

Inference on Consensus Ranking of Distributions

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BU Econometrics Seminar
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“Better”

Definitions and Consensus

“Better”?

Two distributions (of earnings, productivity, ...)

Which is “better”?

“Better”?

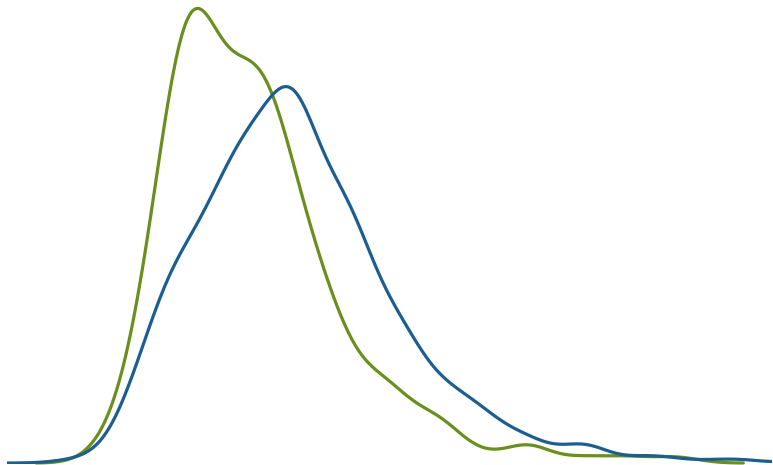
Two distributions (of earnings, productivity, ...)

Which is “better”?

- ▶ Would you prefer to (live there, buy this, use that, ...)

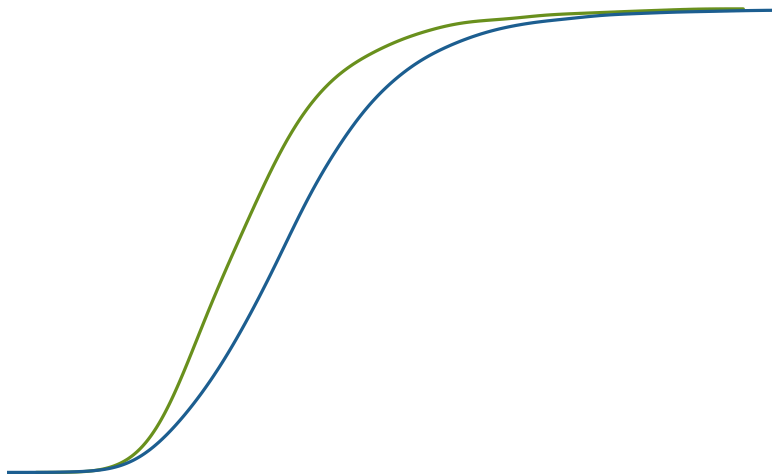
“Better”?

2 PDFs



“Better”?

2 CDFs



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“Deductive”: fix \mathcal{D} , test $SD_{\mathcal{D}}$

“Inductive”: learn about the true \mathcal{D}

CDFs (Atkinson, 1987, §1)

Poverty line: v

Headcount poverty: $F_Y(v)$ and $F_Z(v)$

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Condition I of Atkinson (1987, p. 751)

Deductive (Davidson and Duclos, 2013)

Inductive (Goldman and Kaplan, 2018)

Brief Tangent: Economic Inequality

Literature on measuring inequality, comparing distributions

Similar issue (me/you/all/most), like

- ▶ ϵ of Atkinson (1970, p. 257)
- ▶ α of Cowell and Flachaire (2017, §4.3)

Another Tangent (sorry): Quantiles

~~Expected~~ Quantile utility maximization

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My τ , your τ , ...

All (SD₁): $Q_\tau(Y) \geq Q_\tau(Z)$ for all $\tau \in (0, 1)$

Consensus: $Q_\tau(Y) \geq Q_\tau(Z)$ for $\tau \in \mathcal{T}$

Inference

Learning from Data

Literature: Testing

Two features in common:

- ▶ Single H_0 : all-or-nothing
- ▶ CDF-based

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- ▶ Barrett and Donald (2003), many others
- ▶ Good for testing economic theory that implies SD_1

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$H_0: Y \text{ nonSD}_1 Z$ ($H_1: Y \text{ SD}_1 Z$)

