

# High-order coverage of smoothed Bayesian bootstrap intervals for population quantiles

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## Abstract

We characterize the high-order coverage accuracy of smoothed and unsmoothed Bayesian bootstrap confidence intervals for population quantiles. Although the original (Rubin, 1981) unsmoothed intervals have the same  $O(n^{-1/2})$  coverage error as the standard empirical bootstrap, the smoothed Bayesian bootstrap of Banks (1988) has much smaller  $O(n^{-3/2}[\log(n)]^3)$  coverage error and is exact in special cases, without requiring any smoothing parameter. It automatically removes an error term of order  $1/n$  that other approaches need to explicitly correct for. This motivates further study of the smoothed Bayesian bootstrap in more complex settings and models.

*Keywords and phrases:* continuity correction, credible intervals, fractional order statistics

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## 1 Introduction

We study the frequentist coverage accuracy of certain Bayesian bootstrap confidence intervals for population quantiles. Specifically, we consider percentile method intervals using the Bayesian bootstrap (BB) of Rubin (1981) and the continuity-corrected Bayesian bootstrap (CCBB) of Banks (1988).

Although the BB has the same order of coverage error as the standard bootstrap, the CCBB is substantially more accurate. Specifically, the BB has  $O(n^{-1/2})$  coverage error, whereas CCBB has  $O(n^{-3/2}[\log(n)]^3)$ . Unlike other smoothed bootstraps, the CCBB does not have any smoothing parameter, so this rate depends only on iid sampling and mild smoothness conditions on the population distribution. Further, the CCBB is exact for intervals with order statistic endpoints or when the population distribution is uniform.

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Although other confidence intervals for quantiles achieve the same accuracy, our results motivate further study of the CCBB in other settings to which it may generalize more readily. For quantiles, the accuracy is matched by Ho and Lee (2005) and Goldman and Kaplan (2017), who provide calibrated versions of the order statistic-based intervals in Beran and Hall (1993) and Hutson (1999), respectively. These calibrations explicitly remove an error term of order  $1/n$  that the CCBB automatically avoids. Notwithstanding some extensions of such methods to other related objects of interest (Goldman and Kaplan, 2018a,b), the CCBB may be more readily generalized to other objects of interest or non-iid sampling.

Other papers have examined accuracy of CCBB and related intervals for quantiles, but without our theoretical results on coverage error. Examples include Breth (1979), Banks (1988), and Meeden (1993), who proposes a grid-based alternative to CCBB that maintains a Bayesian interpretation (unlike CCBB).

Section 2 introduces the intervals. Sections 3 and 4 contain theoretical results. Section 5 illustrates the theory with simulations. Appendix A contains proofs.

**Notation** Vectors are bold; e.g., the sample is  $\mathbf{Y} = (Y_1, \dots, Y_n)$ . Also,  $\mathbb{1}\{\cdot\}$  is the indicator function,  $Q_\tau(Y)$  is the  $\tau$ -quantile of random variable  $Y$  whose CDF is  $F_Y(\cdot)$ , and  $P(\cdot)$  is probability; additional notation is defined as needed. Acronyms used include those for Bayesian bootstrap (BB), confidence interval (CI), continuity-corrected Bayesian bootstrap (CCBB), coverage probability (CP), cumulative distribution function (CDF), and probability density function (PDF). Let  $F_\beta(p; a, b)$  and  $f_\beta(p; a, b)$  respectively denote the CDF and PDF of a  $\text{Beta}(a, b)$  distribution evaluated at  $p$ .

## 2 Setup

### 2.1 Assumptions

Assumptions A1 and A2 describe the setting. Assumption A1 describes the sampling and object of interest and is used for all our results. Assumption A2 is an additional smoothness assumption required for our most general results. It is the same as Assumption A2 of Goldman and Kaplan (2017).

**Assumption A1.** Observations  $Y_i$  are iid realizations of random variable  $Y$  with continuous CDF  $F_Y(\cdot)$ , and interest is in the  $\tau$ -quantile,  $Q_\tau(Y) \equiv \inf\{y : F_Y(y) \geq \tau\}$ , for fixed  $\tau \in (0, 1)$ . The sample of size  $n$  contains  $Y_1, \dots, Y_n$ .

**Assumption A2.** i)  $F'_Y(Q_\tau(Y)) > 0$ ; ii)  $F'''_Y(\cdot)$  is continuous in a neighborhood of  $Q_\tau(Y)$ .

Notationally, let the order statistics (i.e., ordered sample values) be

$$Y_{n:1} \leq Y_{n:2} \leq \cdots \leq Y_{n:n}. \quad (1)$$

Given the continuity of  $F_Y(\cdot)$  in A1, the inequalities become strict with probability 1. For notational completion, let  $Y_{n:0}$  and  $Y_{n:n+1}$  denote the lower and upper bounds of the support of  $Y$ , but these only affect CCBB inference on extreme quantiles, which we do not consider.

## 2.2 Bootstrap distributions

The bootstrap distributions are conditional on the data  $\mathbf{Y} = (Y_1, \dots, Y_n)$ , with the randomness coming from a vector of weights. The standard (empirical) bootstrap that resamples with replacement can equivalently be seen as drawing a multinomial-distributed vector of weights (each of which is a non-negative integer) and applying the weights to the original  $Y_1, \dots, Y_n$ , where the weights are independent of the sample values. Instead of focusing on the resampled values, we can think of the empirical distribution corresponding to such values. That is, instead of resampling a bootstrap dataset  $Y_1^*, \dots, Y_n^*$  and computing the corresponding statistic (like the  $\tau$ -quantile), it can be interpreted as randomly drawing multinomial  $(W_1, \dots, W_n)$  and computing the statistic based on the weighted empirical CDF

$$\widehat{F}^*(y) = \frac{1}{n} \sum_{i=1}^n W_i \mathbb{1}\{Y_i \leq y\}.$$

The BB replaces the multinomial weight distribution with a particular Dirichlet distribution. Given sample size  $n$ ,

$$\mathbf{W} \equiv (W_1, \dots, W_n) \sim \text{Dir}(1, \dots, 1), \quad (2)$$

a Dirichlet distribution with all  $n$  parameters equal to 1, i.e., a uniform distribution over the unit  $(n - 1)$ -simplex. The corresponding CDF is

$$\widehat{F}_{*u}(y) = \frac{1}{n} \sum_{i=1}^n W_i \mathbb{1}\{Y_{n:i} \leq y\}, \quad (3)$$

where the  $u$  subscript stands for “unsmoothed.” The  $Y_{n:i}$  could equivalently be replaced by  $Y_i$ , but (3) connects better with the CCBB. For intuition, note  $\widehat{F}_{*u}(Y_{n:k}) = \sum_{i=1}^k W_i$ .

The CCBB applies Dirichlet probability weights to intervals instead of points. There are  $n + 1$  intervals, from  $Y_{n:k-1}$  to  $Y_{n:k}$  for  $k = 1, \dots, n + 1$ ; and thus  $n + 1$  weights,  $(W_1, \dots, W_{n+1}) \sim \text{Dir}(1, \dots, 1)$ , the Dirichlet distribution with all  $n + 1$  parameters equal to 1, i.e., a uniform distribution over the unit  $n$ -simplex. Instead of weight  $W_k$  corresponding

to the  $k$ th point  $Y_{n:k}$ , now  $W_k$  corresponds to the  $k$ th interval, and the probability is spread uniformly over the interval. Instead of (3), this yields

$$\widehat{F}_{*s}(y) = \sum_{i=1}^{n+1} \left[ W_i \mathbb{1}\{Y_{n:i} \leq y\} + W_i \frac{y - Y_{n:i-1}}{Y_{n:i} - Y_{n:i-1}} \mathbb{1}\{Y_{n:i} > y > Y_{n:i-1}\} \right], \quad (4)$$

where subscript  $s$  stands for “smoothed.” Note the bounds  $Y_{n:0}$  and  $Y_{n:n+1}$  do not affect  $\widehat{F}_{*s}(y)$  as long as  $Y_{n:1} \leq y \leq Y_{n:n}$ . For intuition, evaluating (4) at an order statistic yields

$$\widehat{F}_{*s}(Y_{n:k}) = \sum_{i=1}^k W_i.$$

The CDF in (4) linearly interpolates between such points; it is a linear spline with knots at the sample values, which Banks (1988) calls “histospline smoothing” in allusion to Wahba (1975).

