IMPROVED QUANTILE INFERENCE VIA FIXED-SMOOTHING ASYMPTOTICS AND EDGEWORTH EXPANSION: APPENDIX OF PROOFS FOR BIVARIATE CASE

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Contents

Appendix A. Introduction

Recall that in the univariate case, the first-order asymptotic result is

$$\sqrt{n}(X_{n,r} - \xi_p) \xrightarrow{d} N\left(0, \frac{p(1-p)}{[f(\xi_p)]^2}\right),$$

and thus the Studentized test statistic (under the null) for X is

$$T^X_{m,n} = \frac{\sqrt{n}(X_{n,r} - \xi_p)}{S^X_{m,n}\sqrt{p(1-p)}},$$

where

$$S^X_{m,n} = \frac{n}{2m}(X_{n,r+m} - X_{n,r-m}) \xrightarrow{p} g_x(p).$$

For bivariate, assume that there are independent samples of X and Y, each with n observations. The goal is to test if $\xi_{p,x} = \xi_{p,y}$. Under the null hypothesis $H_0 : \xi_{px} = \xi_{py} = \xi_p$, the first-order asymptotic result

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is

$$\sqrt{n}(X_{nr} - \xi_p) - \sqrt{n}(Y_{nr} - \xi_p) \xrightarrow{d} N\left(0, p(1-p)(f_X^{-2} + f_Y^{-2})\right), \tag{1}$$

using the fact that the variance of the sum (or difference) of two independent normals is the sum of the variances. The pivot for the bivariate case is then

$$\frac{\sqrt{n}(X_{nr} - Y_{nr})}{\sqrt{[f_X(\xi_p)]^{-2} + [f_Y(\xi_p)]^{-2}}\sqrt{p(1-p)}} \xrightarrow{d} N(0,1), \tag{2}$$

with the Studentized version using the sample estimates of f_X and f_Y by the same quantile spacing estimator as in the univariate case,

$$T_{m,n} \equiv \frac{\sqrt{n}(X_{nr} - Y_{nr})}{\sqrt{(n/(2m))^2(X_{n,r+m} - X_{n,r-m})^2 + (n/(2m))^2(Y_{n,r+m} - Y_{n,r-m})^2}\sqrt{p(1-p)}}. \tag{3}$$

This is the test statistic I expand below.

For notation, there are now new terms associated with Y, and occasionally the X term will have an "X" added as a sub- or superscript to clarify:

$$\begin{aligned}
f_X &: \text{was just f, the population pdf} \\
f_Y &: \text{the } Y \text{ equivalent of } f_X \\
\xi_{px} &: \text{was just } \xi_p \\
\eta_{px} &: \text{was just } \eta_p \\
\xi_{py} &: \text{the } Y \text{ equivalent of } \xi_{px} \\
\eta_{py} &: \text{the } Y \text{ equivalent of } \eta_{px} \\
a_i' &: \text{the } Y \text{ equivalent of } a_i \\
g_x &: \text{was g(p)} \\
g_x', g_x'' &: \text{were g'(p), g''(p)} \\
g_y &: \text{the } Y \text{ equivalent of } g_x \text{ (so, short for } g_y(p)) \\
g_y', g_y'' &: \text{the } Y \text{ equivalent of } g_x', g_x'' \\
\nabla_i &: \text{the } Y \text{ equivalent of } \Delta_i \\
\daleth_i &: \text{the } Y \text{ equivalent of } D_i = \sqrt{n}\Delta_i \\
S_0 &: \text{no longer just } 1/f(\xi_p)\text{; see below} \\
S_{mn} &: \text{different (includes } Y \text{ now); see below.}
\end{aligned}$$

Note that while p is, of course, the same for X and Y, f_X can be different from f_Y,[1] which means that g_x and its derivatives can be different from those for Y and likewise for the a_i different from a_i'. I do assume, though, that the X and Y are independent, so that $\Delta_i \perp\!\!\!\perp \nabla_i$ and $D_i \perp\!\!\!\perp \daleth_i$. Finally, while the paper uses tildes to distinguish univariate and bivariate results (e.g., $\tilde{S}_0$ and S_0), tildes are omitted here since only the bivariate case is treated.

[1]Assuming exchangeability, they would be the same, and a pooled estimator could be used. Exchangeability (under the null) is a strong assumption but maintained by permutation tests, for example.

Appendix B. Edgeworth Expansion (proof)

Similar to the univariate case, parameterizing $\xi_{py} = \xi_{px} + \gamma/\sqrt{n}$,

$$\begin{aligned}
P(T_{m,n} < z) &= P\left(\frac{\sqrt{n}(X_{nr} - Y_{nr})}{S_{mn}\sqrt{p(1-p)}} < z\right) \\
&= P\left(\frac{\sqrt{n}(X_{nr} - \xi_{px}) - \sqrt{n}(Y_{nr} - \xi_{py}) + \sqrt{n}(\xi_{px} - \xi_{py})}{S_{mn}\sqrt{p(1-p)}} < z\right) \\
&= P\left(\frac{\sqrt{n}(X_{nr} - \xi_{px}) - \sqrt{n}(Y_{nr} - \xi_{py}) - \gamma}{S_{mn}\sqrt{p(1-p)}} < z\right) \\
&= P\left(\frac{\sqrt{n}(X_{nr} - \xi_{px}) - \sqrt{n}(Y_{nr} - \xi_{py}) + \gamma((S_{mn}/S_0) - 1)}{S_{mn}\sqrt{p(1-p)}} < z + \frac{\gamma}{S_0\sqrt{p(1-p)}}\right).
\end{aligned}$$

Define

$$\begin{aligned}
Z &\equiv \sqrt{p(1-p)}\left[\sqrt{n}(X_{nr} - \eta_{px}) - \sqrt{n}(Y_{nr} - \eta_{py}) + \gamma((S_{mn}/S_0) - 1)\right]/\hat{\tau} \\
&= \left[\sqrt{n}(X_{nr} - \eta_{px}) - \sqrt{n}(Y_{nr} - \eta_{py}) + \gamma((S_{mn}/S_0) - 1)\right] \\
&\quad \times \left[(n/(2m))^2(X_{n,r+m} - X_{n,r-m})^2 + (n/(2m))^2(Y_{n,r+m} - Y_{n,r-m})^2\right]^{-1/2} \\
&= [\cdot]S_{mn}^{-1}.
\end{aligned} \tag{4}$$

B.1. **Centering.** As in the univariate case, I first treat the centering issue. Using the univariate results and notation/definitions,

$$\begin{aligned}
\epsilon_n &\equiv \lfloor np \rfloor + 1 - np, \\
\eta_{px} &= \xi_{px} + n^{-1}[\epsilon_n - 1 + \tfrac{1}{2}(1-p)]/f_x(\xi_{px}) + O(n^{-2}), \\
\frac{n}{2m}(X_{n,r+m} - X_{n,r-m}) &= g_x + O_p(m^{-1/2} + m^2/n^2),
\end{aligned}$$

so

$$\begin{aligned}
S_{mn} &= \left[\left(\frac{n}{2m}(X_{n,r+m} - X_{n,r-m})\right)^2 + \left(\frac{n}{2m}(Y_{n,r+m} - Y_{n,r-m})\right)^2\right]^{1/2} \\
&= \left[\left(g_x + O_p(m^{-1/2} + m^2/n^2)\right)^2 + \left(g_y + O_p(m^{-1/2} + m^2/n^2)\right)^2\right]^{1/2} \\
&= \left[g_x^2 + g_y^2 + O_p(m^{-1/2} + m^2/n^2)\right]^{1/2} \\
&= (g_x^2 + g_y^2)^{1/2} + O_p(m^{-1/2} + m^2/n^2) \\
&= S_0 + O_p(m^{-1/2} + m^2/n^2), \\
\sqrt{n}(X_{nr} - \xi_{px}) - \sqrt{n}(Y_{nr} - \xi_{py}) &= \sqrt{n}(X_{nr} - \eta_{px}) - \sqrt{n}(Y_{nr} - \eta_{py}) + \sqrt{n}(\eta_{px} - \xi_{px} - (\eta_{py} - \xi_{py})), \\
(\eta_{px} - \xi_{px}) - (\eta_{py} - \xi_{py}) &= n^{-1}[\epsilon_n - 1 + \tfrac{1}{2}(1-p)](g_x - g_y) + O(n^{-2}), \\
\frac{\sqrt{n}[(\eta_{px} - \xi_{px}) - (\eta_{py} - \xi_{py})]}{\hat{\tau}} &= \frac{\sqrt{n}}{\sqrt{p(1-p)}S_{mn}}\left[n^{-1}[\epsilon_n - 1 + \tfrac{1}{2}(1-p)](g_x - g_y) + O(n^{-2})\right] \\
&= n^{-1/2}\frac{[\epsilon_n - 1 + \frac{1}{2}(1-p)]}{\sqrt{p(1-p)}}\frac{(g_x - g_y)}{S_{mn}} + O(n^{-3/2}) \\
&= n^{-1/2}\frac{[\epsilon_n - 1 + \frac{1}{2}(1-p)]}{\sqrt{p(1-p)}}\frac{(g_x - g_y)}{S_0 + O_p(m^{-1/2} + m^2/n^2)} + O(n^{-3/2}) \\
&= n^{-1/2}\frac{[\epsilon_n - 1 + \frac{1}{2}(1-p)]}{\sqrt{p(1-p)}}\frac{(g_x - g_y)}{S_0} + o_p(m^{-1} + m^2/n^2)
\end{aligned}$$

$$= n^{-1/2} w_n + o_p(m^{-1} + m^2/n^2),$$

with

$$w_n \equiv \frac{[\epsilon_n - 1 + \frac{1}{2}(1-p)](g_x - g_y)}{S_0\sqrt{p(1-p)}}.$$

As in the univariate case, the final distribution will need to subtract $n^{-1/2} w_n \phi(z)$, the only difference being that w_n has a different definition in the bivariate case.

B.2. **Numerator of** Z**.** From the univariate results, $\sqrt{n}(X_{nr} - \eta_{px}) + \gamma((S_{mn}/S_0) - 1) = \sqrt{n}[(\Delta_2 + \Delta_3)a_1 + (1/2)(\Delta_2 + \Delta_3)^2 a_2 + O_p(n^{-3/2})] + \gamma\nu$ in (32) of the main appendix (working paper version), and from (31), $X_{nr} - \eta_{px} = (\Delta_2 + \Delta_3)a_1 + (1/2)(\Delta_2 + \Delta_3)^2 a_2 + O_p(n^{-3/2})$. For the bivariate case, subtracting the Y term, $\sqrt{n}(Y_{nr} - \eta_{py})$, yields

$$\begin{aligned}
&\sqrt{n}(X_{nr} - \eta_{px}) - \sqrt{n}(Y_{nr} - \eta_{py}) + \gamma((S_{mn}/S_0) - 1) \\
&\quad = \sqrt{n}[a_1(\Delta_2 + \Delta_3) - a_1'(\nabla_2 + \nabla_3) + (1/2)(\Delta_2 + \Delta_3)^2 a_2 - (1/2)(\nabla_2 + \nabla_3)^2 a_2'] \\
&\qquad + O_p(n^{-1}) + \gamma((S_{mn}/S_0) - 1),
\end{aligned}$$

where $\gamma((S_{mn}/S_0) - 1)$ is different than the univariate ν but still $o_p(1)$. As derived and defined below,

$$(S_{mn}/S_0) - 1 = \tilde{\nu}/2 - \tilde{\nu}^2/8.$$

B.3. **Denominator of** Z**.** The denominator portion of Z (where Z=Num×Denom) is the term

$$\left[(n/(2m))^2(X_{n,r+m} - X_{n,r-m})^2 + (n/(2m))^2(Y_{n,r+m} - Y_{n,r-m})^2\right]^{-1/2}.$$

Note that

$$\begin{aligned}
(1+x)^{-1/2} &= 1^{-1/2} + x(-(1/2)1^{-3/2}) + (1/2)x^2((3/4)1^{-5/2}) + O(x^3) \\
&= 1 - x/2 + (3/8)x^2 + O(x^3).
\end{aligned}$$

Since the numerator of Z is $O_p(1)$, the denominator has remainder the same as $R = O_p(n^{-1/2}m^{-1/2} + n^{-3/2}m + m^{-3/2} + (m/n)^{2+\epsilon}) = O_p(n^{-\epsilon\eta}[m^{-1} + m^2/n^2])$ as shown in HS88. This means any higher-order terms inside the square root in the denominator will end up in the overall remainder R from $Z = Y + R$ if they are of the same (or smaller) order as R.

From the univariate case ((33) and (34) in working paper main appendix), in the new notation,

$$\begin{aligned}
\frac{n}{2m}(X_{n,r+m} - X_{n,r-m}) = g_x &+ (m/n)^2\frac{g_x''}{6} + O((m/n)^{2+\epsilon} + n^{-1}) - (n/m)(a_1/2)(\Delta_1 + \Delta_2) \\
&- \frac{a_2}{2p}(\Delta_1 + \Delta_2 + 2\Delta_3) + O_p(n^{-1/2}m^{-1/2} + n^{-3/2}m),
\end{aligned}$$

and thus similarly,

$$\begin{aligned}
\frac{n}{2m}(Y_{n,r+m} - Y_{n,r-m}) = g_y &+ (m/n)^2\frac{g_y''}{6} + O((m/n)^{2+\epsilon} + n^{-1}) - (n/m)(a_1'/2)(\nabla_1 + \nabla_2) \\
&- \frac{a_2'}{2p}(\nabla_1 + \nabla_2 + 2\nabla_3) + O_p(n^{-1/2}m^{-1/2} + n^{-3/2}m).
\end{aligned}$$