- ▶ Davidson and Duclos (2013) (and Kaur, Prakasa Rao, and Singh, 1994, et al.)
- ▶ Want stronger evidence for SD_1 (analog: $H_0: \beta = 0$)
- ▶ Actually $\text{SD}_{[v_1, v_2]}$

New Perspectives

~~Single- H_0~~ Multiple testing

- ▶ Goldman and Kaplan (2018)
- ▶ $H_{0v} : F_Y(v) \geq F_Z(v)$ for each $v \in \mathbb{R}$
- ▶ Learn about $\mathcal{V} \equiv \{v : F_Y(v) < F_Z(v)\}$ ($Y \text{ SD}_{\mathcal{V}} Z$)

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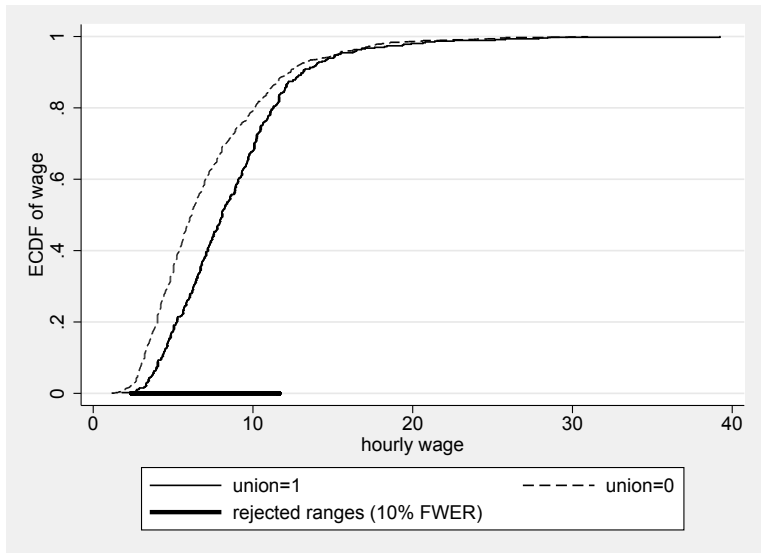
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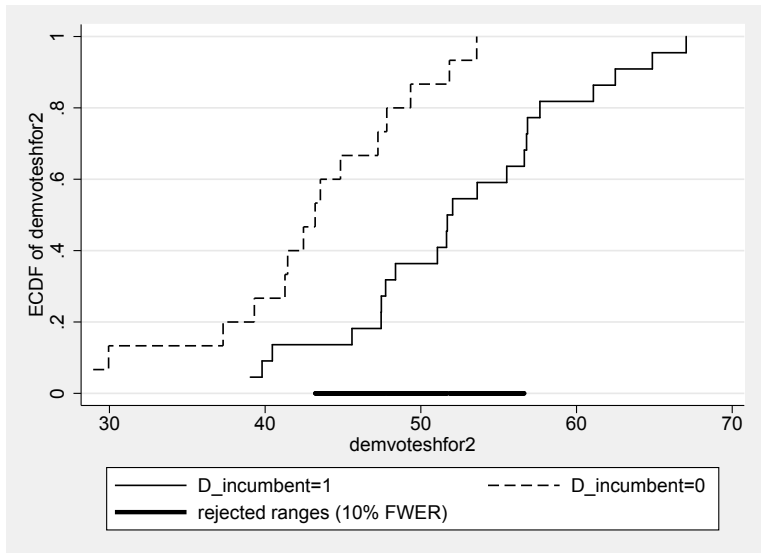
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Quantile: learn about $\mathcal{T} \equiv \{\tau : Q_{\tau}(Y) > Q_{\tau}(Z)\}$