Consider the BB and CCBB distributions of the  $\tau$ -quantile. Let

$$\widehat{Q}_{\tau,u}^* \equiv \inf\{y : \widehat{F}_{*u}(y) \geq \tau\}, \quad \widehat{Q}_{\tau,s}^* \equiv \inf\{y : \widehat{F}_{*s}(y) \geq \tau\}, \quad (5)$$

the  $\tau$ -quantiles of the distributions defined in (3) and (4). Since the BB distributions are discrete,  $\widehat{Q}_{\tau,u}^*$  must equal a sample value  $Y_i$ . In contrast, the CCBB CDF  $\widehat{F}_{*s}$  is continuous and strictly increasing (wp1), so  $\widehat{Q}_{\tau,s}^* = \widehat{F}_{*s}^{-1}(\tau)$ , with a continuous distribution of  $\widehat{Q}_{\tau,s}^*$ . Conditional on the sample  $\mathbf{Y}$  (but not the Dirichlet weights),  $\widehat{Q}_{\tau,u}^*$  and  $\widehat{Q}_{\tau,s}^*$  are random variables with respect to the distribution of the weights.

Distributions of various sums of the  $W_i$  are given by Wilks (1962, pp. 236–238). The CCBB distribution of  $(W_1, \dots, W_{n+1})$  matches the sampling distribution of standard uniform order statistic “spacings,”

$$(U_{n:1}, U_{n:2} - U_{n:1}, \dots, U_{n:n} - U_{n:n-1}, 1 - U_{n:n}) \sim \text{Dir}(1, \dots, 1), \quad (6)$$

where the  $U_{n:k}$  are order statistics based on  $U_i \stackrel{iid}{\sim} \text{Unif}(0, 1)$ ,  $i = 1, \dots, n$ . Further,

$$\begin{aligned} \text{BB: } W_1 + \dots + W_k &\sim \text{Beta}(k, n - k), \\ \text{CCBB: } W_1 + \dots + W_k &\sim \text{Beta}(k, n + 1 - k). \end{aligned} \quad (7)$$

## 2.3 Bootstrap intervals

The percentile method confidence intervals’ endpoints come directly from quantiles of the distributions of the random variables in (5). For a one-sided  $1 - \alpha$  confidence interval, the BB lower endpoint is the  $\alpha$ -quantile of  $\widehat{Q}_{\tau,u}^*$ , and the CCBB lower endpoint is the  $\alpha$ -quantile

of  $\widehat{Q}_{\tau,s}^*$ . The one-sided upper endpoints would instead be the  $(1 - \alpha)$ -quantiles. A two-sided (equal-tailed)  $1 - \alpha$  interval instead takes the  $\alpha/2$  and  $1 - \alpha/2$  quantiles of either  $\widehat{Q}_{\tau,u}^*$  (for BB) or  $\widehat{Q}_{\tau,s}^*$  (for CCBB).

We consider percentile method confidence intervals for two reasons. First, interpreting the BB distribution as a Bayesian posterior given the model of Ferguson (1973) with an improper Dirichlet process prior, the corresponding Bayesian credible interval is exactly the percentile method confidence interval. Second, even for the standard bootstrap, Falk and Kaufmann (1991) show that percentile intervals (the ‘‘backward method’’) perform better for quantiles.

### 3 Results for order statistic endpoints

Theorem 1 characterizes the BB and CCBB probabilities associated with one-sided confidence intervals whose lower endpoint is an order statistic. The unsmoothed part of Theorem 1 is similar to a result from Efron (1982, p. 82, eqn. (10.23)). The proof uses the equivalence  $Y_{n:k} < \widehat{Q}_{\tau}^* \iff \sum_{i=1}^k W_i < \tau$  along with (7). Let  $F_{\beta}(\cdot; a, b)$  denote the CDF of a Beta( $a, b$ ) distribution.

**Theorem 1.** *Under Assumption A1, given (3)–(5), for  $k = 1, \dots, n$ ,*

$$\begin{aligned} \mathrm{P}(Y_{n:k} < \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) &= F_{\beta}(\tau; k, n - k) < \mathrm{P}(Y_{n:k} \leq \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) = F_{\beta}(\tau; k - 1, n + 1 - k), \\ \mathrm{P}(Y_{n:k} < \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) &= \mathrm{P}(Y_{n:k} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) = F_{\beta}(\tau; k, n + 1 - k) = \mathrm{P}(Y_{n:k} \leq Q_{\tau}(Y)). \end{aligned}$$

The main point of Theorem 1 is that the CCBB probability of  $Y_{n:k} \leq \widehat{Q}_{\tau,s}^*$  under  $\widehat{F}_{*s}$  (conditional on the data  $\mathbf{Y}$ ) is identical to the probability of  $Y_{n:k} \leq Q_{\tau}(Y)$  under  $F_Y$ . Thus, if the  $\alpha$ -quantile of the CCBB distribution of  $\widehat{Q}_{\tau,s}^*$  is  $Y_{n:k}$  for some  $k$ , then  $Y_{n:k}$  is the lower endpoint of the one-sided  $1 - \alpha$  CCBB confidence interval for  $Q_{\tau}(Y)$ , and it has exact finite-sample coverage because  $\mathrm{P}(Y_{n:k} \leq Q_{\tau}(Y)) = \mathrm{P}(Y_{n:k} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y})$ . Further, this is not true for BB.

The probabilities in Theorem 1 extend readily to lower one-sided and two-sided intervals because

$$\begin{aligned} \mathrm{P}(Y_{n:k} \geq \widehat{Q}_{\tau}^* \mid \mathbf{Y}) &= 1 - \mathrm{P}(Y_{n:k} < \widehat{Q}_{\tau}^* \mid \mathbf{Y}), \\ \mathrm{P}(Y_{n:k} \leq \widehat{Q}_{\tau}^* \leq Y_{n:r} \mid \mathbf{Y}) &= \mathrm{P}(Y_{n:k} \leq \widehat{Q}_{\tau}^* \mid \mathbf{Y}) - \mathrm{P}(Y_{n:r} < \widehat{Q}_{\tau}^* \mid \mathbf{Y}). \end{aligned} \tag{8}$$

Corollary 1.1 characterizes the difference between the BB and CCBB probabilities in Theorem 1. Let  $f_{\beta}(\cdot; a, b)$  be the Beta( $a, b$ ) PDF.

**Corollary 1.1.** *Given Theorem 1,*

$$\begin{aligned} \mathbb{P}(Y_{n:k} \leq \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) - \mathbb{P}(Y_{n:k} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) &= \frac{1-\tau}{n} f_\beta(\tau; k, n+1-k), \\ \mathbb{P}(Y_{n:k} < \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) - \mathbb{P}(Y_{n:k} < \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) &= (\tau/n) f_\beta(\tau; k, n+1-k). \end{aligned}$$

If  $c_1 \leq k/n \leq c_2$  for constants  $0 < c_1 < c_2 < 1$  as  $n \rightarrow \infty$ , then both differences are  $O(n^{-1/2})$ .

Corollary 1.1 shows the benefit of smoothing. For intervals with an order statistic endpoint  $Y_{n:k}$ , the CCBB probability is exact, whereas the BB probability differs by  $O(n^{-1/2})$  for central order statistics ( $c_1 \leq k/n \leq c_2$ ). If  $\tau$  and  $1 - \alpha$  are fixed, then asymptotically the endpoint will be a central order statistic.