Let (temporarily)

$$\begin{aligned}
A &\equiv \frac{n}{2m}(X_{n,r+m} - X_{n,r-m}) \\
B &\equiv \frac{n}{2m}(Y_{n,r+m} - Y_{n,r-m}) \\
S_{mn}^{-1} &= (A^2 + B^2)^{-1/2},
\end{aligned}$$

then

$$
\begin{aligned}
A^2 &= g_x^2 + O(m^4/n^4) + O((m/n)^{2+\epsilon} + n^{-1})^2 + (n^2/m^2)(a_1^2/4)(\Delta_1+\Delta_2)^2 \\
&\quad + (a_2^2/(4p^2))(\Delta_1+\Delta_2+2\Delta_3)^2 + O_p(n^{-1/2}m^{-1/2} + n^{-3/2}m)^2 \\
&\quad + 2g_x(m/n)^2(g_x''/6) + 2g_x O((m/n)^{2+\epsilon} + n^{-1}) - 2g_x(n/m)(a_1/2)(\Delta_1+\Delta_2) \\
&\quad - 2g_x\frac{a_2}{2p}(\Delta_1+\Delta_2+2\Delta_3) + 2g_x O_p(n^{-1/2}m^{-1/2} + mn^{-3/2}) \\
&\quad + 2(m/n)^2(g_x''/6)O((m/n)^{2+\epsilon} + n^{-1}) - 2(m/n)^2(g_x''/6)(n/m)(a_1/2)(\Delta_1+\Delta_2) \\
&\quad - 2(m/n)^2(g_x''/6)(a_2/(2p))(\Delta_1+\Delta_2+2\Delta_3) + 2(m/n)^2(g_x''/6)O_p(n^{-1/2}m^{-1/2} + mn^{-3/2}) \\
&\quad + 2O((m/n)^{2+\epsilon} + n^{-1})O_p((n/m)(\sqrt{m}/n) - n^{-1/2} + n^{-1/2}m^{-1/2} + mn^{-3/2}) \\
&\quad + 2(n/m)(a_1/2)(\Delta_1+\Delta_2)(a_2/(2p))(\Delta_1+\Delta_2+2\Delta_3) \\
&\quad - 2(n/m)(a_1/2)(\Delta_1+\Delta_2)O_p(n^{-1/2}m^{-1/2} + mn^{-3/2}) \\
&\quad - 2(a_2/(2p))(\Delta_1+\Delta_2+2\Delta_3)O_p(m^{-1/2}n^{-1/2} + mn^{-3/2}) \\
&= g_x^2 + (n/m)^2(a_1^2/4)(\Delta_1+\Delta_2)^2 + (a_2^2/(4p^2))(\Delta_1+\Delta_2+2\Delta_3)^2 \\
&\quad + 2(m/n)^2(g_x g_x''/6) - 2(n/m)g_x(a_1/2)(\Delta_1+\Delta_2) \\
&\quad - 2(a_2/(2p))g_x(\Delta_1+\Delta_2+2\Delta_3) \\
&\quad - 2(m/n)(a_1/2)(g_x''/6)(\Delta_1+\Delta_2) - 2(m/n)^2(a_2/(2p))(g_x''/6)(\Delta_1+\Delta_2+2\Delta_3) \\
&\quad + 2(n/m)(a_1a_2/(4p))(\Delta_1+\Delta_2)(\Delta_1+\Delta_2+2\Delta_3) \\
&\quad + O_p(m^4/n^4 + (m/n)^{4+8\epsilon+\epsilon^2} + n^{-2} + (m/n)^{2+\epsilon}n^{-1} + n^{-1}m^{-1} + m^2n^{-5}) \\
&\quad + O_p(\sqrt{m}n^{-2} + (m/n)^{2+\epsilon} + n^{-1} + n^{-1/2}m^{-1/2}) \\
&\quad + O_p(mn^{-3/2} + m^2n^{-3} + m^{3/2}n^{-5/2} + m^3n^{-7/2}) \\
&\quad + O_p((m/n)^{2+\epsilon}m^{-1/2} + m^{-1/2}n^{-1} + (m/n)^{2+\epsilon}mn^{-3/2} + mn^{-5/2}) \\
&\quad + O_p((n/m)(\sqrt{m}/n)(n^{-1/2}m^{-1/2} + mn^{-3/2}) - m^{-1/2}n^{-1} - mn^{-2}) \\
&= g_x^2 + (n/m)^2(a_1^2/4)(\Delta_1+\Delta_2)^2 + 2(m/n)^2(g_x g_x''/6) - 2(n/m)g_x(a_1/2)(\Delta_1+\Delta_2) \\
&\quad - 2(a_2/(2p))g_x(\Delta_1+\Delta_2+2\Delta_3) + O_p(n^{-1/2}m^{-1/2} + mn^{-3/2} + (m/n)^{2+\epsilon}) \\
&= g_x^2 + A_{ho} + O_p(n^{-1/2}m^{-1/2} + mn^{-3/2} + (m/n)^{2+\epsilon}),
\end{aligned}
$$

where the higher-order terms $A_{ho} = O_p(m^{-1/2}), A_{ho}^2 = O_p(m^{-1})$, but $A_{ho}^3 = O_p(m^{-3/2})$ which will be in the remainder. Thus, we can cut off our expansion of the inverse square root with the cube term: $(1+x)^{-1/2} = 1 - x/2 + (3/8)x^2 + O(x^3)$, since again the numerator of Z is $O_p(1)$.

Which terms in A_{ho}^2 will be kept (i.e., not end up in the remainder)? Since the biggest term in A_{ho} is $O_p(m^{-1/2})$, any given term within A_{ho} will not appear in A_{ho}^2 if the term would be in $m^{1/2}R$:

$$
\begin{aligned}
(n/m)^2(a_1^2/4)(\Delta_1+\Delta_2)^2 &= O_p(m^{-1}), \text{ in } m^{1/2}R \\
2(m/n)^2(g_x g_x''/6) &= O_p(m^2/n^2) = m^{1/2}O_p(mn^{-3/2}\sqrt{m/n}) < m^{1/2}R \\
-2(n/m)(a_1/2)g_x(\Delta_1+\Delta_2) &= O_p(m^{-1/2}) \implies \text{keep in square term} \\
-2(a_2/(2p))g_x(\Delta_1+\Delta_2+2\Delta_3) &= O_p(n^{-1/2}) = m^{1/2}R.
\end{aligned}
$$

Simplifying some again,

$$
\begin{aligned}
A^2 &= g_x^2 + A_{ho} + R \\
&= g_x^2 + 2g_x\left\{(m/n)^2(g_x''/6) - (n/m)(a_1/2)(\Delta_1+\Delta_2) - [a_2/(2p)](\Delta_1+\Delta_2+2\Delta_3)\right\} \\
&\quad + (n/m)^2(a_1^2/4)(\Delta_1+\Delta_2)^2 + R,
\end{aligned}
$$

and symmetrically,

$$B^2 = g_y^2 + 2g_y \left\{(m/n)^2(g_y''/6) - (n/m)(a_1'/2)(\nabla_1 + \nabla_2) - [a_2'/(2p)](\nabla_1 + \nabla_2 + 2\nabla_3)\right\} + (n/m)^2((a_1')^2/4)(\nabla_1 + \nabla_2)^2 + R.$$

Summing,

$$\begin{aligned} A^2 + B^2 = {} & (g_x^2 + g_y^2) \\ & \times \Bigg[1 + 2\frac{g_x}{g_x^2 + g_y^2}\Big((m/n)^2(g_x''/6) - (n/m)(a_1/2)(\Delta_1 + \Delta_2) - (a_2/(2p))(\Delta_1 + \Delta_2 + 2\Delta_3)\Big) \\ & \quad + (n/m)^2(a_1^2/4)(\Delta_1 + \Delta_2)^2/(g_x^2 + g_y^2) \\ & \quad + 2\frac{g_y}{g_x^2 + g_y^2}\Big((m/n)^2(g_y''/6) - (n/m)(a_1'/2)(\nabla_1 + \nabla_2) - (a_2'/(2p))(\nabla_1 + \nabla_2 + 2\nabla_3)\Big) \\ & \quad + (n/m)^2((a_1')^2/4)(\nabla_1 + \nabla_2)^2/(g_x^2 + g_y^2)\Bigg] \\ \equiv {} & (g_x^2 + g_y^2)[1 + \tilde{\nu}], \end{aligned}$$

where $\tilde{\nu}$ is implicitly defined.

Note that $S_{mn} = \sqrt{A^2 + B^2} = \sqrt{(g_x^2 + g_y^2)(1 + \tilde{\nu})} = \sqrt{g_x^2 + g_y^2}\sqrt{1 + \tilde{\nu}}$, and $S_0 \equiv \sqrt{g_x^2 + g_y^2}$, so

$$S_{mn}/S_0 = \sqrt{1 + \tilde{\nu}}.$$

Now,

$$(1 + x)^{1/2} = 1^{1/2} + x((1/2)1^{-1/2}) + (1/2)x^2(-(1/4)1^{-3/2}) + O(x^3) = 1 + x/2 - x^2/8 + O(x^3),$$

so

$$(S_{mn}/S_0) - 1 = \sqrt{1 + \tilde{\nu}} - 1 = \tilde{\nu}/2 - \tilde{\nu}^2/8 + O(\tilde{\nu}^3).$$

Again, $\tilde{\nu}^3 = O_p(m^{-3/2})$ is in the remainder R, and the only term not in R in $\tilde{\nu}^2$ is the $O_p(m^{-1})$ square terms and X-Y cross term:

$$\begin{aligned} \tilde{\nu}^2 = {} & 4\frac{g_x^2}{S_0^4}(n/m)^2(a_1^2/4)(\Delta_1 + \Delta_2)^2 + 4\frac{g_y^2}{S_0^4}(n/m)^2((a_1')^2/4)(\nabla_1 + \nabla_2)^2 \\ & + 8\frac{g_x g_y}{S_0^4}(n/m)^2(a_1 a_1'/4)(\Delta_1 + \Delta_2)(\nabla_1 + \nabla_2) + R, \end{aligned}$$

so

$$\begin{aligned} \frac{S_{mn}}{S_0} - 1 = {} & \tilde{\nu}/2 - \tilde{\nu}^2/8 + O(\tilde{\nu}^3) \\ = {} & (1/2)(n/m)^2(1/S_0)(1/4)[a_1^2(\Delta_1 + \Delta_2)^2 + (a_1')^2(\nabla_1 + \nabla_2)^2] \\ & + (1/S_0^2)\Bigg[(m/n)^2(1/6)(g_x g_x'' + g_y g_y'') - (n/m)(1/2)(a_1(\Delta_1 + \Delta_2)g_x + a_1'(\nabla_1 + \nabla_2)g_y) \\ & \qquad - (1/(2p))(g_x a_2(\Delta_1 + \Delta_2 + 2\Delta_3) + g_y a_2'(\nabla_1 + \nabla_2 + 2\nabla_3))\Bigg] \\ & - S_0^{-4}(1/8)(n^2/m^2) \\ & \times \left[g_x^2 a_1^2(\Delta_1 + \Delta_2)^2 + g_y^2(a_1')^2(\nabla_1 + \nabla_2)^2 + 2g_x g_y a_1 a_1'(\Delta_1 + \Delta_2)(\nabla_1 + \nabla_2)\right] + R. \end{aligned}$$

As opposed to $\sqrt{1+\tilde{\nu}} = 1+\tilde{\nu}/2-\tilde{\nu}^2/8+O(\tilde{\nu}^3)$, recall $(1+\tilde{\nu})^{-1/2} = 1-\tilde{\nu}/2+(3/8)\tilde{\nu}^2-O(\tilde{\nu}^3)$, so

$$S_{mn}^{-1} = (A^2+B^2)^{-1/2} = [(g_x^2+g_y^2)(1+\tilde{\nu})]^{-1/2} \tag{5}$$

$$= \frac{1}{S_0}(1+\tilde{\nu})^{-1/2} = \frac{1}{S_0}(1-\tilde{\nu}/2+(3/8)\tilde{\nu}^2)+R. \tag{6}$$

B.4. **Combining Numerator and Denominator of Z.** Recall

$$\begin{aligned}
\text{Numerator} &= \sqrt{n}[a_1(\Delta_2+\Delta_3)] - a_1'(\nabla_2+\nabla_3) + (1/2)(\Delta_2+\Delta_3)^2 a_2 - (1/2)(\nabla_2+\nabla_3)^2 a_2'] \\
&\quad + \gamma[(\tilde{\nu}/2)-(\tilde{\nu}^2/8)] + O_p(n^{-1}) \\
&= -pn^{1/2}\left[\frac{-a_1}{p}(\Delta_2+\Delta_3) - \frac{-a_1'}{p}(\nabla_2+\nabla_3) - \frac{a_2}{2p}(\Delta_2+\Delta_3)^2 + \frac{a_2'}{2p}(\nabla_2+\nabla_3)^2\right] \\
&\quad + (-pn^{1/2})\frac{\gamma}{-pn^{1/2}}[(\tilde{\nu}/2)-(\tilde{\nu}^2/8)] + O_p(n^{-1}) \\
&\equiv -pn^{1/2}\left(\bar{\Theta} - (\gamma/(pn^{1/2}))[(\tilde{\nu}/2)-(\tilde{\nu}^2/8)]\right) + O_p(n^{-1}),
\end{aligned}$$

where $\bar{\Theta}$ is implicitly defined.

Thus

$$\begin{aligned}
Z &= \left[\sqrt{n}(X_{nr}-\eta_{px}) - \sqrt{n}(Y_{nr}-\eta_{py}) + \gamma((S_{mn}/S_0)-1)\right] \\
&\quad \times \left[(n/(2m))^2(X_{n,r+m}-X_{n,r-m})^2 + (n/(2m))^2(Y_{n,r+m}-Y_{n,r-m})^2\right]^{-1/2} \\
&= -pn^{1/2}\left[\bar{\Theta} - \frac{\gamma}{pn^{1/2}}\left(\frac{\tilde{\nu}}{2}-\frac{\tilde{\nu}^2}{8}\right)\right]\frac{1}{S_0}\left(1-\frac{\tilde{\nu}}{2}+\frac{3}{8}\tilde{\nu}^2\right) + R \\
&= -pn^{1/2}\left[\Theta - \Psi\left(\frac{\tilde{\nu}}{2}-\frac{\tilde{\nu}^2}{8}\right)\right]\left(1-\frac{\tilde{\nu}}{2}+\frac{3}{8}\tilde{\nu}^2\right) + R \\
&= -pn^{1/2}\left[\Theta - \Theta\frac{\tilde{\nu}}{2} + \frac{3}{8}\Theta\tilde{\nu}^2 - \Psi\frac{\tilde{\nu}}{2} + \Psi\frac{\tilde{\nu}^2}{8} + \Psi\frac{\tilde{\nu}^2}{4}\right] + O_p(\tilde{\nu}^3) + R \\
&= -pn^{1/2}\left[\Theta - \frac{\tilde{\nu}}{2}(\Theta+\Psi) + \frac{3}{8}\tilde{\nu}^2(\Theta+\Psi)\right] + R,
\end{aligned}$$

where

$$\begin{aligned}
\Theta &\equiv \bar{\Theta}/S_0 \\
&= \left(\frac{-a_1}{pS_0}\right)(\Delta_2+\Delta_3) - \left(\frac{-a_1'}{pS_0}\right)(\nabla_2+\nabla_3) \\
&\quad + \frac{a_2}{2a_1}\left(\frac{-a_1}{pS_0}\right)(\Delta_2+\Delta_3)^2 - \frac{a_2'}{2a_1'}\left(\frac{-a_1'}{pS_0}\right)(\nabla_2+\nabla_3)^2, \\
\Psi &\equiv \frac{\gamma}{pn^{1/2}S_0}.
\end{aligned}$$

Calculations now must be done to get $\Theta\tilde{\nu}$, $\Psi\tilde{\nu}$, $\Theta\tilde{\nu}^2$, etc. In the univariate case, we had $-a_1/(pg(p)) = 1+O(n^{-1})$, but now $S_0 \neq g_x$, so instead we have

$$-a_1/(pS_0) = -(a_1/(pg_x))(g_x/S_0) = (1+O(n^{-1}))(g_x/S_0) = g_x/S_0 + O(n^{-1}).$$