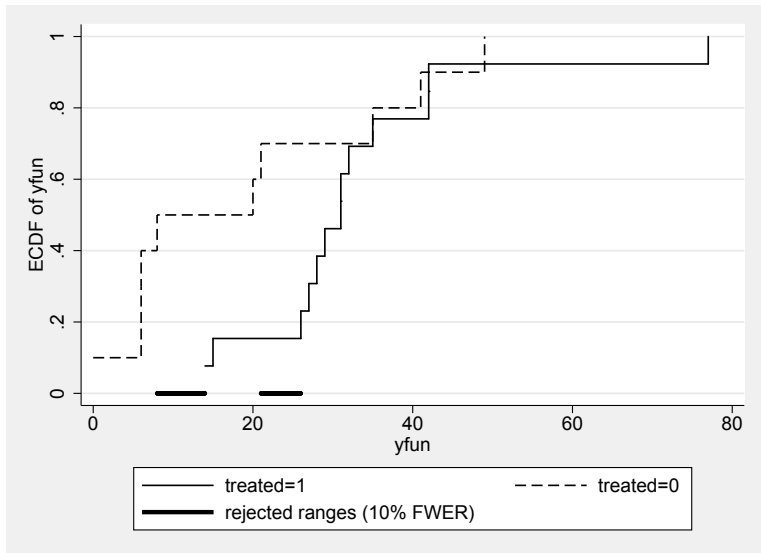
Examples (`distcomp` in Stata)



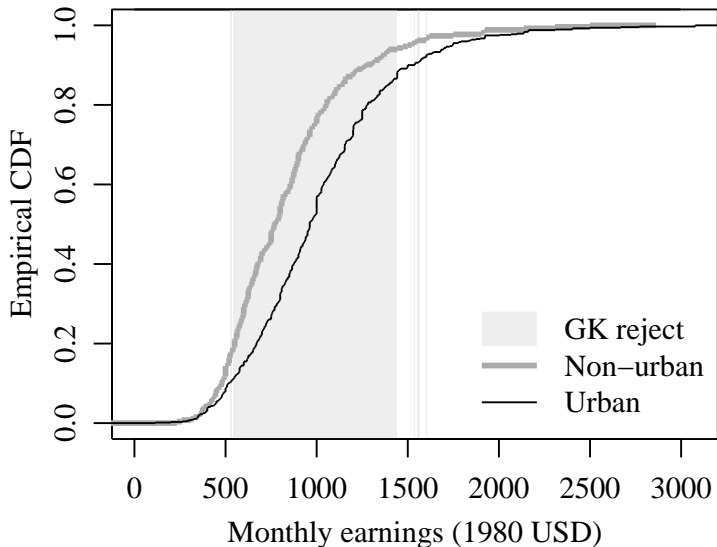
Examples (distcomp RDD)



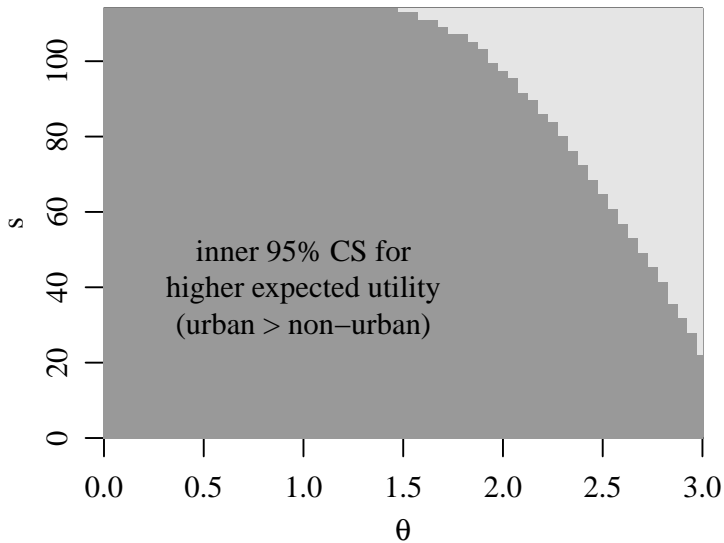
Examples (distcomp experiment)



Examples (CDF)



Examples (utility, CRRA $u(x - s)$)



CDF vs. Utility: Complementary Strengths

Economic interpretation

Top-coding; tail measurement error

Unified framework for SD variants

Choice of \mathcal{U} :

- ▶ Economic restrictions
- ▶ Donsker: Cor. 3.1 of van der Vaart (1996)
- ▶ Computational limits; “sieve”?