## 4 Results for fractional order statistics

We now more generally consider “fractional” order statistic endpoints. This is useful when the desired nominal level (like 95%) is not attainable at an integer order statistic. For integer  $k$  and interpolation weight  $\epsilon \in [0, 1)$ , the  $(k + \epsilon)$ th fractional order statistic is

$$Y_{n:k+\epsilon} \equiv Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}). \quad (9)$$

The special case with  $\epsilon = 0$  yields the usual (integer) order statistics.

Figure 1 illustrates the following intuition for the CCBB distribution of  $\widehat{Q}_{\tau,s}^*$ . Given the piecewise linear CDF smoothing of CCBB, the  $\tau$ -quantile given a particular Dirichlet draw of  $\mathbf{W}$  is also linearly interpolated. Given  $(W_1, \dots, W_{n+1})$ ,

$$\widehat{Q}_{\tau,s}^* = Y_{n:c} + \frac{(\tau - \sum_{i=1}^c W_i)}{W_{c+1}} (Y_{n:c+1} - Y_{n:c}), \quad c \equiv \max\{d : \sum_{i=1}^d W_i \leq \tau\}, \quad (10)$$

i.e.,  $c$  is the largest integer such that  $\sum_{i=1}^c W_i \leq \tau$ . That is, since the CCBB CDF is linearly interpolated between the points

$$\widehat{F}_{*s}(Y_{n:c}) = \sum_{i=1}^c W_i \quad \text{and} \quad \widehat{F}_{*s}(Y_{n:c+1}) = \sum_{i=1}^{c+1} W_i,$$

the CCBB inverse CDF (quantile function)  $\widehat{F}_{*s}^{-1}(\cdot)$  is linearly interpolated between

$$Y_{n:c} = \widehat{F}_{*s}^{-1}\left(\sum_{i=1}^c W_i\right) \quad \text{and} \quad Y_{n:c+1} = \widehat{F}_{*s}^{-1}\left(\sum_{i=1}^{c+1} W_i\right).$$

By inspection, (10) has the same form as (9), with  $k = c$  and  $\epsilon = (\tau - \sum_{i=1}^c W_i)/W_{c+1}$ .

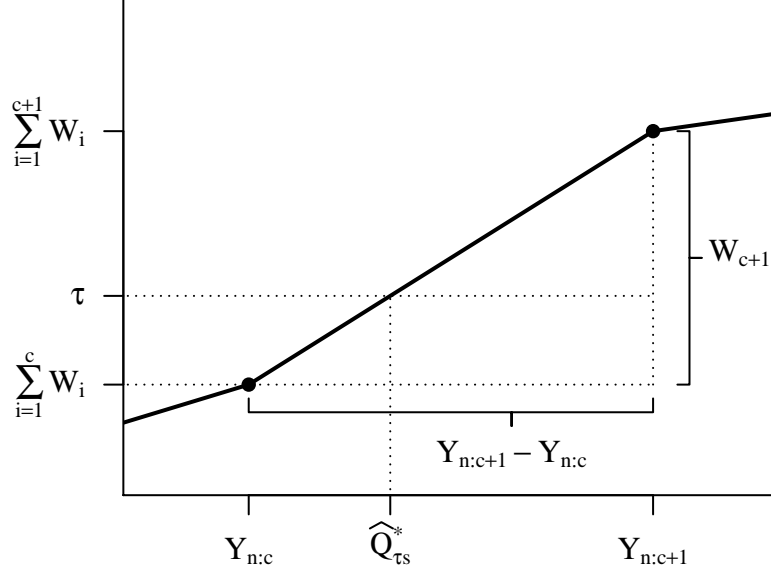


Figure 1: Illustration of (10).

Equation (10) can also be interpreted as moving along the straight line in the quantile function with slope  $(Y_{n:c+1} - Y_{n:c})/W_{c+1}$ , starting at value  $Y_{n:c}$  and moving to the right by  $\tau - \sum_{i=1}^c W_i$ .

We focus on fractional order statistics that would be CI endpoints given  $\tau$ . For central quantiles (asymptotically fixed  $\tau$ ), Lemma 3 of Goldman and Kaplan (2017) says a  $1 - \alpha$  one-sided CI lower endpoint has

$$u \equiv (k + \epsilon)/(n + 1) = \tau - n^{-1/2} z_{1-\alpha} \sqrt{\tau(1-\tau)} + O(n^{-1}), \quad (11)$$

where  $z_{1-\alpha}$  is the  $(1 - \alpha)$ -quantile of the standard normal distribution. Thus, Theorem 2 focuses on  $u = \tau - dn^{-1/2}$ .

Theorem 2 shows that the CCBB probability of fractional order statistic intervals is identical to the coverage probability in certain special cases, and more generally it matches up to a  $O(n^{-3/2}[\log(n)]^3)$  difference. Thus, the  $Y_{n:k+\epsilon}$  that's the  $\alpha$ -quantile of the CCBB distribution of  $\hat{Q}_{\tau,s}^*$  provides a confidence interval with coverage probability  $1 - \alpha + O(n^{-3/2}[\log(n)]^3)$ . Theorem 2 also provides an analytic approximation to the CCBB probability based on the beta distribution, up to the same error term. The BB distribution in Theorem 2 follows immediately from Theorem 1 since the BB distributions have no probability strictly between order statistics. As in Section 3, the results for a lower endpoint CI translate to an upper endpoint and two-sided CIs by (8).

**Theorem 2.** *Let Assumption A1 and (3)–(5) and (9) hold in each of the following.*

(i) If  $(k + \epsilon)/(n + 1) = \tau - dn^{-1/2}$ , then

$$\begin{aligned} \mathbb{P}(Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) &= F_\beta(\tau; k, n - k), \\ \mathbb{P}(Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) &= \mathbb{P}(U_{n:k} + \epsilon(U_{n:k+1} - U_{n:k}) < \tau) \\ &= F_\beta(\tau; k + \epsilon, n + 1 - k - \epsilon) \\ &\quad + n^{-1} \frac{\epsilon(1 - \epsilon)}{\tau(1 - \tau)} \frac{d}{\sqrt{\tau(1 - \tau)}} \phi\left(d/\sqrt{\tau(1 - \tau)}\right) \\ &\quad + O(n^{-3/2}[\log(n)]^3), \end{aligned}$$

where  $\phi(\cdot)$  is the standard normal PDF and (6) is the joint distribution of the  $U_{n:k}$ .

(ii) If  $Y \sim \text{Unif}(a, b)$ , or if  $\epsilon = 0$ , then the CCBB and coverage probabilities are identical:

$$\mathbb{P}(Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) = \mathbb{P}(Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < Q_\tau(Y)).$$

(iii) If Assumption A2 holds, then

$$\begin{aligned} \mathbb{P}(Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) &= \mathbb{P}(Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < Q_\tau(Y)) \\ &\quad + O(n^{-3/2}[\log(n)]^3). \end{aligned}$$

Theorem 2 shows the close connection between CCBB and the confidence intervals from Hutson (1999) and Goldman and Kaplan (2017). Hutson (1999) selects  $k + \epsilon$  to set  $F_\beta(\tau; k + \epsilon, n + 1 - k - \epsilon) = 1 - \alpha$ . This is the same beta CDF in Theorem 2(i). Goldman and Kaplan (2017, Thm. 4) show this leaves an error in coverage probability equal to the  $n^{-1}$  term and remainder in Theorem 2(i), and they suggest adjusting  $\alpha$  to remove the  $n^{-1}$  term. Here, the CCBB probability automatically captures the  $n^{-1}$  term, as well as the beta CDF term. Section 5 illustrates cases where this improved theoretical accuracy translates to better simulation performance, and cases where it does not.

Corollary 2.1 characterizes the difference between the BB and CCBB probabilities with fractional central order statistics, which is again  $O(n^{-1/2})$  as in Corollary 1.1.