To recap:

$$
\begin{aligned}
Z &= -pn^{1/2}[\Theta - (\tilde{\nu}/2)(\Theta+\Psi) + (3/8)\tilde{\nu}^2(\Theta+\Psi)] + R,\\
\Theta &\equiv (-a_1/(pS_0))(\Delta_2+\Delta_3) - (-a_1'/(pS_0))(\nabla_2+\nabla_3)\\
&\quad + (a_2/(2a_1))(-a_1/(pS_0))(\Delta_2+\Delta_3)^2 - (a_2'/(2a_1'))(-a_1'/(pS_0))(\nabla_2+\nabla_3)^2\\
&= [(g_x/S_0)(\Delta_2+\Delta_3) - (g_y/S_0)(\nabla_2+\nabla_3)]\\
&\quad + [(a_2/(2a_1))(g_x/S_0)(\Delta_2+\Delta_3)^2 - (a_2'/(2a_1'))(g_y/S_0)(\nabla_2+\nabla_3)^2] + O_p(n^{-3/2})\\
&\equiv [\Theta_1] + [\Theta_2] + O_p(n^{-3/2}),\\
\tilde{\nu}/2 &= (1/2)(n/m)^2 S_0^{-2}(1/4)[a_1^2(\Delta_1+\Delta_2)^2 + (a_1')^2(\nabla_1+\nabla_2)^2]\\
&\quad + S_0^{-2}[(m/n)^2(1/6)(g_x g_x'' + g_y g_y'') - (n/m)(1/2)(a_1 g_x(\Delta_1+\Delta_2) + a_1' g_y(\nabla_1+\nabla_2))\\
&\qquad - (1/(2p))(g_x a_2(\Delta_1+\Delta_2+2\Delta_3) + g_y a_2'(\nabla_1+\nabla_2+2\nabla_3))]\\
&= S_0^{-2}\Big\{(1/8)(n/m)^2[a_1^2(\Delta_1+\Delta_2)^2 + (a_1')^2(\nabla_1+\nabla_2)^2] + (m/n)^2(1/6)(g_x g_x'' + g_y g_y'')\\
&\qquad - (n/m)(1/2)[a_1 g_x(\Delta_1+\Delta_2) + a_1' g_y(\nabla_1+\nabla_2)]\\
&\qquad - (1/(2p))[g_x a_2(\Delta_1+\Delta_2+2\Delta_3) + g_y a_2'(\nabla_1+\nabla_2+2\nabla_3)]\Big\}\\
&\equiv S_0^{-2}\{\tilde{\nu}_1 + \tilde{\nu}_2 - \tilde{\nu}_3 - \tilde{\nu}_4\},\\
(3/8)\tilde{\nu}^2 &= (3/8)S_0^{-4}(n/m)^2[g_x^2 a_1^2(\Delta_1+\Delta_2)^2 + g_y^2(a_1')^2(\nabla_1+\nabla_2)^2 + 2g_x g_y a_1 a_1'(\Delta_1+\Delta_2)(\nabla_1+\nabla_2)].
\end{aligned}
$$

Note the orders of terms appearing above:

$$
\begin{aligned}
\tilde{\nu}^2 &= O_p(m^{-1})\\
\tilde{\nu}_1 &= O_p(m^{-1})\\
\tilde{\nu}_2 &= O_p(m^2/n^2)\\
\tilde{\nu}_3 &= O_p(m^{-1/2})\\
\tilde{\nu}_4 &= O_p(n^{-1/2})\\
\Theta_1 &= O_p(n^{-1/2})\\
\Theta_2 &= O_p(n^{-1})\\
\Psi &= O_p(n^{-1/2}).
\end{aligned}
$$

Our terms in Z are then

$$
\begin{aligned}
(\tilde{\nu}/2)(\Theta+\Psi) &= S_0^{-2}(\tilde{\nu}_1+\tilde{\nu}_2-\tilde{\nu}_3-\tilde{\nu}_4)(\Theta_1+\Theta_2+\Psi),\ \text{keep if } > n^{-1/2}R,\\
&= S_0^{-2}[(\tilde{\nu}_1(\Theta_1+\Psi) + \tilde{\nu}_2(\Theta_1+\Psi) - \tilde{\nu}_3(\Theta_1+\Psi) - \tilde{\nu}_4(\Theta_1+\Psi)]\\
&\quad + n^{-1/2}O_p(m^{-1}n^{-1/2} + (m^2/n^2)n^{-1/2} + m^{-1/2}n^{-1/2} + n^{-1}),\\
(3/8)\tilde{\nu}^2(\Theta+\Psi) &= (\text{again, } \Theta_2 \to \text{remainder}) = (3/8)\tilde{\nu}^2(\Theta_1+\Psi).
\end{aligned}
$$

Adding together,

$$
\begin{aligned}
&\Theta - \frac{\tilde{\nu}}{2}(\Theta+\Psi) + \frac{3}{8}\tilde{\nu}^2(\Theta+\Psi)\\
&= \frac{g_x}{S_0}(\Delta_2+\Delta_3) - \frac{g_y}{S_0}(\nabla_2+\nabla_3) + \frac{a_2}{2a_1}\frac{g_x}{S_0}(\Delta_2+\Delta_3)^2 - \frac{a_2'}{2a_1'}\frac{g_y}{S_0}(\nabla_2+\nabla_3)^2\\
&\quad - S_0^{-2}\Big\{\frac{1}{8}(n/m)^2[a_1^2(\Delta_1+\Delta_2)^2 + (a_1')^2(\nabla_1+\nabla_2)^2][\frac{g_x}{S_0}(\Delta_2+\Delta_3) - \frac{g_y}{S_0}(\nabla_2+\nabla_3) + \Psi]
\end{aligned}
$$

$$+ (m/n)^2\frac{1}{6}(g_xg_x'' + g_yg_y'')[(g_x/S_0)(\Delta_2 + \Delta_3) - (g_y/S_0)(\nabla_2 + \nabla_3) + \Psi]$$
$$- (1/2)(n/m)[a_1g_x(\Delta_1 + \Delta_2) + a_1'g_y(\nabla_1 + \nabla_2)]$$
$$\times [(g_x/S_0)(\Delta_2 + \Delta_3) - (g_y/S_0)(\nabla_2 + \nabla_3) + \Psi]$$
$$- (1/(2p))[g_xa_2(\Delta_1 + \Delta_2 + 2\Delta_3) + g_ya_2'(\nabla_1 + \nabla_2 + 2\nabla_3)]$$
$$\times [(g_x/S_0)(\Delta_2 + \Delta_3) - (g_y/S_0)(\nabla_2 + \nabla_3) + \Psi]\Big\}$$
$$+ (3/8)S_0^{-4}(n/m)^2[g_x^2a_1^2(\Delta_1 + \Delta_2)^2 + g_y^2(a_1')^2(\nabla_1 + \nabla_2)^2 + 2g_xg_ya_1a_1'(\Delta_1 + \Delta_2)(\nabla_1 + \nabla_2)]$$
$$\times [(g_x/S_0)(\Delta_2 + \Delta_3) - (g_y/S_0)(\nabla_2 + \nabla_3) + \Psi]$$

$$= \frac{g_x}{S_0}(\Delta_2 + \Delta_3) - \frac{g_y}{S_0}(\nabla_2 + \nabla_3) + \frac{a_2}{2a_1}\frac{g_x}{S_0}(\Delta_2 + \Delta_3)^2 - \frac{a_2'}{2a_1'}\frac{g_y}{S_0}(\nabla_2 + \nabla_3)^2$$
$$- S_0^{-2}\Big\{(1/8)(n/m)^2$$
$$\times \Big[a_1^2(g_x/S_0)(\Delta_1 + \Delta_2)^2(\Delta_2 + \Delta_3) - a_1^2(g_y/S_0)(\Delta_1 + \Delta_2)^2(\nabla_2 + \nabla_3) + a_1^2(\Delta_1 + \Delta_2)^2\Psi$$
$$+ (a_1')^2(g_x/S_0)(\nabla_1 + \nabla_2)^2(\Delta_2 + \Delta_3) - (a_1')^2(g_y/S_0)(\nabla_1 + \nabla_2)^2(\nabla_2 + \nabla_3)$$
$$+ (a_1')^2(\nabla_1 + \nabla_2)^2\Psi\Big]$$
$$+ (m/n)^2(1/6)(g_xg_x'' + g_yg_y'')(g_x/S_0)(\Delta_2 + \Delta_3)$$
$$- (m/n)^2(1/6)(g_xg_x'' + g_yg_y'')(g_y/S_0)(\nabla_2 + \nabla_3)$$
$$+ (m/n)^2(1/6)(g_xg_x'' + g_yg_y'')\Psi$$
$$- (1/2)(n/m)\Big[(a_1g_x^2/S_0)(\Delta_1 + \Delta_2)(\Delta_2 + \Delta_3) - (a_1g_xg_y/S_0)(\Delta_1 + \Delta_2)(\nabla_2 + \nabla_3)$$
$$+ a_1g_x(\Delta_1 + \Delta_2)\Psi + (a_1'g_xg_y/S_0)(\nabla_1 + \nabla_2)(\Delta_2 + \Delta_3)$$
$$- (a_1'g_y^2/S_0)(\nabla_1 + \nabla_2)(\nabla_2 + \nabla_3) + a_1'g_y(\nabla_1 + \nabla_2)\Psi\Big]$$
$$- (1/(2p))\Big[(a_2g_x^2/S_0)(\Delta_1 + \Delta_2 + 2\Delta_3)(\Delta_2 + \Delta_3) - (a_2g_xg_y/S_0)(\Delta_1 + \Delta_2 + 2\Delta_3)(\nabla_2 + \nabla_3)$$
$$+ a_2g_x(\Delta_1 + \Delta_2 + 2\Delta_3)\Psi + (a_2'g_xg_y/S_0)(\nabla_1 + \nabla_2 + 2\nabla_3)(\Delta_2 + \Delta_3)$$
$$- (a_2'g_y^2/S_0)(\nabla_1 + \nabla_2 + 2\nabla_3)(\nabla_2 + \nabla_3) + a_2'g_y(\nabla_1 + \nabla_2 + 2\nabla_3)\Psi\Big]\Big\}$$
$$+ (3/8)S_0^{-4}(n/m)^2$$
$$\times \Big[(a_1^2g_x^3/S_0)(\Delta_1 + \Delta_2)^2(\Delta_2 + \Delta_3) - (a_1^2g_x^2g_y/S_0)(\Delta_1 + \Delta_2)^2(\nabla_2 + \nabla_3) + a_1^2g_x^2(\Delta_1 + \Delta_2)^2\Psi$$
$$+ ((a_1')^2g_xg_y^2/S_0)(\nabla_1 + \nabla_2)^2(\Delta_2 + \Delta_3) - ((a_1')^2g_y^3/S_0)(\nabla_1 + \nabla_2)^2(\nabla_2 + \nabla_3)$$
$$+ (a_1')^2g_y^2(\nabla_1 + \nabla_2)^2\Psi$$
$$+ (2a_1a_1'g_x^2g_y/S_0)(\Delta_1 + \Delta_2)(\nabla_1 + \nabla_2)(\Delta_2 + \Delta_3)$$
$$- 2a_1a_1'g_xg_y^2S_0^{-1}(\Delta_1 + \Delta_2)(\nabla_1 + \nabla_2)(\nabla_2 + \nabla_3)$$
$$+ 2a_1a_1'g_xg_y(\Delta_1 + \Delta_2)(\nabla_1 + \nabla_2)\Psi\Big]$$

$$= (g_x/S_0)(\Delta_2 + \Delta_3) - (g_y/S_0)(\nabla_2 + \nabla_3)$$
$$+ (a_2/(2a_1))(g_x/S_0)(\Delta_3^2 + 2\Delta_2\Delta_3) - (a_2'/(2a_1'))(g_y/S_0)(\nabla_3^2 + 2\nabla_2\nabla_3)$$
$$- S_0^{-2}\Big\{(1/8)(n/m)^2$$

$$
\begin{aligned}
&\quad\times \Big[a_1^2 g_x S_0^{-1}(\Delta_1+\Delta_2)^2\Delta_3 - a_1^2 g_y S_0^{-1}(\Delta_1+\Delta_2)^2\nabla_3 + a_1^2(\Delta_1+\Delta_2)^2\Psi \\
&\qquad + (a_1')^2 g_x S_0^{-1}(\nabla_1+\nabla_2)^2\Delta_3 - (a_1')^2 g_y S_0^{-1}(\nabla_1+\nabla_2)^2\nabla_3 + (a_1')^2(\nabla_1+\nabla_2)^2\Psi\Big] \\
&+ (m/n)^2(1/6)(g_x g_x'' + g_y g_y'')[(g_x/S_0)\Delta_3 - (g_y/S_0)\nabla_3 + \Psi] \\
&- (1/2)(n/m)\Big[\,\cdot\,\Big] \\
&- (1/(2p)) \\
&\quad\times \Big[a_2 g_x^2 S_0^{-1}(2\Delta_3^2 + \Delta_1\Delta_3 + 3\Delta_2\Delta_3) - a_2 g_x g_y S_0^{-1}(2\Delta_3\nabla_3 + \Delta_1\nabla_3 + \Delta_2\nabla_3 + 2\Delta_3\nabla_2) \\
&\qquad + a_2 g_x(\Delta_1+\Delta_2+2\Delta_3)\Psi + a_2' g_x g_y S_0^{-1}((\nabla_1+\nabla_2)\Delta_3 + 2\Delta_3\nabla_3 + 2\Delta_2\nabla_3) \\
&\qquad - a_2' g_y^2 S_0^{-1}(2\nabla_3^2 + \nabla_1\nabla_3 + 3\nabla_2\nabla_3) + a_2' g_y(\nabla_1+\nabla_2+2\nabla_3)\Psi\Big]\Big\} \\
&+ (3/8)S_0^{-4}(n/m)^2 \\
&\quad\times \Big[a_1^2 g_x^3 S_0^{-1}(\Delta_1+\Delta_2)^2\Delta_3 - a_1^2 g_x^2 g_y S_0^{-1}(\Delta_1+\Delta_2)^2\nabla_3 + a_1^2 g_x^2(\Delta_1+\Delta_2)^2\Psi \\
&\qquad + (a_1')^2 g_x g_y^2 S_0^{-1}(\nabla_1+\nabla_2)^2\Delta_3 - (a_1')^2 g_y^3 S_0^{-1}(\nabla_1+\nabla_2)^2\nabla_3 + (a_1')^2 g_y^2(\nabla_1+\nabla_2)^2\Psi \\
&\qquad + 2a_1 a_1' g_x^2 g_y S_0^{-1}(\Delta_1+\Delta_2)(\nabla_1+\nabla_2)\Delta_3 - 2a_1 a_1' g_x g_y^2 S_0^{-1}(\Delta_1+\Delta_2)(\nabla_1+\nabla_2)\nabla_3 \\
&\qquad + 2a_1 a_1' g_x g_y(\Delta_1+\Delta_2)(\nabla_1+\nabla_2)\Psi\Big] \\
&+ n^{-1/2}R.
\end{aligned}
$$