Multiple Testing Goal

Multiple testing procedure (MTP)

- ▶ Test $H_{0v}: F_Y(v) \geq F_Z(v)$ for each $v \in \mathbb{R}$
- ▶ $\mathcal{V} \equiv \{v : F_Y(v) < F_Z(v)\}$

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Familywise error rate (FWER)

- ▶ $\text{FWER} \equiv \text{P}(\text{reject any true } H_{0v})$
- ▶ “Strong control”: $\text{FWER} \leq \alpha$ regardless of \mathcal{V}

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Expected utility version

- ▶ Test $H_{0u}: \text{E}[u(Y)] \leq \text{E}[u(Z)]$ for each $u \in \mathcal{U}$

MTP vs. All-or-Nothing Test

If $H_0: Y \text{ SD}_1 Z$ rejected:

- ▶ MTP shows where/why (which v or u)

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If $H_0: Y \text{ SD}_1 Z$ not rejected:

- ▶ MTP shows evidence favoring $Y \text{ SD}_1 Z$ vs. just uncertainty
- ▶ “Reject $H_0: Z \text{ SD}_1 Y$ ” is a crude version of this idea
- ▶ Non-rejection may be type II error if small sample, etc.

“Outer Confidence Set” (CDF)

Usual

- ▶ Object of interest: $\boldsymbol{\theta} \in \mathbb{R}^k$
- ▶ Goal: $1 - \alpha \leq P(\boldsymbol{\theta} \in \hat{\mathcal{C}})$
- ▶ Invert test: $\hat{\mathcal{C}} = \{\mathbf{c} : \text{don't reject } H_0: \boldsymbol{\theta} = \mathbf{c}\}$

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“Outer CS”

- ▶ Goal: $1 - \alpha \leq \mathbb{P}(\mathcal{V} \subseteq \hat{\mathcal{V}})$
- ▶ Invert MTP: $\hat{\mathcal{V}} = \{v : H_{0v} \text{ not rejected}\}$
 $H_{0v}: F_Y(v) \leq F_Z(v) \quad (H_{0v}: v \in \mathcal{V})$
- ▶ $\mathbb{P}(\mathcal{V} \subseteq \hat{\mathcal{V}}) = \mathbb{P}(\text{no true } H_{0v} \text{ rejected}) = 1 - \text{FWER} \geq 1 - \alpha$

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“Outer CS”: $1 - \alpha \leq P(\mathcal{V} \subseteq \hat{\mathcal{V}}) = P(\hat{\mathcal{V}}^c \subseteq \mathcal{V}^c)$

“Inner CS”: $1 - \alpha \leq P(\hat{\mathcal{V}} \subseteq \mathcal{V})$

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- ▶ Or: $\hat{\mathcal{V}}$ is a very conservative estimate of \mathcal{V}

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“2-sided CS”: $1 - \alpha \leq P(\hat{\mathcal{V}}_1 \subseteq \mathcal{V} \subseteq \hat{\mathcal{V}}_2)$

▶ Combine $1 - \alpha/2$ inner & outer (Bonferroni)

Confidence Sets (Expected Utility)

Same arguments but with \mathcal{D} instead of \mathcal{V}

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CS/MTP from Uniform Confidence Band

Uniform confidence band for $\Delta(\cdot) \implies$ CS/MTP

- ▶ $\Delta(v) \equiv F_Z(v) - F_Y(v) \quad \mathcal{V} = \{v : \Delta(v) > 0\}$
- ▶ $\Delta(u) \equiv E[u(Y)] - E[u(Z)] \quad \mathcal{D} = \{u : \Delta(u) > 0\}$
- ▶ Inner CS: values where lower band above zero
- ▶ Outer CS: values where upper band above zero
- ▶ MTP: equivalent to CS like before

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Information vs. comprehension