**Corollary 2.1.** *The difference between the BB and CCBB probabilities in Theorem 2(i) is  $O(n^{-1/2})$  if  $c_1 \leq k/n \leq c_2$  for constants  $0 < c_1 < c_2 < 1$  as  $n \rightarrow \infty$ .*

## 5 Simulations

To illustrate our results, we simulate the coverage probability (CP) of four types of one-sided confidence intervals: BB; CCBB; ‘‘HGK’’ intervals proposed by Hutson (1999) and studied by



Goldman and Kaplan (2017); and “GKc,” the calibration proposed by Goldman and Kaplan (2017). Code in R (R Core Team, 2020) to replicate the results is on the second author’s website.<sup>1</sup>

We consider one-sided intervals (lower endpoint) with nominal level  $1 - \alpha = 0.95$ , setting  $n = 18$  and letting  $\tau$  and  $F_Y$  vary.

The different  $\tau$  are chosen to illustrate different types of situations. Some  $\tau$  are chosen so that HGK has an (integer) order statistic endpoint, in which case it has exact finite-sample CP, regardless of the underlying distribution  $F_Y$ . In such cases, CCBB also has exact CP, per Theorem 2(ii) (up to simulation error). Other  $\tau$  are chosen so the HGK endpoint is roughly halfway between order statistics. This makes HGK coverage error larger because the leading term is proportional to  $\epsilon(1 - \epsilon)$  (Goldman and Kaplan, 2017, Thm. 4(i)), which is maximized at  $\epsilon = 0.5$ . Yet other  $\tau$  are very close to 1, again to make HGK coverage error larger because leading term is also inversely proportional to  $\tau(1 - \tau)$ . One particularly extreme  $\tau$  ( $n\tau > n - 1$ ) is chosen to test the limits of our central quantile results.

The different  $F_Y$  are also chosen to illustrate our theoretical results. Theorem 2(ii) says CCBB should provide exact finite-sample CP when  $F_Y$  is uniform, regardless of  $n$ ,  $\tau$ , or  $1 - \alpha$ . For comparison, normal and exponential distributions are also used. Although no method’s  $k + \epsilon$  depends on  $F_Y$ , the CP does depend on  $F_Y$  if  $\epsilon > 0$ .

## 5.1 Endpoint comparison

Table 1 compares the lower endpoints chosen by each method for each  $\tau$ . Specifically, each confidence interval has the form  $[Y_{n:k+\epsilon}, \infty)$ , with  $Y_{n:k+\epsilon} = Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k})$  as in (9), so Table 1 shows the  $k + \epsilon$  chosen by each method. This  $k + \epsilon$  depends on  $\tau$ , but not  $F_Y$ ; it also depends on  $n$  and  $1 - \alpha$ , but these do not vary across simulations.

Table 1: Fractional order statistic indices ( $k + \epsilon$ ) for endpoints

$n$	$\tau$	BB	CCBB	HGK	GKc
18	0.2000	2.00	1.66	1.53	1.69
18	0.2500	2.00	2.24	2.16	2.25
18	0.4978	6.00	5.99	6.00	6.00
18	0.9000	14.00	14.58	14.44	14.77
18	0.9799	17.00	17.00	17.00	17.00
18	0.9900	17.00	17.77	17.49	18.00

$1 - \alpha = 0.95$ ; 300,000 Dirichlet draws for CCBB.

<sup>1</sup><https://faculty.missouri.edu/kaplandm>

BB always has  $\epsilon = 0$  and sets  $k$  as the largest integer such that  $F_\beta(\tau; k - 1, n + 1 - k) \geq 1 - \alpha$ , using Theorem 1.

HGK and GKc also determine  $k + \epsilon$  with a beta CDF. HGK sets  $1 - \alpha = F_\beta(k + \epsilon, n + 1 - k - \epsilon)$  exactly, per (7) and (8) in Hutson (1999) or (3) and (5) in Goldman and Kaplan (2017). GKc uses the same formula after replacing  $\alpha$  with  $\alpha + n^{-1} z_{1-\alpha} \phi(z_{1-\alpha}) \epsilon (1 - \epsilon) / [\tau(1 - \tau)]$ , per (8) in Goldman and Kaplan (2017). (If the resulting  $k$  is larger than the HGK  $k$ , then GKc sets  $\epsilon = 0$ .)

CCBB requires simulation. Per Theorem 2(ii), it sets  $1 - \alpha = P(U_{n:k} + \epsilon(U_{n:k+1} - U_{n:k}))$ , with probability simulated using (6) (see code for details). On a (sub)standard personal computer, it takes only a few seconds to compute the CCBB  $k + \epsilon$  for all of Table 1.

Table 1 indirectly illustrates Theorems 1 and 2. First, BB is clearly different from the other three methods. Second, as Theorem 2 suggests, the other three methods are relatively similar to each other, especially CCBB and GKc. In special cases, all four methods are identical (up to CCBB simulation error). In other cases, CCBB is between HGK and GKc, usually closer to GKc; BB can be higher or lower than the other three.

## 5.2 Coverage probability

CP is simulated as follows. First, we simulate many samples of  $U_i \stackrel{iid}{\sim} \text{Unif}(0, 1)$ . Second, for the non-uniform  $F_Y$ , samples are generated using  $Y_i = F_Y^{-1}(U_i)$ ; this helps isolate the effect of  $F_Y$  on CP. Third, for each method, the endpoint  $Y_{n:k+\epsilon}$  is computed in each sample using the  $k + \epsilon$  in Table 1. Finally, given a method's  $Y_{n:k+\epsilon}$  endpoints, the simulated CP is the proportion of samples in which  $Y_{n:k+\epsilon} \leq F_Y^{-1}(\tau)$ . The CP does not depend on  $F_Y$  for intervals with  $\epsilon = 0$ , which is always true of BB and in special cases true of CCBB, HGK, and GKc (when Table 1 shows an integer value of  $k + \epsilon$ ).

Table 2 shows the simulated coverage probabilities that illustrate our theoretical results. As a general summary of Theorem 2, Corollary 2.1, and Goldman and Kaplan's (2017) Theorem 4: for central quantiles (fixed  $\tau$  as  $n \rightarrow \infty$ ), BB intervals have CP differ from the nominal level by  $O(n^{-1/2})$ , HGK by  $O(n^{-1})$  and tending toward over-coverage, and GKc and CCBB by  $O(n^{-3/2}[\log(n)]^3)$ . Simulation error seems to be  $\pm 0.001$ , so we ignore such differences in the discussion below.

First, per Theorem 2(ii), Table 2 verifies that CCBB has exact CP when  $\epsilon = 0$  (i.e., for the third and fifth  $\tau$  values, for which the  $k + \epsilon$  value in Table 1 is an integer), which is also true of HGK and GKc (Goldman and Kaplan, 2017, p. 333). This is true regardless of  $F_Y$ . (Somewhat coincidentally, CP is also 0.95 in these cases for BB, even though the BB simulated probability of the interval is strictly above 0.95, per Corollary 1.1.)

Table 2: Simulated coverage probability

$n$	$\tau$	$F_Y$	BB	CCBB	HGK	GKc
18	0.2000	Unif(0, 1)	0.900	0.950	0.961	0.946
18	0.2000	N(0, 1)	0.900	0.956	0.965	0.953
18	0.2000	Exp(1)	0.900	0.947	0.959	0.944
18	0.2500	Unif(0, 1)	0.960	0.950	0.954	0.949
18	0.2500	N(0, 1)	0.960	0.951	0.954	0.950
18	0.2500	Exp(1)	0.960	0.949	0.953	0.948
18	0.4978	Unif(0, 1)	0.950	0.951	0.950	0.950
18	0.4978	N(0, 1)	0.950	0.951	0.950	0.950
18	0.4978	Exp(1)	0.950	0.951	0.950	0.950
18	0.9000	Unif(0, 1)	0.972	0.951	0.959	0.934
18	0.9000	N(0, 1)	0.972	0.947	0.956	0.929
18	0.9000	Exp(1)	0.972	0.945	0.954	0.927
18	0.9799	Unif(0, 1)	0.950	0.950	0.950	0.950
18	0.9799	N(0, 1)	0.950	0.950	0.950	0.950
18	0.9799	Exp(1)	0.950	0.950	0.950	0.950
18	0.9900	Unif(0, 1)	0.986	0.950	0.974	0.835
18	0.9900	N(0, 1)	0.986	0.898	0.954	0.835
18	0.9900	Exp(1)	0.986	0.890	0.946	0.835

$1 - \alpha = 0.95$ ; 300,000 replications.