Note that while HS88 used the form $(1+\delta)(\Delta_2+\Delta_3)$ for the "first-order" term in Y, I will only include the actual first-order terms (δ is higher-order) for clarity. Organizing the terms by order,

$$
\begin{aligned}
Z &= Y + R, \\
Y &= -pn^{1/2}[A+B], \\
A &\equiv (g_x/S_0)(\Delta_2+\Delta_3) - (g_y/S_0)(\nabla_2+\nabla_3), \\
B &\equiv B_1 + B_2 + B_3 + B_4 + B_5,
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 \equiv\ & (a_2/(2a_1))(g_x/S_0)\Delta_3^2 - (a_2'/(2a_1'))(g_y/S_0)\nabla_3^2 \\
& + S_0^{-2}(1/2)(n/m) \\
& \quad\times \Big[a_1 g_x^2 S_0^{-1}(\Delta_1+\Delta_2)\Delta_2 - a_1 g_x g_y S_0^{-1}(\Delta_1+\Delta_2)\nabla_2 + a_1' g_x g_y S_0^{-1}(\nabla_1+\nabla_2)\Delta_2 \\
& \qquad - a_1' g_y^2 S_0^{-1}(\nabla_1+\nabla_2)\nabla_2\Big] \\
& + (2pS_0^2)^{-1}\Big[a_2 g_x^2 S_0^{-1} 2\Delta_3^2 - a_2 g_x g_y S_0^{-1} 2\Delta_3\nabla_3 + 2a_2 g_x\Delta_3\Psi \\
& \qquad + a_2' g_x g_y S_0^{-1} 2\Delta_3\nabla_3 - a_2' g_y^2 S_0^{-1} 2\nabla_3^2 + 2a_2' g_y\nabla_3\Psi\Big], \\
B_2 \equiv\ & (a_2/(2a_1))(g_x/S_0)2\Delta_2\Delta_3 - (a_2'/(2a_1'))(g_y/S_0)(2\nabla_2\nabla_3) \\
& + (2pS_0^2)^{-1} \\
& \quad\times \Big[a_2 g_x^2 S_0^{-1}(\Delta_1\Delta_3 + 3\Delta_2\Delta_3) - a_2 g_x g_y S_0^{-1}(\Delta_1\nabla_3 + \Delta_2\nabla_3 + 2\Delta_3\nabla_2) + a_2 g_x(\Delta_1+\Delta_2)\Psi \\
& \qquad + a_2' g_x g_y S_0^{-1}(2\Delta_2\nabla_3 + \Delta_3(\nabla_1+\nabla_2)) - a_2' g_y^2 S_0^{-1}(\nabla_1\nabla_3 + 3\nabla_2\nabla_3) + a_2' g_y(\nabla_1+\nabla_2)\Psi\Big], \\
B_3 \equiv\ & -(1/8)S_0^{-2}(n/m)^2 \\
& \quad\times \Big[a_1^2 g_x S_0^{-1}(\Delta_1+\Delta_2)^2\Delta_3 - a_1^2 g_y S_0^{-1}(\Delta_1+\Delta_2)^2\nabla_3 + a_1^2(\Delta_1+\Delta_2)^2\Psi
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (a_1')^2 g_x S_0^{-1}(\nabla_1+\nabla_2)^2\Delta_3 - (a_1')g_y S_0^{-1}(\nabla_1+\nabla_2)^2\nabla_3 + (a_1')^2(\nabla_1+\nabla_2)^2\Psi\Big] \\
&+ (1/8)S_0^{-2}(n/m)^2 \\
&\quad \times \Big[3S_0^{-2}a_1^2 g_x^3 S_0^{-1}(\Delta_1+\Delta_2)^2\Delta_3 - 3S_0^{-2}a_1^2 g_x^2 g_y S_0^{-1}(\Delta_1+\Delta_2)^2\nabla_3 + 3a_1^2 g_x^2 S_0^{-2}(\Delta_1+\Delta_2)^2\Psi \\
&\quad + 3S_0^{-2}(a_1')^2 g_x g_y^2 S_0^{-1}(\nabla_1+\nabla_2)^2\Delta_3 - 3S_0^{-2}(a_1')^2 g_y^3 S_0^{-1}(\nabla_1+\nabla_2)^2\nabla_3 \\
&\quad + 3S_0^{-2}(a_1')^2 g_y^2(\nabla_1+\nabla_2)^2\Psi + 3S_0^{-2}2a_1a_1' g_x^2 g_y S_0^{-1}(\Delta_1+\Delta_2)(\nabla_1+\nabla_2)\Delta_3 \\
&\quad - 3S_0^{-2}2a_1a_1' g_x g_y^2 S_0^{-1}(\Delta_1+\Delta_2)(\nabla_1+\nabla_2)\nabla_3 + 3S_0^{-2}2a_1a_1' g_x g_y(\Delta_1+\Delta_2)(\nabla_1+\nabla_2)\Psi\Big], \\
B_4 &\equiv S_0^{-2}(1/2)(n/m)\Big[a_1 g_x^2 S_0^{-1}(\Delta_1+\Delta_2)\Delta_3 - a_1 g_x g_y S_0^{-1}(\Delta_1+\Delta_2)\nabla_3 + a_1 g_x(\Delta_1+\Delta_2)\Psi \\
&\qquad + a_1' g_x g_y S_0^{-1}(\nabla_1+\nabla_2)\Delta_3 - a_1' g_y^2 S_0^{-1}(\nabla_1+\nabla_2)\nabla_3 + a_1' g_y(\nabla_1+\nabla_2)\Psi\Big], \\
B_5 &\equiv -(1/6)S_0^{-2}(m/n)^2(g_x g_x'' + g_y g_y'')(g_x S_0^{-1}\Delta_3 - g_y S_0^{-1}\nabla_3 + \Psi),
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= O_p(n^{-1}), \\
B_2 &= O_p(m^{1/2}n^{-3/2}), \\
B_3 &= O_p(m^{-1}n^{-1/2}), \\
B_4 &= O_p(m^{-1/2}n^{-1/2}), \\
B_5 &= O_p(m^2 n^{-5/2}).
\end{aligned}
$$

Here, clearly $B_1 = o_p(B_4)$, and it turns out that B_4 is the largest of all, which can be shown if B_1 in turn is larger than the other three terms. For this, recall that from HS88, $O(mn^{-3/2})$ was shown to be $o(m^{-1} + (m/n)^2)$, which implies that $O(\sqrt{n}/m) = o(1)$ and $O(m^2 n^{-3/2}) = o(1)$. Thus,

$$
\begin{aligned}
B_2 &= O_p\left(\frac{\sqrt{m}}{\sqrt{n}}n^{-1}\right) = o(1)O_p(B_1), \\
B_3 &= O_p\left(\frac{\sqrt{n}}{m}n^{-1}\right) = o(1)O_p(B_1), \\
B_5 &= O_p(m^2 n^{-3/2}n^{-1}) = o(1)O_p(B_1),
\end{aligned}
$$

so in terms of order, $B_4 > B_1 > B_2, B_3, B_5$.

B.5. **Calculate moments of Y.** As before, the next step is calculating

$$
\begin{aligned}
E[(-p^{-1}Y)^\ell] &= E[n^{\ell/2}(A+B)^\ell] = E[n^{\ell/2}(A^\ell + \ell A^{\ell-1}B + (\ell(\ell-1)/2)A^{\ell-2}B^2 + O(A^{\ell-3}B^3))] \\
&= E[n^{\ell/2}A^\ell] + E[\ell n^{\ell/2}A^{\ell-1}B] + E[(\ell(\ell-1)/2)n^{\ell/2}A^{\ell-2}B^2] + O(n^{\ell/2}E(A^{\ell-3}B^3)) \\
&\equiv z_1(\ell) + z_2(\ell) + z_3(\ell) + o(m^{-1} + (m/n)^2),
\end{aligned}
$$

keeping the same z_i notation as in HS88.

B.6. **$z_1(\ell)$, L and its characteristic function, and the inverse Fourier–Stieltjes transform thereof.** Looking ahead, all the operations the rest of the proof are linear, so this term will remain additively separable. In the next step, it will become the univariate L term. The cumulant generating function of L is then expanded to approximate the characteristic function of L, and then the inverse Fourier–Stieltjes transform is taken. To get the proper higher-order terms (and check the remainder), four moments of L need to be calculated.

Similar to the univariate case, and using notation $\text{ᗡ}_i = \sqrt{n}\nabla_i$, define

$$\begin{aligned} L &\equiv -[p/(1-p)]^{1/2} n^{1/2} A \\ &= -[p/(1-p)]^{1/2} n^{1/2} [(g_x/S_0)(\Delta_2+\Delta_3) - (g_y/S_0)(\nabla_2+\nabla_3)] \\ &= -[p/(1-p)]^{1/2} [(g_x/S_0)(D_2+D_3) - (g_y/S_0)(\text{ᗡ}_2+\text{ᗡ}_3)], \end{aligned}$$

so that (below) the characteristic function of L is the first additive part of the characteristic function of K (defined below).

First moment,

$$E(L) = -[p/(1-p)]^{1/2} E[(g_x/S_0)(D_2+D_3) - (g_y/S_0)(\text{ᗡ}_2+\text{ᗡ}_3)] = 0$$

since the D_i and ᗡ_i are all mean zero.

Second moment,

$$\begin{aligned} E(L^2) &= \{-[p/(1-p)]^{1/2}\}^2 E[((g_x/S_0)(D_2+D_3) - (g_y/S_0)(\text{ᗡ}_2+\text{ᗡ}_3))^2] \\ &= \frac{p}{1-p} nE[(g_x^2/S_0^2)(\Delta_2+\Delta_3)^2 + (g_y^2/S_0^2)(\nabla_2+\nabla_3)^2 - 2(g_x g_y/S_0^2)(\Delta_2+\Delta_3)(\nabla_2+\nabla_3)] \\ &= \frac{p}{1-p} n[(g_x^2/S_0^2)((1-p)/(np) + O(n^{-2})) + (g_y^2/S_0^2)((1-p)/(np) + O(n^{-2})) - 0] \\ &= (p/(1-p))[(1-p)/p]\frac{g_x^2+g_y^2}{S_0^2} + O(n^{-1}) \\ &= 1 + O(n^{-1}), \end{aligned}$$

using the result from the univariate case that $E[(\Delta_2+\Delta_3)^2] = (1-p)/(np) + O(n^{-2})$, the fact that $E(\Delta_i\nabla_j) = 0$ for any i,j due to independence and having (individual) means of zero, the fact that the ᗡ_i have the same moments as D_i, and the definition $S_0 \equiv \sqrt{g_x^2+g_y^2}$.

Third moment, using the result from the univariate proof that $E[(D_2+D_3)^3] = n^{-1/2}(1-p)(1+p)/p^2 + O(n^{-3/2})$, and again the properties of independence and mean zero and equivalent moments of D_i and ᗡ_i,

$$\begin{aligned} E(L^3) &= \{-[p/(1-p)]^{1/2}\}^3 E[((g_x/S_0)(D_2+D_3) - (g_y/S_0)(\text{ᗡ}_2+\text{ᗡ}_3))^3] \\ &= -[p/(1-p)]^{3/2}\{(g_x^3/S_0^3)E[(D_2+D_3)^3] - 3(g_x^2 g_y/S_0^3)E[(D_2+D_3)^2(\text{ᗡ}_2+\text{ᗡ}_3)] \\ &\qquad + 3(g_x g_y^2/S_0^3)E[(D_2+D_3)(\text{ᗡ}_2+\text{ᗡ}_3)^2] - (g_y^3/S_0^3)E[(\text{ᗡ}_2+\text{ᗡ}_3)^3]\} \\ &= -[p/(1-p)]^{3/2}[(g_x^3/S_0^3)(n^{-1/2}(1-p)(1+p)/p^2 + O(n^{-3/2})) - 0 + 0 \\ &\qquad - (g_y^3/S_0^3)(n^{-1/2}(1-p)(1+p)/p^2 + O(n^{-3/2}))] \\ &= -[p/(1-p)]^{3/2}\frac{g_x^3-g_y^3}{S_0^3} n^{-1/2}\frac{(1-p)(1+p)}{p^2} + O(n^{-3/2}) \\ &= -n^{-1/2}\frac{1+p}{\sqrt{p(1-p)}}\frac{g_x^3-g_y^3}{S_0^3} + O(n^{-3/2}). \end{aligned}$$

Fourth moment, using prior result that $E[(D_2+D_3)^4] = 3(1-p)^2p^{-2} + O(n^{-1})$,

$$\begin{aligned} E(L^4) &= \{-[p/(1-p)]^{1/2}\}^4 E[((g_x/S_0)(D_2+D_3) - (g_y/S_0)(\text{ᗡ}_2+\text{ᗡ}_3))^4] \\ &= p^2(1-p)^{-2}\{(g_x^4/S_0^4)E[(D_2+D_3)^4] - 4(g_x^3 g_y/S_0^4)E[(D_2+D_3)^3(\text{ᗡ}_2+\text{ᗡ}_3)] \\ &\qquad + 6(g_x^2 g_y^2/S_0^4)E[(D_2+D_3)^2(\text{ᗡ}_2+\text{ᗡ}_3)^2] - 4(g_x g_y^3/S_0^4)E[(D_2+D_3)(\text{ᗡ}_2+\text{ᗡ}_3)^3] \\ &\qquad + (g_y^4/S_0^4)E[(\text{ᗡ}_2+\text{ᗡ}_3)^4]\} \\ &= p^2(1-p)^{-2}\Big\{\frac{g_x^4+g_y^4}{S_0^4}[3(1-p)^2p^{-2} + O(n^{-1})] - 4(0+0) \\ &\qquad + 6\frac{g_x^2 g_y^2}{S_0^4}[(1-p)/p + O(n^{-1})][(1-p)/p + O_p(n^{-1})]\Big\} \end{aligned}$$

$$= p^2(1-p)^{-2}3(1-p)^2p^{-2}\left[\frac{g_x^4+g_y^4}{S_0^4}+\frac{2g_x^2g_y^2}{S_0^4}\right]+O(n^{-1})$$
$$= 3+O(n^{-1}).$$

Writing κ_i for the ith cumulant and μ_i' for the ith moment (and using their relationships), in a manner similar to the univariate case,

$$\begin{aligned}
\ln E(e^{itL}) &= \sum_{j=1}^{\infty}\frac{(it)^j}{j!}\kappa_j \\
&= (it)\kappa_1-(t^2/2)\kappa_2+\frac{(it)^3}{6}\kappa_3+O(\kappa_4) \\
&= (it)\mu_1'-(t^2/2)(\mu_2'-(\mu_1')^2)+\frac{(it)^3}{6}\kappa_3+O(\kappa_4) \\
&= 0-(t^2/2)(1+O(n^{-1})-0)+\frac{(it)^3}{6}\kappa_3+O(\kappa_4),
\end{aligned}$$

so

$$\begin{aligned}
E(e^{itL}) &= e^{-t^2/2}\exp\left\{\frac{(it)^3}{6}\kappa_3+O(\kappa_4)\right\} \\
&= e^{-t^2/2}\Bigg[1+\frac{(it)^3}{6}(\mu_3'-3\mu_2'\mu_1'+2(\mu_1')^3) \\
&\qquad +O(\mu_4'-4\mu_3'\mu_1'-3(\mu_2')^2+12(\mu_2')(\mu_1')^2-6(\mu_1')^4)+O(n^{-1})\Bigg] \\
&= e^{-t^2/2}\Bigg[1+\frac{(it)^3}{6}\left(-n^{-1/2}\frac{1+p}{\sqrt{p(1-p)}}\frac{g_x^3-g_y^3}{S_0^3}+O(n^{-3/2})\right) \\
&\qquad +O(3+O(n^{-1})-3(1+O(n^{-1}))^2)+O(n^{-1})+O((n^{-1/2})^2)\Bigg] \\
&= e^{-t^2/2}\left[1-n^{-1/2}\frac{1}{6}\frac{1+p}{\sqrt{p(1-p)}}\left(\frac{g_x^3-g_y^3}{S_0^3}\right)(it)^3\right]+O(n^{-1}).
\end{aligned}$$

The RHS is the Fourier–Stieltjes transform of

$$\Phi(z)+n^{-1/2}\frac{1}{6}\frac{1+p}{\sqrt{p(1-p)}}\left(\frac{g_x^3-g_y^3}{S_0^3}\right)(z^2-1)\phi(z)+O(n^{-1}), \tag{7}$$

which will be part of the final, higher-order distribution of the test statistic. Note that the first-order term is $\Phi(z)$, as it should be. To this, $z_2(\ell)$ and $z_3(\ell)$ will add other higher-order terms.