- ▶ EU band more informative, CS/MTP easier to comprehend

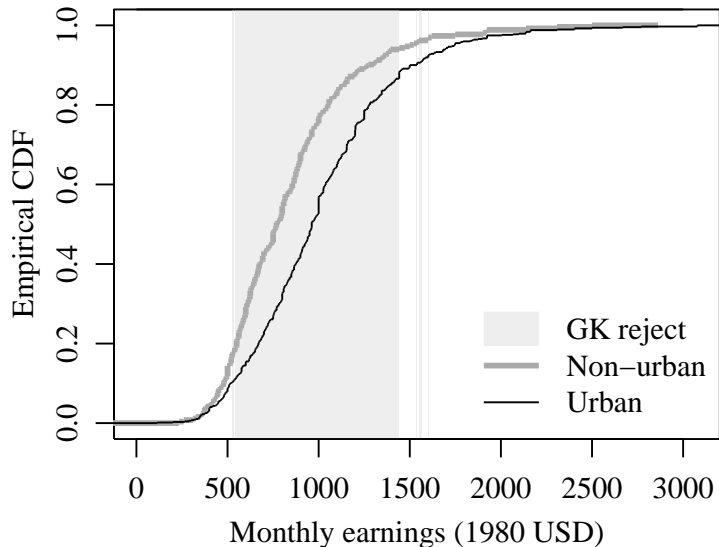
Availability

- ▶ CDF diff: asymptotic band, but finite-sample CS/MTP

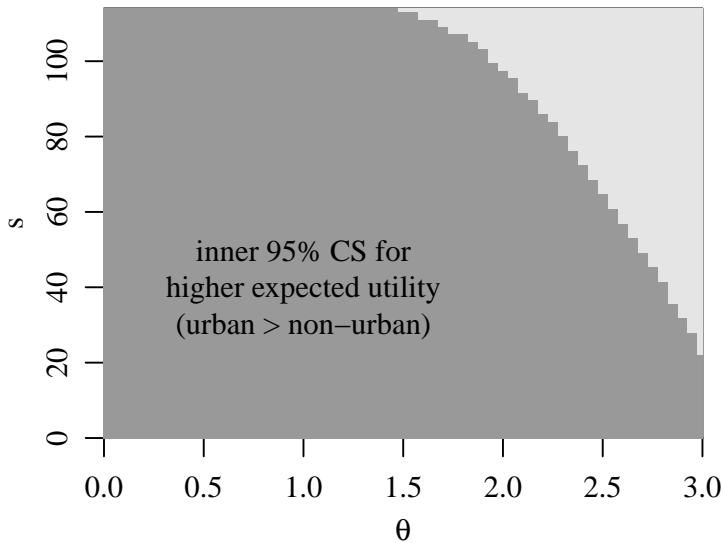
Refinements

- ▶ Stepdown, pre-test, ...

CDF-based CS/MTP



Utility-based CS/MTP



Details

Theoretical & Otherwise

CDF: KS vs. Probability Integral Transform

Kolmogorov–Smirnov MTP/CS

- ▶ Reject H_{0v} when $\hat{F}_Y(v) - \hat{F}_Z(v)$ exceeds KS critical value
- ▶ Prop. 3 of Goldman and Kaplan (2018)

KS: well-known low tail power

- ▶ `ks.test(c(1:15/21), 10^6+1:5), punif)`
D = 0.25, p-value = 0.1376

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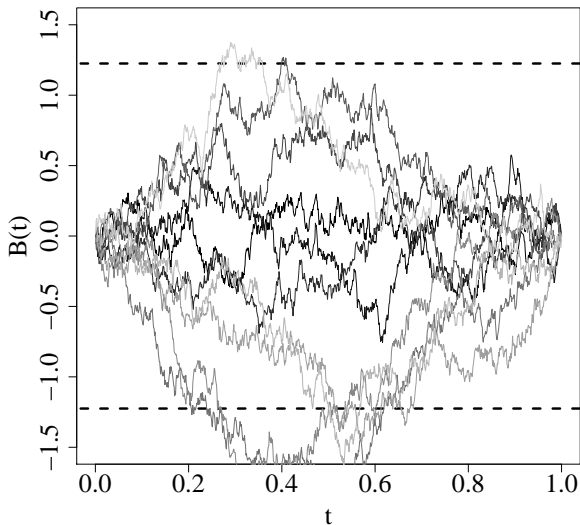
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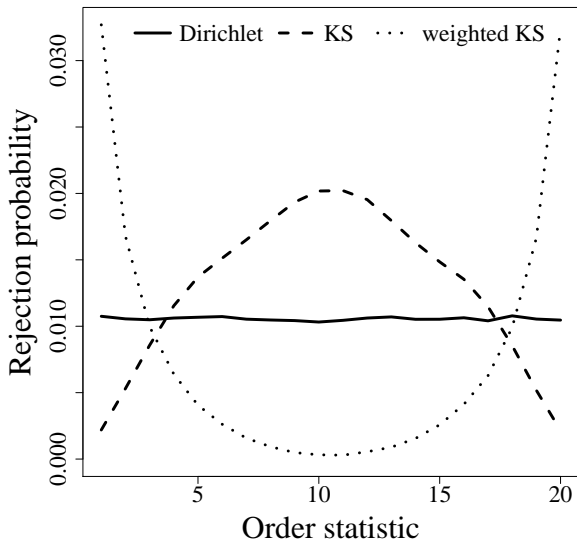
If cts, $F_Y(Y_i) \sim \text{Unif}(0, 1)$