Second, also per Theorem 2(ii), CCBB has exact CP when  $F_Y$  is uniform. This is true even for the most challenging  $\tau = 0.99$ .

Third, CCBB is more accurate than BB. For all but the largest (most extreme)  $\tau$ , CCBB has CP between 0.945 and 0.956, whereas BB's CP varies between 0.900 and 0.972. With  $\tau = 0.99$ , CCBB is exact when  $F_Y$  is uniform, but when  $F_Y$  is normal or exponential, CCBB has CP below 0.90, whereas BB has 0.986 CP. This illustrates how our central quantile results in Theorem 2 are less accurate for extreme  $\tau$  (here  $n\tau = 17.82 > n - 1$ ).

Fourth, CCBB is generally more accurate than HGK, as well as GKc, although there are important caveats. It is somewhat unfair to compare them when  $F_Y$  is uniform because we picked the uniform distribution precisely because CCBB is exact in that case. Nonetheless, CCBB usually has CP closer to nominal with the other  $F_Y$ , too. For all but the largest (most extreme)  $\tau$ , for normal and exponential  $F_Y$ , CCBB has CP in the range  $[0.945, 0.956]$ , while the HGK range is  $[0.950, 0.965]$ , and GKc is  $[0.927, 0.953]$ . As noted by Goldman and Kaplan (2017), HGK rarely undercovers because the leading  $n^{-1}$  term in its coverage error is always positive (regardless of  $F_Y$ , interval type, and  $\tau$ ), whereas CCBB and GKc can have CP above or below the nominal level. This is particularly notable with the most extreme  $\tau = 0.99$ ,

where CCBB significantly undercovers (for the non-uniform  $F_Y$ ) and GKc undercovers even more, whereas HGK has CP in the range  $[0.946, 0.974]$  among the three  $F_Y$ . Again, the  $\tau = 0.99$  results are partly explained by the central quantile asymptotic framework breaking down at such an extreme quantile.

## 6 Conclusion

For population quantiles of a continuous distribution, we have established the high-order accuracy of confidence intervals from the smoothed Bayesian bootstrap of Banks (1988), with exact finite-sample coverage in special cases. The accuracy is orders of magnitude better than that of the unsmoothed Bayesian bootstrap (Rubin, 1981). These results motivate future study of the smoothed Bayesian bootstrap with non-iid sampling and/or other objects of interest.

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## A Proofs

### A.1 Proof of Theorem 1

*Proof.* First, for either  $\widehat{Q}_\tau^* = \widehat{Q}_{\tau,u}^*$  or  $\widehat{Q}_\tau^* = \widehat{Q}_{\tau,s}^*$ ,

$$Y_{n:k} < \widehat{Q}_\tau^* \iff \sum_{i=1}^k W_i < \tau.$$

Thus,

$$P(Y_{n:k} < \widehat{Q}_\tau^* \mid \mathbf{Y}) = P\left(\sum_{i=1}^k W_i < \tau \mid \mathbf{Y}\right).$$

For the BB, by (7),  $\sum_{i=1}^k W_i \mid \mathbf{Y} \sim \text{Beta}(k, n - k)$ . Since the beta distribution is

continuous,

$$\mathbb{P}\left(\sum_{i=1}^k W_i < \tau \mid \mathbf{Y}\right) = \mathbb{P}\left(\sum_{i=1}^k W_i \leq \tau \mid \mathbf{Y}\right) = F_\beta(\tau; k, n - k). \quad (12)$$

Since the BB  $\widehat{F}_{*u}(\cdot)$  includes only discrete distributions with support  $\{Y_i\}_{i=1}^n$ ,

$$Y_{n:k} \leq \widehat{Q}_{\tau,u}^* \iff Y_{n:k-1} < \widehat{Q}_{\tau,u}^*.$$

Combining this with (12),

$$\mathbb{P}(Y_{n:k} \leq \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) = \mathbb{P}(Y_{n:k-1} < \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) = F_\beta(\tau; k - 1, n - (k - 1)).$$

For the CCBB, the same steps hold as for (12), but with  $\sum_{i=1}^k W_i \mid \mathbf{Y} \sim \text{Beta}(k, n + 1 - k)$  from (7), so

$$\mathbb{P}(Y_{n:k} < \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) = \mathbb{P}\left(\sum_{i=1}^k W_i < \tau \mid \mathbf{Y}\right) = \mathbb{P}\left(\sum_{i=1}^k W_i \leq \tau \mid \mathbf{Y}\right) = F_\beta(\tau; k, n + 1 - k).$$

Since the CCBB distribution of  $\widehat{Q}_{\tau,s}^*$  is continuous,  $\mathbb{P}(Y_{n:k} < \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) = \mathbb{P}(Y_{n:k} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y})$ .

With continuous  $F_Y(\cdot)$ , using Wilks (1962, **8.7.4**),

$$\mathbb{P}(Y_{n:k} \leq Q_\tau(Y)) = \mathbb{P}(F_Y(Y_{n:k}) \leq \tau) = F_\beta(\tau; k, n + 1 - k) = \mathbb{P}(Y_{n:k} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}). \quad \square$$

## A.2 Proof of Corollary 1.1

*Proof.* The connection between beta and binomial distributions is used below. The PDF of a Beta( $a, b$ ) distribution is (Abramowitz and Stegun, 1972, **6.2.2**, **26.1.33**)

$$f_\beta(p; a, b) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} p^{a-1} (1 - p)^{b-1},$$

where  $\Gamma(\cdot)$  is the gamma function. If  $m$  is an integer, then  $\Gamma(m) = (m - 1)!$  (Abramowitz and Stegun, 1972, **6.1.6**). Thus, if  $a$  and  $b$  are integers,

$$f_\beta(p; a, b) = \frac{(a + b - 1)!}{(a - 1)!(b - 1)!} p^{a-1} (1 - p)^{b-1}.$$

The probability mass function of a Binomial( $m, p$ ) distribution is (Abramowitz and Stegun, 1972, **26.1.20**)

$$f_B(d; m, p) = \binom{m}{d} p^d (1 - p)^{m-d} = \frac{m!}{d!(m - d)!} p^d (1 - p)^{m-d}.$$

With  $d = k - 1$ ,  $m = n - 1$ , and  $p = \tau$ ,

$$f_B(k - 1; n - 1, \tau) = \frac{(n - 1)!}{(k - 1)!(n - k)!} \tau^{k-1} (1 - \tau)^{n-k}.$$

With  $p = \tau$ ,  $a = k$ , and  $b = n + 1 - k$ ,

$$f_\beta(\tau; k, n + 1 - k) = \frac{(n)!}{(k - 1)!(n - k)!} \tau^{k-1} (1 - \tau)^{n-k} = n f_B(k - 1; n - 1, \tau). \quad (13)$$

The beta and binomial CDFs can both be written in terms of the regularized incomplete beta function  $I$ . The beta CDF is  $F_\beta(p; a, b) = I_p(a, b)$  (David and Nagaraja, 2003, eqn. (1.3.2)). Applying **26.5.24** in Abramowitz and Stegun (1972),

$$1 - F_B(d; m, p) = \overbrace{\sum_{s=d+1}^m f_B(s; m, p)}^{\text{from 26.5.24}} = I_p(d + 1, m - d) = F_\beta(p; d + 1, m - d). \quad (14)$$

From Theorem 1, applying (14) with  $p = \tau$  and either  $(d = k - 2, m = n - 1)$  or  $(d = k - 1, m = n)$ ,

$$\begin{aligned} & \mathbb{P}(Y_{n:k} \leq \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) - \mathbb{P}(Y_{n:k} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) \\ &= F_\beta(\tau; k - 1, n + 1 - k) - F_\beta(\tau; k, n + 1 - k) \\ &= [1 - F_B(k - 2; n - 1, \tau)] - [1 - F_B(k - 1; n, \tau)] \\ &= F_B(k - 1; n, \tau) - F_B(k - 2; n - 1, \tau). \end{aligned}$$