B.7. **Remainder terms for $E[(-p^{-1}Y)^\ell]$.** For the remainder, since the biggest term in B is $B_4 = O_p(m^{-1/2}n^{-1/2})$ (as shown above), note that

$$\begin{aligned}
n^{\ell/2}E[A^{\ell-3}B^3] &= O(n^{\ell/2}(n^{-1/2})^{\ell-3}(m^{-1/2}n^{-1/2})^3) \\
&= O(n^{3/2}m^{-3/2}n^{-3/2}) = O(m^{-3/2}) = o(m^{-1}).
\end{aligned}$$

B.8. $z_3(\ell)$**.** From above, $z_3(\ell) = (1/2)\ell(\ell-1)n^{\ell/2}E[A^{\ell-2}B^2]$. The biggest term in B^2 is $B_4^2 = O_p(m^{-1}n^{-1})$. If only the Δ_3 and ∇_3 from A are kept, then $n^{\ell/2}A^{\ell-2}B^2 = O(nO_p(m^{-1}n^{-1})) = O_p(m^{-1})$, which should be kept (i.e., not in the remainder).

However, if there is even one $\Delta_2\Delta_3^{\ell-3}$ term in the A part, then

$$\begin{aligned} n^{\ell/2}A^{\ell-2}B^2 &= O_p(n^{\ell/2}n^{-(\ell-3)/2}m^{1/2}n^{-1}B^2) \\ &= O_p(n^{3/2}n^{-1}m^{1/2}m^{-1}n^{-1}) \\ &= O_p(m^{-1/2}n^{-1/2}) = o_p(m^{-1}), \end{aligned}$$

which will end up in the remainder. Thus, from the A part, we will only keep the Δ_3 and ∇_3.

The next-biggest term in B^2 would be $B_4B_1 = O_p(m^{-1/2}n^{-1/2}n^{-1}) = O_p(m^{-1/2}n^{-3/2})$. If only using the Δ_3 and ∇_3 from A, then $E[n^{\ell/2}A^{\ell-2}B_4B_1] = O(nm^{-1/2}n^{-3/2}) = O(m^{-1/2}n^{-1/2}) = o(m^{-1})$, into the remainder. Thus, all smaller terms in B^2 go into the remainder also.

To recap: first, only keep the Δ_3 and ∇_3 from A; second, keep only the B_4^2 from B^2.

Now, defining

$$\bar{\Delta}_i \equiv (g_x/S_0)\Delta_i, \tag{8}$$

$$\bar{\nabla}_i \equiv (g_y/S_0)\nabla_i, \tag{9}$$

$$\bar{D}_3 \equiv (g_x/S_0)D_3 - (g_y/S_0)\reflectbox{D}_3, \tag{10}$$

we have

$$\begin{aligned} z_3(\ell) &= (1/2)\ell(\ell-1)n^{\ell/2}E[((g_x/S_0)\Delta_3 - (g_y/S_0)\nabla_3)^{\ell-2}B_4^2] + o(m^{-1} + (m/n)^2) \\ &= (1/2)\ell(\ell-1)nE[\bar{D}_3^{\ell-2}B_4^2] + R' \\ &= (1/2)\ell(\ell-1)n \\ &\quad \times E\Big\{\bar{D}_3^{\ell-2}[(1/4)(n^2/m^2)S_0^{-4}] \\ &\quad\quad \times \Big[a_1^2g_x^4S_0^{-2}(\Delta_1^2+\Delta_2^2)\Delta_3^2 + a_1^2g_x^2g_y^2S_0^{-2}(\Delta_1^2+\Delta_2^2)\nabla_3^2 + a_1^2g_x^2(\Delta_1^2+\Delta_2^2)\Psi^2 \\ &\quad\quad\quad + (a_1')^2g_x^2g_y^2S_0^{-2}(\nabla_1^2+\nabla_2^2)\Delta_3^2 + (a_1)^2g_y^4S_0^{-2}(\nabla_1^2+\nabla_2^2)\nabla_3^2 + (a_1')^2g_y^2(\nabla_1^2+\nabla_2^2)\Psi^2 \\ &\quad\quad\quad - 2a_1^2g_x^3g_yS_0^{-2}(\Delta_1^2+\Delta_2^2)\Delta_3\nabla_3 + 2a_1^2g_x^3S_0^{-1}(\Delta_1^2+\Delta_2^2)\Delta_3\Psi + 0 - 0 + 0 \\ &\quad\quad\quad - 2a_1^2g_x^2g_yS_0^{-1}(\Delta_1^1+\Delta_2^2)\nabla_3\Psi - 0 + 0 - 0 + 0 - 0 + 0 \\ &\quad\quad\quad - 2(a_1')^2g_xg_y^3S_0^{-2}(\nabla_1^2+\nabla_2^2)\Delta_3\nabla_3 + 2(a_1')^2g_xg_y^2S_0^{-1}(\nabla_1^2+\nabla_2^2)\Delta_3\Psi \\ &\quad\quad\quad - 2(a_1')^2g_y^3S_0^{-1}(\nabla_1^2+\nabla_2^2)\nabla_3\Psi\Big]\Big\} + R' \\ &= \left[\frac{1}{2}\ell(\ell-1)n\frac{1}{4}\frac{n^2}{m^2}S_0^{-4}\right] \\ &\quad \times E\Big\{\bar{D}_3^{\ell-2}\Big[\Psi^2(2mn^{-2}p^{-2} + O(m^2n^{-3}))(a_1^2g_x^2 + (a_1')^2g_y^2) \\ &\quad\quad + \Delta_3^2(2mn^{-2}p^{-2} + O(m^2n^{-3}))g_x^2S_0^{-2}(a_1^2g_x^2 + (a_1')^2g_y^2) \\ &\quad\quad + \nabla_3^2(2mn^{-2}p^{-2} + O(m^2n^{-3}))g_y^2S_0^{-2}(a_1^2g_x^2 + (a_1')^2g_y^2) \\ &\quad\quad + 2\Delta_3\Psi(2mn^{-2}p^{-2} + O(m^2n^{-3}))g_xS_0^{-1}(a_1^2g_x^2 + (a_1')^2g_y^2) \\ &\quad\quad - 2\nabla_3\Psi(2mn^{-2}p^{-2} + O(m^2n^{-3}))g_yS_0^{-1}(a_1^2g_x^2 + (a_1')^2g_y^2) \\ &\quad\quad - 2\Delta_3\nabla_3(2mn^{-2}p^{-2} + O(m^2n^{-3}))g_xg_yS_0^{-2}(a_1^2g_x^2 + (a_1')^2g_y^2)\Big]\Big\} + R' \\ &= \left[\frac{1}{8}\ell(\ell-1)S_0^{-4}\frac{n^2}{m^2}n\right][a_1^2g_x^2 + (a_1')^2g_y^2]\frac{2m}{n^2p^2} \\ &\quad \times E\Big\{\bar{D}_3^{\ell-2}\Big[\Psi^2 + g_x^2S_0^{-2}\Delta_3^2 + g_y^2S_0^{-2}\nabla_3^2 + 2g_xS_0^{-1}\Delta_3\Psi - 2g_yS_0^{-1}\nabla_3\Psi - 2g_xg_yS_0^{-2}\Delta_3\nabla_3\Big]\Big\} + R' \end{aligned}$$

$$= \frac{1}{4} m^{-1}\ell(\ell-1)\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4}$$
$$\times E\Big\{\bar{D}_3^{\ell-2}\Big[\Psi^2 n + 2\Psi\sqrt{n}\bar{D}_3 + \bar{D}_3^2\Big]\Big\} + R'$$
$$= \frac{1}{4} m^{-1}\ell(\ell-1)\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4}\left[E(\bar{D}_3^{\ell-2})n\Psi^2 + 2\Psi\sqrt{n}E(\bar{D}_3^{\ell-1}) + E(\bar{D}_3^{\ell})\right] + R'.$$

B.9. $z_2(\ell)$**.** First, determine which terms will end up in the remainder and which need to be kept. If there are two or more Δ_2 or ∇_2 from the A term, it will end up in the remainder. Take B_1, the second-largest term:

$$n^{\ell/2}E(\Delta_2^2\Delta_3^{\ell-3}B_1) = O(n^{3/2}mn^{-2}D_3^{\ell-3}n^{-1}) = O(mn^{-3/2}) = o(m^{-1} + (m/n)^2).$$

Now we only need to check B_4. The terms in B_4 with a single Δ_1 or ∇_1 will zero out due to independence/mean zero. The other terms are the product of Δ_2 (or ∇_2) and some $O_p(n^{-1/2})$ term $(\Delta_3, \nabla_3, \Psi)$. If there is a single Δ_2 from B_4, there has to be at least one other Δ_2 from the A part, either Δ_2^2 or $\Delta_2\nabla_2$; but if the latter, then there is a single ∇_2, which will zero out the whole term due to independence/mean zero again. Thus in order to keep the single Δ_2 (or ∇_2) from B_4, there must be Δ_2^2 (or ∇_2^2) from the A part. This yields

$$n^{\ell/2}E(\Delta_2^2\Delta_3^{\ell-3}(\Delta_2 O_p(n^{-1/2}))) = O(nE(\Delta_2^3)) = O(nmn^{-3}) = o(n^{-1}),$$

which is clearly in the remainder, using the result from the univariate case that $E(\Delta_2^3) = O(mn^{-3})$. Thus, there will not be two (or more) Δ_2 coming from the A part, and thus

$$z_2(\ell) = \ell n^{\ell/2}E\left[((g_x/S_0)(\Delta_2+\Delta_3) - (g_y/S_0)(\nabla_2+\nabla_3))^{\ell-1}B\right]$$
$$= \ell n^{1/2}E(\bar{D}_3^{\ell-1}B) + \ell(\ell-1)nE\left[(\bar{\Delta}_2 - \bar{\nabla}_2)\bar{D}_3^{\ell-2}B\right] \equiv z_{2,1}(\ell) + z_{2,2}(\ell). \tag{11}$$

B.9.1. $z_{2,1}(\ell)$. Here, since there are only Δ_3 and ∇_3 from the A part, single $\Delta_1, \Delta_2, \nabla_1, \nabla_2$ will zero out. B_2 and B_4 disappear completely. For B_5,

$$\ell n^{1/2}E(\bar{D}_3^{\ell-1}B_5) = -(1/6)m^2n^{-2}S_0^{-2}\ell(g_x g_x'' + g_y g_y'')[E(\bar{D}_3^{\ell}) + \Psi\sqrt{n}E(\bar{D}_3^{\ell-1})].$$

B_3 is left with

$$(1/8)S_0^{-2}(n/m)^2$$
$$\times \Big[a_1^2 g_x S_0^{-1}(\Delta_1^2+\Delta_2^2)\Delta_3(3g_x^2S_0^{-2}-1) - a_1^2 g_y S_0^{-1}(\Delta_1^2+\Delta_2^2)\nabla_3(3g_x^2S_0^{-2}-1)$$
$$+ a_1^2(\Delta_1^2+\Delta_2^2)\Psi(3g_x^2S_0^{-2}-1) + (a_1')^2 g_x S_0^{-1}(\nabla_1^2+\nabla_2^2)\Delta_3(3g_y^2S_0^{-2}-1)$$
$$- (a_1')^2 g_y S_0^{-1}(\nabla_1^2+\nabla_2^2)\nabla_3(3g_y^2S_0^{-2}-1) + (a_1')^2(\nabla_1^2+\nabla_2^2)\Psi(3g_y^2S_0^{-2}-1)\Big]$$
$$+0+0+0$$
$$= (1/8)S_0^{-2}n^2m^{-2}(2mn^{-2}p^{-2} + O(m^2n^{-3}))$$
$$\times \Big[(3a_1^2g_x^2S_0^{-2} - a_1^2 + 3(a_1')^2g_y^2S_0^{-2} - (a_1')^2)(g_xS_0^{-1}\Delta_3 - g_yS_0^{-1}\nabla_3 + \Psi)\Big],$$

so

$$\ell n^{1/2}E(\bar{D}_3^{\ell-1}B_3) = (1/4)m^{-1}\ell S_0^{-2}p^{-2}E[\bar{D}_3^{\ell-1}(\bar{D}_3 + \Psi\sqrt{n})(3a_1^2g_x^2S_0^{-2} - a_1^2 + 3(a_1')^2g_y^2S_0^{-2} - (a_1')^2)]$$
$$= (1/4)m^{-1}\ell S_0^{-2}p^{-2}(3a_1^2g_x^2S_0^{-2} - a_1^2 + 3(a_1')^2g_y^2S_0^{-2} - (a_1')^2)$$
$$\times [E(\bar{D}_3^{\ell}) + \Psi\sqrt{n}E(\bar{D}_3^{\ell-1})]$$