- ▶ Retain finite-sample properties
- ▶ Power more even than KS across distribution
- ▶ Goldman and Kaplan (2018): two-sample MTP, RDD, computation

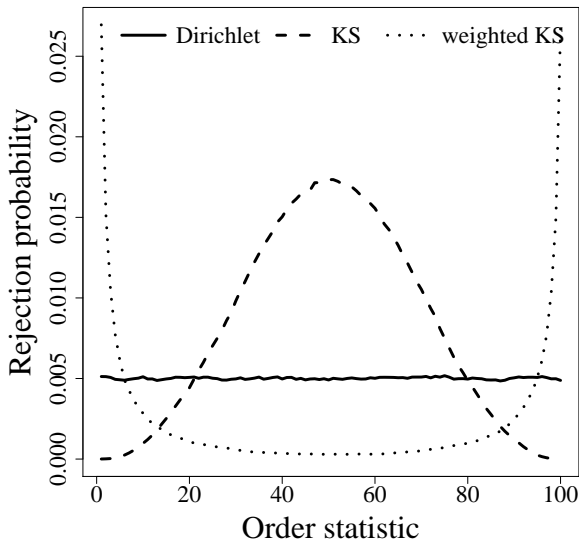
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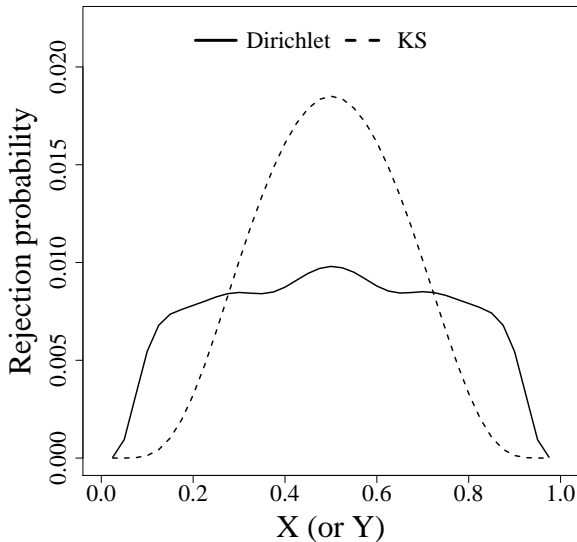
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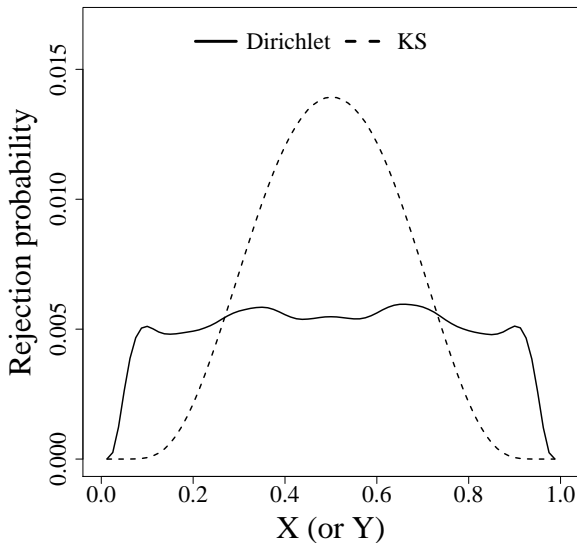
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Expected Utility: Asymptotics

CDF:

- ▶ $\Delta(v) \equiv F_Z(v) - F_Y(v) = \mathbb{E}[\mathbf{1}\{Z \leq v\}] - \mathbb{E}[\mathbf{1}\{Y \leq v\}]$
- ▶ $\Delta(f) = \mathbb{E}[f(Z)] - \mathbb{E}[f(Y)]$, $f(x) = \mathbf{1}\{x \leq v\}$
- ▶ $\{f_v(\cdot) : f_v(t) = \mathbf{1}\{t \leq v\}, v \in \mathbb{R}\}$ is Donsker
- ▶ $\hat{\Delta}(\cdot)$: Gaussian limit