The two binomial CDFs may be expressed in terms of  $B_i \stackrel{iid}{\sim} \text{Bernoulli}(\tau)$ :

$$F_B(k - 2; n - 1, \tau) = \mathbb{P}\left(\sum_{i=1}^{n-1} B_i \leq k - 2\right), \quad F_B(k - 1; n, \tau) = \mathbb{P}\left(\sum_{i=1}^n B_i \leq k - 1\right).$$

Since  $\sum_{i=1}^n B_i = B_n + \sum_{i=1}^{n-1} B_i$  and  $B_n \leq 1$ ,

$$\sum_{i=1}^{n-1} B_i \leq k - 2 \implies B_n + \sum_{i=1}^{n-1} B_i \leq 1 + k - 2 \implies \sum_{i=1}^n B_i \leq k - 1.$$

Thus, the CDF difference equals the probability that the latter inequality holds but the former does not, which only occurs when  $\sum_{i=1}^{n-1} B_i = k - 1$  and  $B_n = 0$ :

$$F_B(k - 1; n, \tau) - F_B(k - 2; n - 1, \tau) = \mathbb{P}\left(\sum_{i=1}^n B_i \leq k - 1 \text{ and } \sum_{i=1}^{n-1} B_i > k - 2\right)$$

$$\begin{aligned}
& \overbrace{\phantom{= P\left(\sum_{i=1}^{n-1} B_i = k-1 \text{ and } B_n = 0\right)}}^{\text{use } \{B_i\}_{i=1}^{n-1} \perp B_n} \\
&= P\left(\sum_{i=1}^{n-1} B_i = k-1 \text{ and } B_n = 0\right) \\
&= P\left(\sum_{i=1}^{n-1} B_i = k-1\right) P(B_n = 0) \\
&= \overbrace{f_B(k-1; n-1, \tau)}^{\text{apply (13)}} (1-\tau) \\
&= n^{-1} f_\beta(\tau; k, n+1-k)(1-\tau).
\end{aligned}$$

The other probability difference is derived using the same steps. From Theorem 1, applying (14) with  $p = \tau$  and either  $(d = k-1, m = n)$  or  $(d = k-1, m = n-1)$ , and again using the  $B_i$  defined earlier,

$$\begin{aligned}
& P(Y_{n:k} < \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) - P(Y_{n:k} < \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) \\
&= F_\beta(\tau; k, n-k+1) - F_\beta(\tau; k, n-k) \\
&= [1 - F_B(k-1; n, \tau)] - [1 - F_B(k-1; n-1, \tau)] \\
&= F_B(k-1; n-1, \tau) - F_B(k-1; n, \tau) \\
&= P\left(\sum_{i=1}^{n-1} B_i \leq k-1\right) - P\left(\sum_{i=1}^n B_i \leq k-1\right) \\
&= P\left(\sum_{i=1}^{n-1} B_i = k-1 \text{ and } B_n = 1\right) \\
&= P\left(\sum_{i=1}^{n-1} B_i = k-1\right) P(B_n = 1) \\
&= f_B(k-1; n-1, \tau)\tau \\
&= \frac{\tau}{n} f_\beta(\tau; k, n+1-k).
\end{aligned}$$

The mode of a Beta( $a, b$ ) distribution is  $(a-1)/(a+b-2)$  for  $a, b > 1$ . Thus for  $1 < k < n$ , the mode of a Beta( $k, n+1-k$ ) distribution is  $(k-1)/(n-1)$ , i.e.,

$$\arg \sup_{0 \leq t \leq 1} f_\beta(t; k, n+1-k) = (k-1)/(n-1).$$

Thus, the maximum value of the beta PDF is  $f_\beta((k-1)/(n-1); k, n+1-k)$ . Applying the Stirling-type approximation

$$m! = \sqrt{2\pi m} m^{m+(1/2)} e^{-m} e^{r_m}, \quad \frac{1}{12m+1} < r_m < \frac{1}{12m} \quad (15)$$



of Robbins (1955) and evaluating (13) at the mode  $\tau = (k-1)/(n-1)$ ,

$$\begin{aligned}
& f_\beta((k-1)/(n-1); k, n+1-k) \\
&= \frac{n!}{(k-1)!(n-k)!} [(k-1)/(n-1)]^{k-1} \overbrace{\left[1 - (k-1)/(n-1)\right]^{n-k}}{=[n-1-(k-1)]/(n-1)} \\
&= \frac{n[(n-1)!]}{(k-1)!(n-k)!} [(k-1)/(n-1)]^{k-1} [(n-k)/(n-1)]^{n-k} \\
&= n \frac{(n-1)!}{(k-1)!(n-k)!} (k-1)^{k-1} (n-k)^{n-k} [1/(n-1)]^{(k-1)+(n-k)} \\
&= n \frac{(n-1)!}{(n-1)^{n-1}} \frac{(k-1)^{k-1}}{(k-1)!} \frac{(n-k)^{n-k}}{(n-k)!} \\
&= n \frac{\sqrt{2\pi}(n-1)^{1/2} e^{-(n-1)} e^{r_{n-1}}}{2\pi(k-1)^{1/2}(n-k)^{1/2} e^{-(k-1)} e^{-(n-k)} e^{r_{k-1}} e^{r_{n-k}}} \\
&= n(2\pi)^{-1/2} \sqrt{\frac{n-1}{(k-1)(n-k)}} \underbrace{\exp\{-(n-1) + (k-1) + (n-k)\}}_{=1} \exp\{r_{n-1} - r_{k-1} - r_{n-k}\}.
\end{aligned}$$

For the approximation error, since  $r_{k-1}, r_{n-k} > 0$ ,

$$\exp\{r_{n-1} - r_{k-1} - r_{n-k}\} \leq \exp\{r_{n-1}\} \leq \exp\{1/[12(n-1)]\} = 1 + O(n^{-1}).$$

Thus,

$$f_\beta((k-1)/(n-1); k, n+1-k) \leq n(2\pi)^{-1/2} \sqrt{\frac{n-1}{(k-1)(n-k)}} [1 + O(n^{-1})]. \quad (16)$$

If  $c_1 \leq k/n \leq c_2$  for constants  $0 < c_1 < c_2 < 1$  as  $n \rightarrow \infty$ , i.e., if  $k$  is of order  $n$ , then the upper bound rate of (16) simplifies further. Specifically, this implies  $k-1 \geq c_1 n - 1$  and  $n-k \geq n - c_2 n = n(1-c_2)$ , so

$$\sqrt{\frac{n-1}{(k-1)(n-k)}} \leq \sqrt{\frac{n}{(c_1 n - 1)n(1-c_2)}} = [(c_1 n - 1)(1-c_2)]^{-1/2} = O(n^{-1/2}). \quad (17)$$

Replacing  $n(2\pi)^{-1/2}$  with  $O(n)$  in (16), the probability difference is

$$n^{-1} \overbrace{f_\beta((k-1)/(n-1); k, n+1-k)}^{\text{use (16)}} = n^{-1} O(n) O(n^{-1/2}) [1 + O(n^{-1})] = O(n^{-1/2}).$$

If  $k$  is not required to have rate  $n$ , then this bound on the difference is larger. If  $k$  or  $n-k$  is fixed as  $n \rightarrow \infty$  (i.e.,  $Y_{n:k}$  is an ‘‘extreme order statistic’’), then (17) becomes  $O(1)$  instead of  $O(n^{-1/2})$ , so the overall difference becomes  $O(1)$  instead of  $O(n^{-1/2})$ . If  $c_1 n^r \leq k \leq c_2 n^r$

for  $0 < r < 1$ , then (17) becomes

$$\sqrt{\frac{n-1}{(k-1)(n-k)}} \leq \sqrt{\frac{n}{(c_1 n^r - 1)(n - c_2 n^r)}} = O(n^{-r/2}).$$