B_1 is left with

$$
\begin{aligned}
&(a_2/(2a_1))g_x S_0^{-1}\Delta_3^2 - (a_2'/(2a_1'))g_y S_0^{-1}\nabla_3^2 \\
&\quad + (1/2)S_0^{-2}(n/m)[a_1 g_x^2 S_0^{-1}\Delta_2^2 - a_1' g_y^2 S_0^{-1}\nabla_2^2] \\
&\quad + p^{-1}S_0^{-2}[a_2 g_x^2 S_0^{-1}\Delta_3^2 - a_2 g_x g_y S_0^{-1}\Delta_3\nabla_3 + a_2 g_x \Delta_3 \Psi \\
&\qquad\qquad + a_2' g_x g_y S_0^{-1}\Delta_3\nabla_3 - a_2' g_y^2 S_0^{-1}\nabla_3^2 + a_2' g_y \nabla_3 \Psi] \\
&= (1/2)S_0^{-3}(n/m)[a_1 g_x^2 \Delta_2^2 - a_1' g_y^2 \nabla_2^2] + \Delta_3^2[(a_2/(2a_1))g_x S_0^{-1} + a_2 g_x^2 p^{-1} S_0^{-3}] \\
&\quad - \nabla_3^2[(a_2'/(2a_1'))g_y S_0^{-1} + a_2' g_y^2 p^{-1} S_0^{-3}] + \Delta_3\nabla_3[g_x g_y/(pS_0^3)](a_2' - a_2) \\
&\quad + \Psi p^{-1} S_0^{-2}(a_2 g_x \Delta_3 + a_2' g_y \nabla_3)
\end{aligned}
$$

so

$$
\begin{aligned}
\ell n^{1/2} E(\bar{D}_3^{\ell-1} B_1) &= \ell \Big\{ (1/2)S_0^{-3}(n/m)n^{1/2} \\
&\qquad \times (a_1 g_x^2(mn^{-2}p^{-2} + O(m^2 n^{-3})) - a_1' g_y^2(mn^{-2}p^{-2} + O(m^2 n^{-3})))E(\bar{D}_3^{\ell-1}) \\
&\quad + (g_x S_0^{-1}\Delta_3)^2[(a_2/(2a_1))S_0 g_x^{-1} + a_2/(pS_0)] \\
&\quad - (g_y S_0^{-1}\nabla_3)^2[(a_2'/(2a_1'))S_0 g_y^{-1} + a_2'/(pS_0)] \\
&\quad + \bar{\Delta}_3\bar{\nabla}_3(pS_0)^{-1}(a_2' - a_2) + \Psi(pS_0)^{-1}(a_2\bar{\Delta}_3 + a_2'\bar{\nabla}_3) \Big\} \\
&= \ell\Big[\Big\{ n^{-1/2}(2p^2 S_0)^{-1}[a_1 g_x^2 S_0^{-2} - a_1' g_y^2 S_0^{-2}]E(\bar{D}_3^{\ell-1}) \Big\} \\
&\quad + E\Big(\Big\{ (pS_0)^{-1}(\bar{\Delta}_3 - \bar{\nabla}_3)(a_2\bar{\Delta}_3 + a_2'\bar{\nabla}_3) + (pS_0)^{-1}\Psi(a_2\bar{\Delta}_3 + a_2'\bar{\nabla}_3) \\
&\qquad + \bar{\Delta}_3^2(a_2/(2a_1))S_0 g_x^{-1} - \bar{\nabla}_3^2(a_2'/(2a_1'))S_0 g_y^{-1} \Big\} \bar{D}_3^{\ell-1} n^{1/2} \Big) \Big] \\
&= n^{-1/2}\ell(pS_0)^{-1} \\
&\quad \times \Big\{ (2p)^{-1}(a_1 g_x^2 S_0^{-2} - a_1' g_y^2 S_0^{-2})E(\bar{D}_3^{\ell-1}) + \sqrt{n}E[\bar{D}_3^{\ell}(a_2\bar{\Delta}_3 + a_2'\bar{\nabla}_3)] \\
&\qquad + \Psi\sqrt{n}\sqrt{n}E[\bar{D}_3^{\ell-1}(a_2\bar{\Delta}_3 + a_2'\bar{\nabla}_3)] \\
&\qquad + pS_0 n E[\bar{D}_3^{\ell-1}(\bar{\Delta}_3^2(a_2/(2a_1))S_0 g_x^{-1} - \bar{\nabla}_3^2(a_2'/(2a_1'))S_0 g_y^{-1})] \Big\}.
\end{aligned}
$$

B.9.2. $z_{2,2}(\ell)$. Here, all but the B_4 term end up in the remainder:

$$
\begin{aligned}
nE[(\bar{\Delta}_2 - \bar{\nabla}_2)\bar{D}_3^{\ell-2}B_1] &= O(n^2 m^{-1} E[\bar{D}_3^{\ell-2}(\Delta_2^3 - \nabla_2^3)]) \\
&= O(n^2 m^{-1}(2mn^{-3}p^{-3} + O(m^2 n^{-4}))E(\bar{D}_3^{\ell-2})) \\
&= O(n^{-1}), \\
nE[(\bar{\Delta}_2 - \bar{\nabla}_2)\bar{D}_3^{\ell-2}B_2] &= O(nE\{\bar{D}_3^{\ell-2}[\Delta_2^2\Delta_3 + \nabla_2^2\nabla_3 - \Delta_2^2\nabla_3 + \nabla_2^2\Delta_3 + \Delta_2^2\Psi - \nabla_2^2\Psi]\}) \\
&= O(nmn^{-2}p^{-2}(1 + O(m/n))E\{\bar{D}_3^{\ell-2}[\Delta_3 + \Psi + \nabla_3]\}) \\
&= O(mn^{-3/2}) = o(m^{-1} + (m/n)^2), \\
nE[(\bar{\Delta}_2 - \bar{\nabla}_2)\bar{D}_3^{\ell-2}B_3] &= O(nm^{1/2}n^{-1}m^{-1}n^{-1/2}) = O(m^{-1/2}n^{-1/2}) = o(m^{-1}), \\
nE[(\bar{\Delta}_2 - \bar{\nabla}_2)\bar{D}_3^{\ell-2}B_5] &= O(nm^{1/2}n^{-1}m^2 n^{-5/2}) = O(m^2 n^{-2} m^{1/2} n^{-1/2}) = o((m/n)^2).
\end{aligned}
$$

The B_4 term becomes

$$\begin{aligned}
&\ell(\ell-1)nE[\bar{D}_3^{\ell-2}(\bar{\Delta}_2-\bar{\nabla}_2)B_4]\\
&\quad=\ell(\ell-1)n^2m^{-1}(2S_0^2)^{-1}\\
&\qquad\times E[\bar{D}_3^{\ell-2}(g_xS_0^{-1}\Delta_2^2a_1g_x\bar{\Delta}_3-g_xS_0^{-1}\Delta_2^2a_1g_x\bar{\nabla}_3+a_1g_x^2S_0^{-1}\Delta_2^2\Psi\\
&\qquad\qquad-g_yS_0^{-1}\nabla_2^2a_1'g_y\bar{\Delta}_3+g_yS_0^{-1}\nabla_2^2a_1'g_y\bar{\nabla}_3-a_1'g_y^2S_0^{-1}\nabla_2^2\Psi)]\\
&\quad=\ell(\ell-1)(2S_0^2)^{-1}n^2m^{-1}mn^{-2}p^{-2}\\
&\qquad\times E\{\bar{D}_3^{\ell-2}[\Psi(a_1g_x^2-a_1'g_y^2)S_0^{-1}+\bar{\Delta}_3S_0^{-1}(a_1g_x^2-a_1'g_y^2)\\
&\qquad\qquad-\bar{\nabla}_3S_0^{-1}(a_1g_x^2-a_1'g_y^2)]\}+O(mn^{-3/2})\\
&\quad=n^{-1/2}\ell(\ell-1)(2p^2S_0^2)^{-1}(a_1g_x^2-a_1'g_y^2)S_0^{-1}E[\bar{D}_3^{\ell-2}\sqrt{n}(\bar{\Delta}_3-\bar{\nabla}_3+\Psi)]+R'\\
&\quad=n^{-1/2}\ell(\ell-1)\frac{a_1g_x^2-a_1'g_y^2}{2p^2S_0^3}[E(\bar{D}_3^{\ell-1})+\Psi\sqrt{n}E(\bar{D}_3^{\ell-2})]+o(m^{-1}+(m/n)^2).
\end{aligned}$$

Altogether,

$$\begin{aligned}
z_2(\ell)&=n^{-1/2}\ell(pS_0)^{-1}\Big\{(2p)^{-1}(a_1g_x^2S_0^{-2}-a_1'g_y^2S_0^{-2})E(\bar{D}_3^{\ell-1})+\sqrt{n}E[\bar{D}_3^{\ell}(a_2\bar{\Delta}_3+a_2'\bar{\nabla}_3)]\\
&\qquad+\Psi\sqrt{n}\sqrt{n}E[\bar{D}_3^{\ell-1}(a_2\bar{\Delta}_3+a_2'\bar{\nabla}_3)]\\
&\qquad+pS_0nE[\bar{D}_3^{\ell-1}(\bar{\Delta}_3^2a_2(2a_1)^{-1}S_0g_x^{-1}-\bar{\nabla}_3^2a_2'(2a_1')^{-1}S_0g_y^{-1})]\Big\}\\
&\quad+(1/4)m^{-1}\ell(p^2S_0^2)^{-1}[3a_1^2g_x^2S_0^{-2}-a_1^2+3(a_1')^2g_y^2S_0^{-2}-(a_1')^2][E(\bar{D}_3^{\ell})+\Psi\sqrt{n}E(\bar{D}_3^{\ell-1})]\\
&\quad-(1/6)(m/n)^2\ell S_0^{-2}(g_xg_x''+g_yg_y'')[E(\bar{D}_3^{\ell})+\Psi\sqrt{n}E(\bar{D}_3^{\ell-1})]\\
&\quad+n^{-1/2}\ell(\ell-1)\frac{a_1g_x^2-a_1'g_y^2}{2p^2S_0^3}[E(\bar{D}_3^{\ell-1})+\Psi\sqrt{n}E(\bar{D}_3^{\ell-2})]\\
&\quad+o(m^{-1}+(m/n)^2),\\
z_3(\ell)&=\frac{1}{4}m^{-1}\ell(\ell-1)\frac{a_1^2g_x^2+(a_1')^2g_y^2}{p^2S_0^4}\left[E(\bar{D}_3^{\ell-2})n\Psi^2+2\Psi\sqrt{n}E(\bar{D}_3^{\ell-1})+E(\bar{D}_3^{\ell})\right]+o(m^{-1}+(m/n)^2).
\end{aligned}$$

B.10. **Moments of $\bar{D}_3$.** Recall that D_3 (from the univariate proof) is asymptotically $N(0,(1-p)/p)$, and thus $\reflectbox{D}_3$ is also, and that $D_3 \perp\!\!\!\perp \reflectbox{D}_3$. Thus, $(g_x/S_0)D_3 \to N(0,(g_x^2/S_0^2)(1-p)/p)$, and $(g_y/S_0)\reflectbox{D}_3 \to N(0,(g_y^2/S_0^2)(1-p)/p)$, and then

$$\bar{D}_3=(g_x/S_0)D_3-(g_y/S_0)\reflectbox{D}_3\to N(0,((g_x^2+g_y^2)/S_0^2)(1-p)/p),$$

so $\bar{D}_3$ has the same asymptotic distribution as D_3 from the univariate case, i.e. $N(0,(1-p)/p)$.

Since $D_3 \perp\!\!\!\perp \reflectbox{D}_3$, the moments converge at the same rate. Consider the moment generating functions (which exist in this case). Let the standard normal mgf be $M_N(t)$. The mgf of $\bar{D}_3=(g_x/S_0)D_3-(g_y/S_0)\reflectbox{D}_3$ is, due to independence, $M_{D_3}(tg_x/S_0)M_{\reflectbox{D}_3}(-tg_y/S_0)$. As a sufficient bound on the error, recall that the moments of D_3 have error no bigger than $O(n^{-1/2})$, so we can write the mgf for $\bar{D}_3$ as

$$\begin{aligned}
&[M_N(tg_xS_0^{-1}[(1-p)/p]^{1/2})+O(n^{-1/2})]\times[M_N(-tg_yS_0^{-1}[(1-p)/p]^{1/2})+O(n^{-1/2})]\\
&\quad=[M_N(tg_xS_0^{-1}[(1-p)/p]^{1/2})][M_N(-tg_yS_0^{-1}[(1-p)/p]^{1/2})]+O(n^{-1/2}),
\end{aligned}$$

so the moments will be those of a standard normal plus error $O(n^{-1/2})$, which will always end in the overall remainder since we are already dealing with higher-order terms here.

B.11. **Cross moments with $\bar{D}_3$ and $\bar{\Delta}_3$ and $\bar{\nabla}_3$.** This is needed for some of the $n^{-1/2}$ terms in $z_2(\ell)$. Isserlis' Theorem (Wick's Theorem) can be used when noting that $\bar{D}_3$ and $\bar{\Delta}_3$ are asymptotically bivariate normal, as are $\bar{D}_3$ and $\bar{\nabla}_3$, and thus

$$
\begin{aligned}
&\sqrt{n}\,E[\bar{D}_3^{\ell}(a_2\bar{\Delta}_3 + a_2'\bar{\nabla}_3)] + \Psi\sqrt{n}\sqrt{n}\,E[\bar{D}_3^{\ell-1}(a_2\bar{\Delta}_3 + a_2'\bar{\nabla}_3)] \\
&+pS_0 n E[\bar{D}_3^{\ell-1}(a_2(2a_1)^{-1}S_0 g_x^{-1}\bar{\Delta}_3^2 - a_2'(2a_1')^{-1}S_0 g_y^{-1}\bar{\nabla}_3^2)] \\
&(\text{if } \ell = 2k) \\
&= \Psi\sqrt{n}\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k}(a_2 g_x^2 S_0^{-2} - a_2' g_y^2 S_0^{-2}) + O(n^{-1}), \\
&(\text{if } \ell = 2k-1) \\
&= \left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k} \\
&\quad\times \Big[a_2 g_x^2 S_0^{-2} + a_2 g_x^2 S_0^{-2}(1/2)(pg_x/a_1)(2k-2)/(2k-1) \\
&\quad\quad + (a_2/2)(pg_x/a_1)/(2k-1) - a_2' g_y^2 S_0^{-2} \\
&\quad\quad - a_2' g_y^2 S_0^{-2}(1/2)(pg_y/a_1')(2k-2)/(2k-1) - (a_2'/2)(pg_y/a_1')/(2k-1)\Big] \\
&= \left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k} \\
&\quad\times \Big[a_2 g_x^2 S_0^{-2}[1-(k-1)/(2k-1)] - (a_2/2)/(2k-1) \\
&\quad\quad - a_2' g_y^2 S_0^{-2}[1-(k-1)/(2k-1)] + (a_2'/2)/(2k-1)\Big] \\
&= \left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k} \frac{kS_0^{-2}(a_2 g_x^2 - a_2' g_y^2) - (1/2)(a_2 - a_2')}{2k-1}.
\end{aligned}
$$

For $\bar{D}_3$ and $\sqrt{n}\,\bar{\Delta}_3$ to be bivariate normal, it is sufficient to show that all linear combinations of the two are (univariate) normal:

$$
\begin{aligned}
a(\sqrt{n}\,\bar{\Delta}_3 - \sqrt{n}\,\bar{\nabla}_3) + b\sqrt{n}\,\bar{\Delta}_3 &= (a+b)\sqrt{n}\,\bar{\Delta}_3 - a\sqrt{n}\,\bar{\nabla}_3, \\
(a+b)\sqrt{n}\,\bar{\Delta}_3 &\to_d N(0, (a+b)^2 g_x^2 S_0^{-2}(1-p)/p), \\
-a\sqrt{n}\,\bar{\nabla}_3 &\to_d N(0, a^2 g_y^2 S_0^{-2}(1-p)/p),
\end{aligned}
$$

and thus the sum is also asymptotically normal since $\bar{\Delta}_3 \perp\!\!\!\perp \bar{\nabla}_3$.

Isserlis' Theorem states that for a multivariate normal vector with elements X_i,

$$
\begin{aligned}
E(X_1 X_2 \ldots X_{2k-1}) &= 0, \\
E(X_1 X_2 \ldots X_{2k}) &= \Sigma\Pi E(X_i X_j),
\end{aligned}
$$

where $\Sigma\Pi$ is summing over all distinct partitions of $X_1 \ldots X_{2k}$ into pairs, such as the example partition $E(X_1X_2)E(X_3X_4)\ldots E(X_{2k-1}X_{2k})$.

