, C) = \frac{\#A}{\sqrt{2\pi}} \int_{-\infty}^{C'} e^{-t^2/2} dt + O\left( M \left( \frac{\#A}{\#B \log \log M} \right)^{1/2} \right) + o(N).$$

Using the estimates (8), (9) and (11) together with the fact that $M \asymp N$, the result follows for the case that $\alpha > 1$. 
If \( \alpha < 1 \), put \( t = \lceil \alpha^{-1} \rceil \) and \( N_j = (N - j)/t \) for \( j = 0, \ldots, t - 1 \). Then,

\[
F(N, C) = \sum_{j=0}^{t-1} F_j,
\]

where

\[
F_j = \#\{ m \leq N_j : \omega(\lfloor \alpha m + \alpha j + \beta \rfloor) - \log \log N \leq C(\log \log N)^{1/2} \}
\]

for \( j = 0, \ldots, t - 1 \). For each \( j \), let \( C_j \) be chosen to satisfy the relation

\[
\log \log N + C(\log \log N)^{1/2} = \log \log N_j + C_j (\log \log N_j)^{1/2}.
\]

Since \( N_j \asymp N \), we have \( C_j = C(1 + o(1)) \), and thus

\[
\int_{-\infty}^{C_j} e^{-t^2/2} dt = \int_{-\infty}^{C} e^{-t^2/2} dt + o(1).
\]

Applying the preceding argument with the irrational number \( \alpha t > 1 \), it follows that

\[
F(N, C) = \sum_{j=0}^{t-1} \left( \frac{N_j}{\sqrt{2\pi}} \int_{-\infty}^{C_j} e^{-t^2/2} dt + o(N_j) \right)
\]

\[
= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{C} e^{-t^2/2} dt + o(N),
\]

and this completes the proof. \( \square \)

**Theorem 4.** Let \( \alpha, \beta \in \mathbb{R} \) be fixed, where \( \alpha \) is positive and irrational. Then,

\[
\sum_{n \leq N} \omega(\lfloor \alpha n + \beta \rfloor) = (1 + o(1))N \log \log N.
\]

**Proof.** First, we consider the case that \( \alpha > 1 \). Put

\[
K = \lceil N(\log \log N)^{-1} \rceil \quad \text{and} \quad \Delta = (\log \log N)^{-1},
\]

and let \( \gamma, c, N, N^c, K, A, B \) and \( M \) be defined as in the proof of Theorem 1.

For every \( k \in K \), we have

\[
\sum_{n \leq N} \omega(\lfloor \alpha n + \beta \rfloor) = \sum_{n \leq N} \omega(\lfloor \alpha(n + k) + \beta \rfloor) + \sum_{n \leq k} \omega(\lfloor \alpha n + \beta \rfloor) - \sum_{N < n \leq N + k} \omega(\lfloor \alpha n + \beta \rfloor).
\]
By Lemma 3 we have
\[
\sum_{n \leq k} \omega([\alpha n + \beta]) \leq \sum_{m \leq \alpha K + \beta} \omega(m) \ll K \log \log K
\]
and
\[
\sum_{n < N \leq N+k} \omega([\alpha n + \beta]) \ll \sum_{\alpha N + \beta < m \leq \alpha(N+K) + \beta} \omega(m) \ll K \log \log N.
\]
Therefore,
\[
\sum_{n \leq N} \omega([\alpha n + \beta]) = \sum_{n \leq N} \omega([\alpha(n + k) + \beta]) + O(K \log \log N),
\]
and it follows that
\[
\sum_{n \leq N} \omega([\alpha n + \beta]) = \frac{U}{\#\mathcal{K}} + \frac{V}{\#\mathcal{K}} + O(K \log \log N),
\]
where
\[
U = \sum_{n \in \mathcal{N}} \sum_{k \in \mathcal{K}} \omega([\alpha(n + k) + \beta]),
\]
\[
V = \sum_{n \in \mathcal{N}} \sum_{k \in \mathcal{K}} \omega([\alpha(n + k) + \beta]).
\]

For every \( n \leq N \), Lemma 3 can be used again to obtain the bound
\[
\sum_{k \in \mathcal{K}} \omega([\alpha(n + k) + \beta]) \ll \#\mathcal{K} \log \log N.
\]
In particular, it follows that
\[
U \ll \#\mathcal{N}^c \#\mathcal{K} \log \log N = o(N \#\mathcal{K} \log \log N),
\]
where we have used (3) in the second step. Substituting this estimate into (12), we have
\[
\sum_{n \leq N} \omega([\alpha n + \beta]) = \frac{V}{\#\mathcal{K}} + o(N \log \log N).
\]
Finally, as in the proof of Theorem 1, we have
\[
[\alpha(n + k) + \beta] = [\alpha n + \beta - \gamma] + [\alpha k + \gamma] \quad (n \in \mathcal{N}, \ k \in \mathcal{K}).
\]
Consequently,

\[ V = \sum_{n \in \mathbb{N}} \sum_{k \in \mathcal{K}} \omega(\lfloor \alpha n + \beta - \gamma \rfloor + \lfloor \alpha k + \gamma \rfloor) = \sum_{a \in A} \sum_{b \in B} \omega(a + b). \]

Applying Lemma 6, it follows that

\[ V = (1 + o(1)) \# A \# B \log \log M = (1 + o(1)) N \# \mathcal{K} \log \log N. \]

Substituting this estimate into (13), we obtain the desired result in the case that \( \alpha > 1 \).

If \( \alpha < 1 \), we put \( t = \lceil \alpha^{-1} \rceil \) and write

\[ \sum_{n \leq N} \omega(\lfloor \alpha n + \beta \rfloor) = \sum_{j=0}^{t-1} \sum_{m \leq (N-j)/t} \omega(\lfloor \alpha tm + \alpha j + \beta \rfloor). \]

Applying the preceding argument with the irrational number \( \alpha t > 1 \), the result follows.

Similar arguments combined with Lemma 8 give the following result:

**Theorem 5.** Let \( \alpha, \beta \in \mathbb{R} \) be fixed, where \( \alpha \) is positive and irrational. Then, for any fixed \( \varepsilon > 0 \) and uniformly for \( \exp((\log N)^{2/3+\varepsilon}) < Q \leq N \), we have

\[ \{ n \leq N : P(\lfloor \alpha n + \beta \rfloor) \leq Q \} = (1 + o(1)) \rho(u) N, \]

where \( u = (\log N) / \log Q \).

Similarly, using Lemma 9, we have:

**Theorem 6.** Let \( \alpha, \beta \in \mathbb{R} \) be fixed, where \( \alpha \) is positive and irrational. Then,

\[ \sum_{n \leq N} (-1)^{\sigma_2(\lfloor \alpha n + \beta \rfloor)} = o(N). \]

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**References**