The same holds if instead  $c_1 n^r \leq n - k \leq c_2 n^r$ . In either case, the overall difference also becomes  $O(n^{-r/2})$ .  $\square$

### A.3 Proof of Theorem 2

*Proof.* For Theorem 2(i), the BB result follows immediately from Theorem 1 since

$$\mathbb{P}(Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) = \mathbb{P}(Y_{n:k} < \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}).$$

For the CCBB result in Theorem 2(i), first we show that (conditional on  $\mathbf{Y}$ )

$$\overbrace{Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < \widehat{Q}_{\tau,s}^*}^{\text{event } A} \iff \overbrace{\sum_{i=1}^k W_i + \epsilon W_{k+1} < \tau}^{\text{event } B}. \quad (18)$$

Write  $\widehat{Q}_{\tau,s}^*$  with  $c \equiv \max\{d : \sum_{i=1}^d W_i \leq \tau\}$  as in (10). We show (18) holds for  $c < k$ ,  $c > k$ , and  $c = k$ . If  $c < k$ , implying  $\sum_{i=1}^k W_i > \tau$ , then neither  $A$  nor  $B$  occurs:

$$\begin{aligned} \text{for } A, \quad & \overbrace{\widehat{Q}_{\tau,s}^* < Y_{n:c+1}}^{\text{by (10)}} \leq \underbrace{Y_{n:k}}_{\text{by } c < k} \leq Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}); \\ & \text{by } \epsilon \geq 0 \\ \text{for } B, \quad & \overbrace{\sum_{i=1}^k W_i + \epsilon W_{k+1}}^{\text{by } \epsilon \geq 0} \geq \underbrace{\sum_{i=1}^k W_i}_{\text{implied by } c < k} > \tau. \end{aligned}$$

If  $c > k$ , implying  $c \geq k + 1$  and thus  $\sum_{i=1}^{k+1} W_i \leq \tau$ , then both  $A$  and  $B$  occur:

$$\begin{aligned} \text{for } A, \quad & \overbrace{\widehat{Q}_{\tau,s}^* \geq Y_{n:c}}^{\text{by (10)}} \geq \underbrace{Y_{n:k+1}}_{\text{by } c > k} > Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}); \\ & \text{by } \epsilon < 1 \\ \text{for } B, \quad & \overbrace{\sum_{i=1}^k W_i + \epsilon W_{k+1}}^{\text{by } \epsilon < 1} < \underbrace{\sum_{i=1}^{k+1} W_i}_{\text{implied by } c > k} \leq \tau. \end{aligned}$$

If  $c = k$ , then event  $A$  in (18) becomes

$$\begin{aligned}
& \overbrace{Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < \widehat{Q}_{\tau,s}^*}^{\text{event } A \text{ in (18)}} = \underbrace{Y_{n:k} + \frac{(\tau - \sum_{i=1}^k W_i)}{W_{k+1}}(Y_{n:k+1} - Y_{n:k})}_{\text{from (10) with } c=k} \\
& \iff \epsilon < \frac{\tau - \sum_{i=1}^k W_i}{W_{k+1}} \\
& \iff \underbrace{\sum_{i=1}^k W_i + \epsilon W_{k+1}}_{\text{event } B \text{ in (18)}} < \tau.
\end{aligned}$$

Since  $A \iff B$  in (18), the corresponding CCBB probabilities are equal:

$$\mathbb{P}\{\overbrace{Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < \widehat{Q}_{\tau,s}^*}^A \mid \mathbf{Y}\} = \mathbb{P}\{\overbrace{\sum_{i=1}^k W_i + \epsilon W_{k+1} < \tau}^B \mid \mathbf{Y}\}. \quad (19)$$

The right-hand side of (19) can then be approximated by Theorem 2(i) of Goldman and Kaplan (2017). Their theorem concerns the distribution of (linear combinations of) fractional order statistics. Recall from (6) that the joint Dirichlet distribution of  $(W_1, \dots, W_{n+1})$  is the same as the joint sampling distribution of standard uniform order statistic spacings  $(U_{n:1}, U_{n:2} - U_{n:1}, \dots, 1 - U_{n:n})$  given iid sampling. The sum of the first  $k$  spacings is  $U_{n:k}$ , and the  $(k+1)$ th spacing is  $U_{n:k+1} - U_{n:k}$ , so the CCBB distribution of  $\sum_{i=1}^k W_i + \epsilon W_{k+1}$  equals the sampling distribution of  $U_{n:k+\epsilon} \equiv U_{n:k} + \epsilon(U_{n:k+1} - U_{n:k})$ , i.e., the fractional  $(k + \epsilon)$ th order statistic. That is, for any value  $0 \leq r \leq 1$ ,

$$\mathbb{P}\left(\sum_{i=1}^k W_i + \epsilon W_{k+1} \leq r \mid \mathbf{Y}\right) = \mathbb{P}\{U_{n:k} + \epsilon(U_{n:k+1} - U_{n:k}) \leq r\}. \quad (20)$$

Theorem 2(i) of Goldman and Kaplan (2017) approximates the right-hand side of (20).

The following shows specifically how to apply Theorem 2(i) of Goldman and Kaplan (2017) to  $U_{n:k+\epsilon}$ . Their Assumption 1 is iid sampling, as in our case. In their notation,  $J = 1$ ,  $\psi_1 = 1$ , and  $F(\cdot)$  is the standard uniform CDF: for  $0 \leq t \leq 1$ ,  $F(t) = t$ ,  $F^{-1}(t) = t$ , and PDF  $f(t) = 1$ . Thus, their Assumption 2 is satisfied for any  $0 < u < 1$  since  $f(u) > 0$  and  $f''(u) = 0$  is continuous in  $u$ . Their  $u_1$  and  $\epsilon_1$  are the same as  $u$  and  $\epsilon$  here. So far, this makes their general fractional  $L$ -statistic  $L^L$  equal to our  $U_{n:k+\epsilon}$ . Also, their  $\mathbb{X}_0$  equals our  $u$ , and their  $\mathcal{V}_\psi$  equals our  $u(1 - u)$ . Their  $L^J$  follows a  $\text{Beta}((n+1)u, (n+1)(1-u)) = \text{Beta}(k + \epsilon, n + 1 - k - \epsilon)$  distribution. Thus, setting their  $\mathbb{X}_0 + n^{-1/2}K$  equal to our  $\tau$ , their

Theorem 2(i) says in their notation that

$$\mathbb{P}(L^L < \mathbb{X}_0 + n^{-1/2}K) - \mathbb{P}(L^I < \mathbb{X}_0 + n^{-1/2}K) = O(n^{-1}),$$

which in our notation is

$$\mathbb{P}(U_{n:k+\epsilon} < \tau) - F_\beta(k + \epsilon, n + 1 - k - \epsilon) = O(n^{-1}). \quad (21)$$

Theorem 2(i) of Goldman and Kaplan (2017) further provides an analytic expression for the  $n^{-1}$  term. In their notation, setting  $\mathbb{X}_0 = u$  and  $\mathbb{X}_0 + n^{-1/2}K = \tau$ , solving for their  $K$  yields  $K = \sqrt{n}(\tau - u)$ . Given  $u = \tau - dn^{-1/2}$ ,  $K = \sqrt{n}(\tau - u) = d$ . Plugging into their Theorem 2(i) yields, first in their notation and then translated into ours with  $K = d$ ,  $\mathcal{V}_\psi = u(1 - u)$ ,  $\psi_1 = 1$ , and  $\epsilon_1 = \epsilon$ ,

$$\begin{aligned} & \mathbb{P}(U_{n:k+\epsilon} < \tau) - F_\beta(k + \epsilon, n + 1 - k - \epsilon) \\ &= n^{-1} \frac{K \exp\{-K^2/(2\mathcal{V}_\psi)\}}{\sqrt{2\pi\mathcal{V}_\psi^3}} \underbrace{\psi_1}_{=1} \underbrace{\epsilon_1(1 - \epsilon_1)}{=\epsilon(1-\epsilon)} \underbrace{[f(F^{-1}(u))]^{-2}}_{=1} + O(n^{-3/2}[\log(n)]^3) \\ &= n^{-1} \frac{d \exp\{-d^2/[2u(1 - u)]\}}{\sqrt{2\pi[u(1 - u)]^3}} \epsilon(1 - \epsilon) + O(n^{-3/2}[\log(n)]^3) \\ &= n^{-1} \frac{\epsilon(1 - \epsilon)}{u(1 - u)} \frac{d}{\sqrt{u(1 - u)}} \frac{1}{\sqrt{2\pi}} \exp\{-(1/2)[d/\sqrt{u(1 - u)}]^2\} + O(n^{-3/2}[\log(n)]^3) \\ &= n^{-1} \frac{\epsilon(1 - \epsilon)}{\tau(1 - \tau)} \frac{d}{\sqrt{\tau(1 - \tau)}} \phi\left(d/\sqrt{\tau(1 - \tau)}\right) + O(n^{-3/2}[\log(n)]^3), \end{aligned} \quad (22)$$

using the fact that  $u = \tau + O(n^{-1/2})$  and writing  $\phi(\cdot)$  as the standard normal PDF.