First examine $\sqrt{n}\,E[\bar{D}_3^{\ell}(a_2\bar{\Delta}_3 + a_2'\bar{\nabla}_3)]$. If $\ell = 2k$, there will be $2k+1$ terms in the expectation (i.e., ℓ from $\bar{D}_3$ and one from Δ_3), and thus the expectation is zero asymptotically, or rather $O(n^{-1/2})$. If $\ell = 2k-1$ (odd), we want $E(\bar{D}_3\bar{D}_3 \cdots \bar{D}_3\sqrt{n}\,\bar{\Delta}_3)$, where there are $2k-1$ occurrences of $\bar{D}_3$ and one of $\sqrt{n}\,\bar{\Delta}_3$, and thus $2k$ altogether. Then there will be $(2k)![k!2^k]^{-1}$ total unique partitions, and the value of each will be the same since it will be the product of $k-1$ terms of $E(\bar{D}_3^2)$ and one term of $E(\bar{D}_3\sqrt{n}\,\bar{\Delta}_3)$, which is equal to $[(1-p)/p]^{k-1}\{[(1-p)/p](g_x^2/S_0^2)\} + O(n^{-1}) = (g_x^2/S_0^2)[(1-p)/p]^k + O(n^{-1})$. Multiplying the number of partitions by the value of each partition, we get the result that for $\ell = 2k-1$,

$$
\sqrt{n}\,E(\bar{D}_3^{\ell} a_2\bar{\Delta}_3) = a_2(g_x^2/S_0^2)\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k} + O(n^{-1}).
$$

For the $\bar{\nabla}_3$ part, $E(\bar{D} - 3\sqrt{n}\,\bar{\nabla}_3) = E(-(g_y^2/S_0^2)\reflectbox{D}_3^2) = -(g_y^2/S_0^2)[(1-p)/p] + O(n^{-1})$, and thus the result is that for $\ell = 2k-1$,

$$\sqrt{n}\,E(\bar{D}_3^\ell a_2' \bar{\nabla}_3) = -a_2'(g_y^2/S_0^2)\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k} + O(n^{-1}),$$

so

$$\sqrt{n}\,E[\bar{D}_3^\ell(a_2\bar{\Delta}_3 + a_2'\bar{\nabla}_3)] = [a_2(g_x^2/S_0^2) - a_2'(g_y^2/S_0^2)]\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k} + O(n^{-1})$$

Second, for $\Psi\sqrt{n}\sqrt{n}\,E[\bar{D}_3^{\ell-1}(a_2\bar{\Delta}_3 + a_2'\bar{\nabla}_3)]$, the same reasoning will apply, except that for $\ell = 2k-1$ it will be zero (or, $O[n^{-1/2}]$) while for $\ell = 2k$ we get

$$\Psi\sqrt{n}\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k}\left(a_2 g_x^2 S_0^{-2} - a_2' g_y^2 S_0^{-2}\right) + O(n^{-1}).$$

Third, for $pS_0 nE[\bar{D}_3^{\ell-1}(a_2(2a_1)^{-1}S_0 g_x^{-1}\bar{\Delta}_3^2 - a_2'(2a_1')^{-1}S_0 g_y^{-1}\bar{\nabla}_3^2)]$, for $\ell = 2k$ it will be zero (or, $O[n^{-1/2}]$). For $\ell = 2k-1$, there are now two different classes of partitions we must consider: those with one $E(n\bar{\Delta}_3^2)$ term, and those with two $E(\bar{D}_3\sqrt{n}\,\bar{\Delta}_3^2)$ terms. When there is the lone $\bar{\Delta}_3^2$, the number of unique partitions is just the number of partitions you can make from the other $2k-2$ variables (all $\bar{D}_3$), which is $(2k-3)!! = (2k-2)![(k-1)!2^{k-1}]^{-1}$. Thus these partitions contribute

$$\frac{g_x^2}{S_0^2}\left(\frac{1-p}{p}\right)^k \frac{(2k-2)!}{(k-1)!2^{k-1}} = \frac{g_x^2}{S_0^2}\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k}\frac{1}{2k-1}$$

in total. The number of partitions left is then just the total minus the number used, or

$$\frac{(2k)!}{k!2^k} - \frac{(2k-2)!}{(k-1)!2^{k-1}} = \frac{(2k)!}{k!2^k}\frac{2k-2}{2k-1}.$$

The value is

$$\begin{aligned}\underbrace{E(\bar{D}_3^2)\ldots E(\bar{D}_3^2)}_{k-2\text{ terms}} E(\bar{D}_3\sqrt{n}\,\bar{\Delta}_3)E(\bar{D}_3\sqrt{n}\,\bar{\Delta}_3) &= \left[\left(\frac{1-p}{p}\right)^{k-2} + O(n^{-1})\right]\left[E\left(g_x^2 S_0^{-2} D_3^2\right)\right]^2 \\ &= \frac{g_x^4}{S_0^4}\left(\frac{1-p}{p}\right)^k + O(n^{-1}).\end{aligned}$$

So if $\ell = 2k-1$, then

$$\begin{aligned}E(\bar{D}_3^{\ell-1}\bar{\Delta}_3^2) &= \frac{g_x^2}{S_0^2}\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k}\frac{1}{2k-1} + \frac{g_x^4}{S_0^4}\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k}\frac{2k-2}{2k-1} \\ &= \frac{g_x^2}{S_0^2}\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k}\left[\frac{1}{2k-1} + \frac{g_x^2}{S_0^2}\frac{2k-2}{2k-1}\right],\end{aligned}$$

and thus

$$\begin{aligned}pS_0 nE\left[\bar{D}_3^{\ell-1}\left(\frac{a_2}{2a_1}\frac{S_0}{g_x}\bar{\Delta}_3^2\right)\right] &= pS_0\frac{a_2}{2a_1}\frac{S_0}{g_x}\frac{g_x^2}{S_0^2}\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k}\left(\frac{(g_x^2/S_0^2)(2k-2)+1}{2k-1}\right) \\ &= \frac{a_2}{2a_1}pg_x\left(\frac{(g_x^2/S_0^2)(2k-2)+1}{2k-1}\right)\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k} + O(n^{-1}).\end{aligned}$$

For the terms with $\bar{\nabla}_3^2$, $E(\bar{D}_3^{\ell-1}(-\bar{\nabla}_3^2)) = -E(\bar{D}_3^{\ell-1}\bar{\nabla}_3^2)$, and terms with $E(\bar{\nabla}_3^2)$ are $g_y^2 S_0^{-2}(1-p)/p + O(n^{-1})$, and

$$\begin{aligned}E(\bar{D}_3\sqrt{n}\,\bar{\nabla}_3) &= E\left(-\frac{g_y}{S_0}\reflectbox{D}_3\frac{g_y}{S_0}\reflectbox{D}_3\right) \\ &= -g_y^2 S_0^{-2}E(\reflectbox{D}_3^2) = -g_y^2 S_0^{-2}(1-p)/p + O(n^{-1}),\end{aligned}$$

and with two such terms,

$$\frac{g_y^4}{S_0^4}\left(\frac{1-p}{p}\right)^2 + O(n^{-1})$$

altogether. This is essentially the same as for $\bar{\Delta}_3^2$ but negative and with g_y and a_i' instead of g_x and a_i, so

$$pS_0 nE\left[\bar{D}_3^{\ell-1}\left(-\frac{a_2'}{2a_1'}\frac{S_0}{g_y}\bar{\nabla}_3^2\right)\right] = -\frac{a_2'}{2a_1'}pg_y\left(\frac{(g_y^2/S_0^2)(2k-2)+1}{2k-1}\right)\left(\frac{1-p}{p}\right)^k\frac{(2k)!}{k!2^k} + O(n^{-1}),$$

leading to the result at the beginning of this subsection.

B.12. **Characteristic function of K.** Define

$$K \equiv [p(1-p)]^{-1/2}Y. \tag{12}$$

From above,

$$E[(-p^{-1}Y)^\ell] = z_1(\ell) + z_2(\ell) + z_3(\ell) + R',$$

so

$$\begin{aligned} E(K^\ell)(it)^\ell/\ell! &= E[(\{p(1-p)\}^{-1/2}Y)^\ell](it)^\ell/\ell! = [-\{p/(1-p)\}^{1/2}]^\ell E[(-p^{-1}Y)^\ell](it)^\ell/\ell! \\ &= E(L^\ell)(it)^\ell/\ell! \\ &\quad + (-\{p/(1-p)\}^{1/2})^\ell[(it)^\ell/\ell!][z_2(\ell) + z_3(\ell) + R']. \end{aligned}$$

The characteristic function of K is the sum from 1 to ∞ of the LHS. The characteristic function of L has been approximated earlier in this proof. The remainder of the proof will be working out the infinite sum of the higher-order terms on the RHS, and then taking the inverse Fourier–Stieltjes transform.

There are some results from the univariate proof that will be helpful here:

$$\sum_{\ell=1}^{\infty}\frac{(it)^\ell}{\ell!}(-1)^\ell[p/(1-p)]^{\ell/2}\ell(\ell+1)E(D_3^\ell) = e^{-t^2/2}[(it)^4 + 3(it)^2] \tag{13}$$

$$\sum_{\ell=1}^{\infty}\frac{(it)^\ell}{\ell!}(-1)^\ell[p/(1-p)]^{\ell/2}\ell^2E(D_3^{\ell-1}) = e^{-t^2/2}[-\sqrt{p/(1-p)}][(it)^3 + (it)] \tag{14}$$

$$\sum_{\ell=1}^{\infty}\frac{(it)^\ell}{\ell!}(-1)^\ell[p/(1-p)]^{\ell/2}\ell(\ell-1)E(D_3^{\ell-2}) = e^{-t^2/2}[p/(1-p)][(it)^2] \tag{15}$$

$$\sum_{\ell=1}^{\infty}\frac{(it)^\ell}{\ell!}(-1)^\ell[p/(1-p)]^{\ell/2}\ell E(D_3^\ell) = e^{-t^2/2}[(it)^2] \tag{16}$$

$$\sum_{\ell=1}^{\infty}\frac{(it)^\ell}{\ell!}(-1)^\ell[p/(1-p)]^{\ell/2}\ell(\ell+1)E(D_3^{\ell-1}) = e^{-t^2/2}[-\sqrt{p/(1-p)}][(it)^3 + 2(it)] \tag{17}$$

$$\sum_{\ell=1}^{\infty}\frac{(it)^\ell}{\ell!}(-1)^\ell[p/(1-p)]^{\ell/2}\ell E(D_3^{\ell-1}) = e^{-t^2/2}[-\sqrt{p/(1-p)}][(it)]. \tag{18}$$

Additionally,

$$\begin{aligned}
&\sum_{k=1}^{\infty}(2k-1)(-1)^{2k-1}[p/(1-p)]^{(2k-1)/2}[(1-p)/p]^k\frac{(2k)!}{k!2^k}\frac{k}{2k-1}\frac{(it)^{2k-1}}{(2k-1)!}\\
&\quad=\sum_{k=1}^{\infty}(-1)\frac{k}{k!2^k}\frac{2k-1}{2k-1}\frac{(2k)!}{(2k-1)!}(it)^{2k-1}[(1-p)/p]^{1/2}\\
&\quad=[(1-p)/p]^{1/2}\sum_{k=1}^{\infty}(01)\frac{1}{(k-1)!2^k}(2k)(it)^{2k-1}\\
&\quad=(-1)\sqrt{(1-p)/p}\sum_{k=1}^{\infty}\frac{(it)^{2k-1}}{(k-1)!2^{k-1}}k\\
&\quad=(-1)\sqrt{(1-p)/p}\sum_{k=0}^{\infty}(it)^{2k+1}\left[\frac{k}{k!2^{k-1}2}+\frac{1}{k!2^k}\right]\\
&\quad=(-1)\sqrt{(1-p)/p}\left\{\sum_{k=1}^{\infty}\frac{(it)^3}{2}\frac{(it)^{2k-2}}{(k-1)!2^{k-1}}+\sum_{k=0}^{\infty}(it)\frac{(it)^{2k}}{k!2^k}\right\}\\
&\quad=(-1)\sqrt{(1-p)/p}\frac{(it)^3}{2}e^{-t^2/2}-\sqrt{(1-p)/p}(it)e^{-t^2/2},\\
&\sum_{k=1}^{\infty}(2k-1)(-1)[p/(1-p)]^{(2k-1)/2}[(1-p)/p]^k\frac{(2k)!}{k!2^k}\frac{1}{2k-1}\frac{(it)^{2k-1}}{(2k-1)!}\\
&\quad=\sum_{k=1}^{\infty}(-1)\sqrt{(1-p)/p}\frac{2k-1}{2k-1}\frac{(2k)!}{k!2^k}\frac{1}{(2k-1)!}(it)^{2k-1}\\
&\quad=\sum_{k=1}^{\infty}(-1)\sqrt{(1-p)/p}\frac{2k}{k!2^k}(it)^{2k-1}\\
&\quad=-\sqrt{(1-p)/p}\sum_{k=1}^{\infty}(it)\frac{(it)^{2k-2}}{(k-1)!2^{k-1}}\\
&\quad=-\sqrt{(1-p)/p}(it)e^{-t^2/2}.
\end{aligned}$$

Recall the earlier result that

$$\begin{aligned}
z_2(\ell)=&\,n^{-1/2}\ell(pS_0)^{-1}\Big\{(2p)^{-1}(a_1g_x^2S_0^{-2}-a_1'g_y^2S_0^{-2})E(\bar{D}_3^{\ell-1})+\sqrt{n}E[\bar{D}_3^{\ell}(a_2\bar{\Delta}_3+a_2'\bar{\nabla}_3)]\\
&+\Psi\sqrt{n}\sqrt{n}E[\bar{D}_3^{\ell-1}(a_2\bar{\Delta}_3+a_2'\bar{\nabla}_3)]\\
&+pS_0nE[\bar{D}_3^{\ell-1}(\bar{\Delta}_3^2a_2(2a_1)^{-1}S_0g_x^{-1}-\bar{\nabla}_3^2a_2'(2a_1')^{-1}S_0g_y^{-1})]\Big\}\\
&+(1/4)m^{-1}\ell(p^2S_0^2)^{-1}[3a_1^2g_x^2S_0^{-2}-a_1^2+3(a_1')^2g_y^2S_0^{-2}-(a_1')^2][E(\bar{D}_3^{\ell})+\Psi\sqrt{n}E(\bar{D}_3^{\ell-1})]\\
&-(1/6)(m/n)^2\ell S_0^{-2}(g_xg_x''+g_yg_y'')[E(\bar{D}_3^{\ell})+\Psi\sqrt{n}E(\bar{D}_3^{\ell-1})]\\
&+n^{-1/2}\ell(\ell-1)\frac{a_1g_x^2-a_1'g_y^2}{2p^2S_0^3}[E(\bar{D}_3^{\ell-1})+\Psi\sqrt{n}E(\bar{D}_3^{\ell-2})]\\
&+o(m^{-1}+(m/n)^2),\\
z_3(\ell)=&\,\frac{1}{4}m^{-1}\ell(\ell-1)\frac{a_1^2g_x^2+(a_1')^2g_y^2}{p^2S_0^4}\left[E(\bar{D}_3^{\ell-2})n\Psi^2+2\Psi\sqrt{n}E(\bar{D}_3^{\ell-1})+E(\bar{D}_3^{\ell})\right]+o(m^{-1}+(m/n)^2).
\end{aligned}$$