EU:

- ▶ $\Delta(u) \equiv \mathbb{E}[u(Y)] - \mathbb{E}[u(Z)]$
- ▶ $\hat{\Delta}(\cdot)$: Gaussian limit and bootstrap consistency
if Donsker \mathcal{U}

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- ▶ $\hat{\Delta}(\cdot)$: Gaussian limit and bootstrap consistency if Donsker \mathcal{U}

Cor. 3.1 of van der Vaart (1996): \mathcal{U} Donsker if

- ▶ non-decreasing
- ▶ bounded from below (or above)
- ▶ $2 + \delta$ moments of envelope function

Expected Utility: MTP

$$H_{0u}: \Delta(u) \equiv E[u(Y)] - E[u(Z)] \leq 0, \text{ each } u \in \mathcal{U}$$

Define pointwise t -statistics:

- ▶ $\hat{T}_u = [\hat{\Delta}(u) - \Delta(u)] / \widehat{SE}_u$
- ▶ $\hat{T}_u^0 = \hat{\Delta}(u) / \widehat{SE}_u$

Bootstrap cv: $1 - \alpha$ quantile of $\sup_{u \in \mathcal{U}} \hat{T}_u$

$$\text{FWER} = P(\text{reject any true}) \leq P(\sup_u \hat{T}_u > \text{cv}) \rightarrow \alpha$$

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Bootstrap cv: $1 - \alpha$ quantile of $\sup_{u \in \mathcal{U}} \hat{T}_u$

FWER = $P(\text{reject any true}) \leq P(\sup_u \hat{T}_u > cv) \rightarrow \alpha$

Stepdown (Holm, 1979)

- ▶ \leq maybe very conservative if many $\Delta(u) > 0$
- ▶ Re-compute bootstrap cv using only non-rejected u
- ▶ Iterate: bounded by oracle test using true $\{u : H_{0u} \text{ true}\}$

Can also pre-test to remove $\hat{\Delta}(u) \ll 0$, etc.

Expected Utility: CS

Invert MTP to get CS

Simulation

Performance of New Methods

Setup

$$Y_i \stackrel{iid}{\sim} \log N(0, 1) + 0.1, i = 1, \dots, n$$

$$Z_i \stackrel{iid}{\sim} \log N(\mu, \sigma) + 0.1, i = 1, \dots, n$$

\mathcal{U} : CRRA w/ risk aversion $\theta \in [0, 3]$

Band for $\Delta(u) = E[u(Y)] - E[u(Z)]$

▶ Equivalently: $\Delta(\theta)$ on $\theta \in [0, 3]$

CSs for $\mathcal{D} \equiv \{u : E[u(Y)] > E[u(Z)]\}$

▶ Equivalently: \mathcal{D} is subset of $\theta \in [0, 3]$

Results: $n = 40$

σ	μ	$\{\theta : u_\theta \in \mathcal{D}\}$	Coverage ($1 - \alpha = 0.9$)			
			band	2s CS	inner	outer
0.7	-0.3	[0.0, 2.8]	0.873	0.960	0.968	0.992
0.7	0.0	[0.0, 1.1]	0.865	0.972	0.990	0.982
0.7	0.3	[]	0.855	0.998	0.998	1.000
1.0	-0.3	[0.0, 3.0]	0.920	0.999	1.000	0.999
1.0	0.0	[]	0.938	0.972	0.972	1.000
1.0	0.3	[]	0.922	0.995	0.995	1.000
1.3	-0.3	[0.2, 3.0]	0.896	0.965	0.967	0.998
1.3	0.0	[1.2, 3.0]	0.883	0.976	0.988	0.988
1.3	0.3	[2.5, 3.0]	0.861	0.962	0.994	0.968

Results: $n = 100$

σ	μ	$\{\theta : u_\theta \in \mathcal{D}\}$	Coverage ($1 - \alpha = 0.9$)			
			band	2s CS	inner	outer
0.7	-0.3	[0.0, 2.8]	0.907	0.968	0.975	0.993
0.7	0.0	[0.0, 1.1]	0.897	0.977	0.993	0.984
0.7	0.3	[]	0.908	0.999	0.999	1.000
1.0	-0.3	[0.0, 3.0]	0.934	1.000	1.000	1.000
1.0	0.0	[]	0.929	0.965	0.965	1.000
1.0	0.3	[]	0.922	1.000	1.000	1.000
1.3	-0.3	[0.2, 3.0]	0.901	0.974	0.979	0.995
1.3	0.0	[1.2, 3.0]	0.900	0.983	0.987	0.996
1.3	0.3	[2.5, 3.0]	0.887	0.964	0.992	0.972