For Theorem 2(ii), the  $\epsilon = 0$  result follows immediately from Theorem 1 and the result from Wilks (1962) that  $F_Y(Y_{n:k}) \stackrel{d}{=} U_{n:k} \sim \text{Beta}(k, n + 1 - k)$  if  $F_Y(\cdot)$  is continuous, so

$$\mathbb{P}(Y_{n:k} < Q_\tau(Y)) = \mathbb{P}\{F_Y(Y_{n:k}) < F_Y(Q_\tau(Y))\} = \mathbb{P}\{U_{n:k} < \tau\} = F_\beta(\tau; k, n + 1 - k).$$

The other result comes from (20), which directly gives the equivalence for  $\text{Unif}(0, 1)$  that can be extended to  $\text{Unif}(a, b)$  as follows. Let  $V_i \stackrel{iid}{\sim} \text{Unif}(a, b)$ , with order statistics  $V_{n:k}$  and  $Q_\tau(V) = a + (b - a)\tau$ . Then,  $(V_i - a)/(b - a) \stackrel{iid}{\sim} \text{Unif}(0, 1)$ , and

$$\begin{aligned} & \mathbb{P}\{V_{n:k} + \epsilon(V_{n:k+1} - V_{n:k}) < a + (b - a)\tau\} \\ &= \mathbb{P}\{V_{n:k} - a + \epsilon[(V_{n:k+1} - a) - (V_{n:k} - a)] < (b - a)\tau\} \\ &= \mathbb{P}\left\{\frac{V_{n:k} - a}{b - a} + \epsilon\left[\frac{V_{n:k+1} - a}{b - a} - \frac{V_{n:k} - a}{b - a}\right] < \tau\right\} \end{aligned}$$

$$= \mathbb{P}\{U_{n:k} + \epsilon(U_{n:k+1} - U_{n:k}) < \tau\}.$$

For Theorem 2(iii), plugging  $d = z_{1-\alpha}\sqrt{\tau(1-\tau)} + O(n^{-1/2})$  from (11) into (22),

$$\begin{aligned} & \mathbb{P}\{Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}\} - F_\beta(k + \epsilon, n + 1 - k - \epsilon) \\ &= n^{-1} \frac{\epsilon(1-\epsilon)}{\tau(1-\tau)} \frac{d}{\sqrt{\tau(1-\tau)}} \phi\left(d/\sqrt{\tau(1-\tau)}\right) + O(n^{-3/2}[\log(n)]^3) \\ &= n^{-1} \frac{\epsilon(1-\epsilon)z_{1-\alpha}}{\tau(1-\tau)} \phi(z_{1-\alpha}) + O(n^{-3/2}[\log(n)]^3), \end{aligned}$$

for any  $0 < \alpha < 1$ . This matches the coverage probability in Theorem 4(i) from Goldman and Kaplan (2017), noting that their  $p$  is our  $\tau$ . That is, their Theorem 4(i) says that if A1 and A2 and (11) hold, then

$$\begin{aligned} & \mathbb{P}\{Y_{n:k} + \epsilon(Y_{n:k+1} - Y_{n:k}) < Q_\tau(Y)\} - F_\beta(k + \epsilon, n + 1 - k - \epsilon) \\ &= n^{-1} \frac{\epsilon(1-\epsilon)z_{1-\alpha}}{\tau(1-\tau)} \phi(z_{1-\alpha}) + O(n^{-3/2}[\log(n)]^3). \end{aligned}$$

More directly, the  $n^{-1}$  term in their Theorem 4(i) does not depend on the underlying distribution (other than satisfying A2). Since by Theorem 2(ii) the CCBB probability equals the coverage probability when the underlying distribution is uniform, it also equals the coverage probability given any other distribution satisfying A2 up to the  $O(n^{-3/2}[\log(n)]^3)$  remainder.  $\square$

## A.4 Proof of Corollary 2.1

*Proof.* For  $\epsilon = 0$ , Corollary 1.1 gives the  $O(n^{-1/2})$  difference. The difference cannot be smaller over the larger range of  $\epsilon \in [0, 1)$ ; below we show it is not bigger, either.

For  $0 < \epsilon < 1$ , the CCBB probability is bounded by

$$\mathbb{P}(Y_{n:k+1} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) < \mathbb{P}(Y_{n:k+\epsilon} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) < \mathbb{P}(Y_{n:k} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}).$$

From Theorems 1 and 2, this implies

$$F_\beta(\tau; k + 1, n - k) < \mathbb{P}(Y_{n:k+\epsilon} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) < F_\beta(\tau; k, n + 1 - k). \quad (23)$$

The difference between the unsmoothed BB probability  $F_\beta(\tau; k, n - k)$  and the bounds in (23) can be computed using Corollary 1.1. The upper bound difference comes directly from

Corollary 1.1:

$$F_\beta(\tau; k+1, n-k) - F_\beta(\tau; k, n-k) = \frac{\tau}{n} f_\beta(\tau; k, n+1-k). \quad (24)$$

Replacing  $k$  with  $k+1$  in Theorem 1 yields

$$\mathrm{P}(Y_{n:k+1} \leq \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) = F_\beta(\tau; k, n-k), \quad \mathrm{P}(Y_{n:k+1} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) = F_\beta(\tau; k+1, n-k). \quad (25)$$

Further applying Corollary 1.1 after replacing  $k$  with  $k+1$ ,

$$F_\beta(\tau; k, n-k) - F_\beta(\tau; k+1, n-k) = \frac{1-\tau}{n} f_\beta(\tau; k+1, n-k). \quad (26)$$

This is the difference between the BB probability and the lower bound in (23).

As  $\epsilon$  goes from 0 to 1, the CCBB probability goes from the upper bound to the lower bound in (23). Thus, the BB–CCBB difference goes from

$$-\frac{\tau}{n} f_\beta(\tau; k, n+1-k) \text{ to } \frac{1-\tau}{n} f_\beta(\tau; k+1, n-k). \quad (27)$$

That is,

$$\begin{aligned} -\frac{\tau}{n} f_\beta(\tau; k, n+1-k) &\leq \mathrm{P}(Y_{n:k+\epsilon} \leq \widehat{Q}_{\tau,u}^* \mid \mathbf{Y}) - \mathrm{P}(Y_{n:k+\epsilon} \leq \widehat{Q}_{\tau,s}^* \mid \mathbf{Y}) \\ &\leq \frac{1-\tau}{n} f_\beta(\tau; k+1, n-k). \end{aligned} \quad (28)$$

Since the lower bound is negative and the upper bound positive, and the difference varies continuously with  $\epsilon$ , there is some magic  $\epsilon$  for which the BB and CCBB actually agree. However, since  $\epsilon$  is determined by the nominal level and sample size (i.e., not chosen independently), in general the difference is larger. As stated in Corollary 1.1, with  $c_1 \leq k/n \leq c_2$ , the bounds in (28) are  $O(n^{-1/2})$ .  $\square$