Results: $n = 250$

σ	μ	$\{\theta : u_\theta \in \mathcal{D}\}$	Coverage ($1 - \alpha = 0.9$)			
			band	2s CS	inner	outer
0.7	-0.3	[0.0, 2.8]	0.920	0.978	0.983	0.995
0.7	0.0	[0.0, 1.1]	0.912	0.981	0.995	0.986
0.7	0.3	[]	0.893	0.998	0.998	1.000
1.0	-0.3	[0.0, 3.0]	0.920	1.000	1.000	1.000
1.0	0.0	[]	0.937	0.968	0.968	1.000
1.0	0.3	[]	0.942	1.000	1.000	1.000
1.3	-0.3	[0.2, 3.0]	0.927	0.976	0.978	0.998
1.3	0.0	[1.2, 3.0]	0.902	0.979	0.988	0.991
1.3	0.3	[2.5, 3.0]	0.892	0.974	0.994	0.980

Bonus Material

Time Permitting

Quantile Utility Maximization

Maximize $Q_\tau(u(X))$ instead of $E[u(X)]$

- ▶ Manski (1988)
- ▶ Rostek (2010): axiomatization
- ▶ de Castro and Galvao (2019): dynamic

Quantile Utility Maximization

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$Q_\tau(u(X)) = u(Q_\tau(X)) \implies u(\cdot)$ doesn't matter (!?)

Set of preferences \iff set of τ

- ▶ Learn about $\mathcal{T} = \{\tau : Q_\tau(Y) \geq Q_\tau(Z)\}$

Uniform Confidence Band

Object of interest: $\Delta(u) = E[u(Y)] - E[u(Z)]$ over $u \in \mathcal{U}$

Goal: $1 - \alpha = P\{\hat{b}_1(u) \leq \Delta(u) \leq \hat{b}_2(u) \text{ for all } u \in \mathcal{U}\}$

Asymptotically: $\sqrt{n}(\hat{\Delta}(\cdot) - \Delta(\cdot))$ Gaussian

Alg. 3 of Chernozhukov, Fernández-Val, and Melly (2013):

- ▶ Bootstrap to get std dev and t -stat for each u
- ▶ Bootstrap absolute sup t -stat to get critical value
- ▶ Band: $\hat{\Delta}(u) \pm cv_{1-\alpha} \hat{\sigma}(u) / \sqrt{n}$

Test of Restricted (non)SD

$H_0: Z \text{ SD}_{\mathcal{D}} Y \iff E[u(Y)] - E[u(Z)] \leq 0 \text{ for all } u \in \mathcal{D}$

$H_1: Z \text{ nonSD}_{\mathcal{D}} Y$

- ▶ Reject $\text{SD}_{\mathcal{D}} \implies$ reject SD
- ▶ Reject when sup t -stat exceeds bootstrap sup- t cv
- ▶ (Least favorable null: all zero)

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$H_0: Z \text{ nonSD}_{\mathcal{D}} Y$ vs. $H_1: Z \text{ SD}_{\mathcal{D}} Y$

- ▶ Utility version of Davidson and Duclos (2013)
- ▶ Reject when all t -stats below $\Phi^{-1}(\alpha) < 0$
- ▶ Least favorable null: $E[u^*(Y)] - E[u^*(Z)] \downarrow 0$ for single $u^* \in \mathcal{D}$; for $u \neq u^*$ $E[u(Y)] - E[u(Z)] \ll 0$

Epilogue

Past & Future

Conclusion

“Better”: restricted stochastic dominance based on

- ▶ CDF
- ▶ Expected utility

Inference on set of:

- ▶ values with lower CDF
- ▶ utility functions with higher expected utility

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- ▶ non-iid, improve power, implement richer utility family
- ▶ economic inequality
- ▶ restricted stochastic monotonicity
- ▶ other ideas?

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Thank you / further questions & comments appreciated

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