Then we want

$$(-1)^{\ell}[p/(1-p)]^{\ell/2}[(it)^{\ell}/\ell!][z_2(\ell)+z_3(\ell)+R']$$

$$
\begin{aligned}
&= n^{-1/2}(pS_0)^{-1}(2p)^{-1}(a_1 g_x^2 S_0^{-2} - a_1' g_y^2 S_0^{-2}) e^{-t^2/2}[-\sqrt{p/(1-p)}][(it)] \\
&\quad + n^{-1/2}(pS_0)^{-1}\Big[\Psi\sqrt{n}(a_2 g_x^2 S_0^{-2} - a_2' g_y^2 S_0^{-2})(it)^2 e^{-t^2/2} \\
&\qquad\qquad + S_0^{-2}(a_2 g_x^2 - a_2' g_y^2)[-(it)^3/2 - (it)]\sqrt{(1-p)/p}\, e^{-t^2/2} \\
&\qquad\qquad - (1/2)(a_2 - a_2')[-(it)]\sqrt{(1-p)/p}\, e^{-t^2/2}\Big] \\
&\quad + (1/4)m^{-1}(p^2 S_0^2)^{-1}[3a_1^2 g_x^2 S_0^{-2} - a_1^2 + 3(a_1')^2 g_y^2 S_0^{-2} - (a_1')^2] \\
&\qquad \times [(it)^2 e^{-t^2/2} + \Psi\sqrt{n}(it)(-1)\sqrt{p/(1-p)}\, e^{-t^2/2}] \\
&\quad - (1/6)(m/n)^2 S_0^{-2}(g_x g_x'' + g_y g_y'')[(it)^2 e^{-t^2/2} + \Psi\sqrt{n}(it)(-1)\sqrt{p/(1-p)}\, e^{-t^2/2}] \\
&\quad + n^{-1/2}\frac{a_1 g_x^2 - a_1' g_y^2}{2p^2 S_0^3}\Big[e^{-t^2/2}[(it)^3 + (it)](-1)\sqrt{p/(1-p)} - e^{-t^2/2}[-\sqrt{p/(1-p)}][(it)] \\
&\qquad\qquad + \Psi\sqrt{n}[p/(1-p)](it)^2 e^{-t^2/2}\Big] \\
&\quad + \frac{1}{4}m^{-1}\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4}\Big[\Psi^2 n[p/(1-p)](it)^2 e^{-t^2/2} \\
&\qquad\qquad + 2\Psi\sqrt{n}\, e^{-t^2/2}(-1)\sqrt{p/(1-p)}[(it)^3 + (it) - (it)] \\
&\qquad\qquad + e^{-t^2/2}[(it)^4 + 3(it)^2 - 2(it)^2]\Big] \\
&\quad + o(m^{-1} + (m/n)^2) \\
&= n^{-1/2}e^{-t^2/2}(pS_0)^{-1}\Big[\Psi\sqrt{n}(a_2 g_x^2 S_0^{-2} - a_2' g_y^2 S_0^{-2})(it)^2 \\
&\qquad\qquad - S_0^{-2}(a_2 g_x^2 - a_2' g_y^2)[(it)^3/2 + (it)]\sqrt{(1-p)/p} \\
&\qquad\qquad + (1/2)(a_2 - a_2')(it)\sqrt{(1-p)/p}\Big] \\
&\quad + n^{-1/2}e^{-t^2/2}\frac{a_1 g_x^2 - a_1' g_y^2}{2p^2 S_0^3}\Big[\Psi\sqrt{n}[p/(1-p)](it)^2 - [(it)^3 + (it)]\sqrt{p/(1-p)}\Big] \\
&\quad + (1/4)m^{-1}e^{-t^2/2}\left[3\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4} - (p^2 S_0^2)^{-1}(a_1^2 + (a_1')^2)\right][(it)^2 - \Psi\sqrt{n}(it)\sqrt{p/(1-p)}] \\
&\quad + \frac{1}{4}m^{-1}e^{-t^2/2}\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4}\Big[\Psi^2 n[p/(1-p)](it)^2 - 2\Psi\sqrt{n}\sqrt{p/(1-p)}[(it)^3] + (it)^4 + (it)^2\Big] \\
&\quad - (1/6)(m/n)^2 e^{-t^2/2} S_0^{-2}(g_x g_x'' + g_y g_y'')[(it)^2 - \Psi\sqrt{n}(it)\sqrt{p/(1-p)}] \\
&\quad + o(m^{-1} + (m/n)^2) \\
&= n^{-1/2}e^{-t^2/2}\Big[\Psi\sqrt{n}(a_2 g_x^2 - a_2' g_y^2)(pS_0^3)^{-1}(it)^2 \\
&\qquad\qquad - (a_2 g_x^2 - a_2' g_y^2)\sqrt{(1-p)/p}\,(pS_0^3)^{-1}[(it)^3/2] \\
&\qquad\qquad - (a_2 g_x^2 - a_2' g_y^2)\sqrt{(1-p)/p}\,(pS_0^3)^{-1}[(it)] \\
&\qquad\qquad + (1/2)(a_2 - a_2')\sqrt{(1-p)/p}\,(pS_0)^{-1}(it) \\
&\qquad\qquad + \Psi\sqrt{n}[p/(1-p)](a_1 g_x^2 - a_1' g_y^2)(2p^2 S_0^3)^{-1}(it)^2 \\
&\qquad\qquad - (a_1 g_x^2 - a_1' g_y^2)\sqrt{p/(1-p)}\,(2p^2 S_0^3)^{-1}(it)^3 \\
&\qquad\qquad - (a_1 g_x^2 - a_1' g_y^2)\sqrt{p/(1-p)}\,(2p^2 S_0^3)^{-1}(it)\Big] \\
&\quad + (1/4)m^{-1}e^{-t^2/2}\left[3\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4} - (p^2 S_0^2)^{-1}(a_1^2 + (a_1')^2)\right](it)^2
\end{aligned}
$$

$$
\begin{aligned}
&- (1/4)m^{-1}e^{-t^2/2}\left[3\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4} - (p^2S_0^2)^{-1}(a_1^2 + (a_1')^2)\right]\Psi\sqrt{n}\sqrt{p/(1-p)}(it)\\
&+ \frac{1}{4}m^{-1}e^{-t^2/2}\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4}\left[\Psi^2 n[p/(1-p)](it)^2 + (it)^2\right]\\
&+ \frac{1}{4}m^{-1}e^{-t^2/2}\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4}\left[-2\Psi\sqrt{n}\sqrt{p/(1-p)}[(it)^3] + (it)^4\right]\\
&- (1/6)(m/n)^2 e^{-t^2/2}(g_x g_x'' + g_y g_y'')S_0^{-2}[(it)^2 - \Psi\sqrt{n}\sqrt{p/(1-p)}(it)]\\
&+ o(m^{-1} + (m/n)^2)\\
= n^{-1/2}e^{-t^2/2}\Big[&-\left[(a_2 g_x^2 - a_2' g_y^2)(1-p) + (a_1 g_x^2 - a_1' g_y^2)\right][p/(1-p)]^{1/2}(2p^2S_0^3)^{-1}(it)^3\\
&\quad + \left[2(a_2 g_x^2 - a_2' g_y^2) + (a_1 g_x^2 - a_1' g_y^2)/(1-p)\right](2pS_0^3)^{-1}\Psi\sqrt{n}(it)^2\\
&\quad + \left[(1-p)S_0^2(a_2 - a_2') - 2(1-p)(a_2 g_x^2 - a_2' g_y^2) - (a_1 g_x^2 - a_1' g_y^2)\right]\\
&\qquad \times [p/(1-p)]^{1/2}(2p^2S_0^3)^{-1}(it)\Big]\\
&+ \frac{1}{4}m^{-1}e^{-t^2/2}\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4}\left[(it)^4 - 2\Psi\sqrt{n}\sqrt{p/(1-p)}(it)^3\right]\\
&+ \frac{1}{4}m^{-1}e^{-t^2/2}(p^2S_0^2)^{-1}\left\{\left[\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{S_0^2}\right][4 + \Psi^2 np/(1-p)] - (a_1^2 + (a_1')^2)\right\}(it)^2\\
&- \frac{1}{4}m^{-1}e^{-t^2/2}(p^2S_0^2)^{-1}\left[3\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{S_0^2} - (a_1^2 + (a_1')^2)\right]\Psi\sqrt{n}\sqrt{p/(1-p)}(it)\\
&- \frac{1}{6}(m/n)^2 e^{-t^2/2}(g_x g_x'' + g_y g_y'')S_0^{-2}\left[(it)^2 - \Psi\sqrt{n}\sqrt{p/(1-p)}(it)\right]\\
&+ o(m^{-1} + (m/n)^2).
\end{aligned}
$$

B.13. **Inverse Fourier–Stieltjes transform of characteristic function of K.** The Fourier–Stieltjes transforms of various derivatives of the standard normal distribution, $\Phi(z)$, are

$$
\begin{aligned}
\Phi(z) &\to e^{-t^2/2}\\
\Phi'(z) = \phi(z) &\to -(it)e^{-t^2/2}\\
\Phi''(z) = -z\phi(z) &\to (it)^2 e^{-t^2/2}\\
\Phi'''(z) = \phi(z)(-1 + z^2) &\to -(it)^3 e^{-t^2/2}\\
\Phi''''(z) = \phi(z)(3z - z^3) &\to (it)^4 e^{-t^2/2}.
\end{aligned}
$$

Thus, the inverse Fourier–Stieltjes transform of the higher-order terms in the characteristic function of K is

$$
\begin{aligned}
n^{-1/2}\Big[&-\left[(a_2 g_x^2 - a_2' g_y^2)(1-p) + (a_1 g_x^2 - a_1' g_y^2)\right][p/(1-p)]^{1/2}(2p^2S_0^3)^{-1}(-\phi(z)(-1 + z^2))\\
&+ \left[2(a_2 g_x^2 - a_2' g_y^2) + (a_1 g_x^2 - a_1' g_y^2)/(1-p)\right](2pS_0^3)^{-1}\Psi\sqrt{n}(-z\phi(z))\\
&+ \left[(1-p)S_0^2(a_2 - a_2') - 2(1-p)(a_2 g_x^2 - a_2' g_y^2) - (a_1 g_x^2 - a_1' g_y^2)\right]\\
&\quad \times [p/(1-p)]^{1/2}(2p^2S_0^3)^{-1}(-\phi(z))\Big]\\
+ \frac{1}{4}m^{-1}&\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4}\left[(\phi(z)(3z - z^3)) - 2\Psi\sqrt{n}\sqrt{p/(1-p)}(-\phi(z)(-1 + z^2))\right]\\
+ \frac{1}{4}m^{-1}&(p^2S_0^2)^{-1}\left\{\left[\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{S_0^2}\right][4 + \Psi^2 np/(1-p)] - (a_1^2 + (a_1')^2)\right\}(-z\phi(z))
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{4} m^{-1} (p^2 S_0^2)^{-1} \left[3 \frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{S_0^2} - (a_1^2 + (a_1')^2) \right] \Psi \sqrt{n} \sqrt{p/(1-p)} (-\phi(z)) \\
&- \frac{1}{6} (m/n)^2 (g_x g_x'' + g_y g_y'') S_0^{-2} \left[(-z\phi(z)) - \Psi \sqrt{n} \sqrt{p/(1-p)} (-\phi(z)) \right] \\
&+ o(m^{-1} + (m/n)^2) \\
= n^{-1/2} \Big[& \left[(a_2 g_x^2 - a_2' g_y^2)(1-p) + (a_1 g_x^2 - a_1' g_y^2) \right] [p/(1-p)]^{1/2} (2p^2 S_0^3)^{-1} (z^2) \\
& \quad - \left[2(a_2 g_x^2 - a_2' g_y^2) + (a_1 g_x^2 - a_1' g_y^2)/(1-p) \right] (2p S_0^3)^{-1} \Psi \sqrt{n} (z) \\
& \quad - \left[(a_2 g_x^2 - a_2' g_y^2)(1-p) + (a_1 g_x^2 - a_1' g_y^2) \right] [p/(1-p)]^{1/2} (2p^2 S_0^3)^{-1} \\
& \quad - \left[(1-p) S_0^2 (a_2 - a_2') - 2(1-p)(a_2 g_x^2 - a_2' g_y^2) - (a_1 g_x^2 - a_1' g_y^2) \right] \\
& \qquad \times [p/(1-p)]^{1/2} (2p^2 S_0^3)^{-1} \Big] \phi(z) \\
&+ \frac{1}{4} m^{-1} \frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4} \left[(-z^3) + 2\Psi \sqrt{n} \sqrt{p/(1-p)} (z^2) \right] \phi(z) \\
&+ \frac{1}{4} m^{-1} \frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4} \left[(3z) \right] \phi(z) \\
&- \frac{1}{4} m^{-1} (p^2 S_0^2)^{-1} \left\{ \left[\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{S_0^2} \right] \left[4 + \Psi^2 np/(1-p) \right] - (a_1^2 + (a_1')^2) \right\} (z\phi(z)) \\
&+ \frac{1}{4} m^{-1} \frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4} \left[-2\Psi \sqrt{n} \sqrt{p/(1-p)} \right] \phi(z) \\
&+ \frac{1}{4} m^{-1} (p^2 S_0^2)^{-1} \left[3 \frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{S_0^2} - (a_1^2 + (a_1')^2) \right] \Psi \sqrt{n} \sqrt{p/(1-p)} (\phi(z)) \\
&+ \frac{1}{6} (m/n)^2 (g_x g_x'' + g_y g_y'') S_0^{-2} \left[z - \Psi \sqrt{n} \sqrt{p/(1-p)} \right] \phi(z) \\
&+ o(m^{-1} + (m/n)^2) \\
= n^{-1/2} \phi(z) \Big[& \left[(a_2 g_x^2 - a_2' g_y^2)(1-p) + (a_1 g_x^2 - a_1' g_y^2) \right] [p/(1-p)]^{1/2} (2p^2 S_0^3)^{-1} (z^2) \\
& \quad - \left[2(a_2 g_x^2 - a_2' g_y^2) + (a_1 g_x^2 - a_1' g_y^2)/(1-p) \right] (2p S_0^3)^{-1} \Psi \sqrt{n} (z) \\
& \quad + \left[(a_2 g_x^2 - a_2' g_y^2) - S_0^2 (a_2 - a_2') \right] [p(1-p)]^{1/2} (2p^2 S_0^3)^{-1} \Big] \\
&- \frac{1}{4} m^{-1} \frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{p^2 S_0^4} \left[z^3 - 2\Psi \sqrt{n} \sqrt{p/(1-p)} (z^2) \right] \phi(z) \\
&- \frac{1}{4} m^{-1} (p^2 S_0^2)^{-1} \left\{ \frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{S_0^2} - (a_1^2 + (a_1')^2) + \frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{S_0^2} \Psi^2 np/(1-p) \right\} z\phi(z) \\
&+ \frac{1}{4} m^{-1} (p^2 S_0^2)^{-1} \left[\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{S_0^2} - (a_1^2 + (a_1')^2) \right] \Psi \sqrt{n} \sqrt{p/(1-p)} (\phi(z)) \\
&+ \frac{1}{6} (m/n)^2 (g_x g_x'' + g_y g_y'') S_0^{-2} \left[z - \Psi \sqrt{n} \sqrt{p/(1-p)} \right] \phi(z) \\
&+ o(m^{-1} + (m/n)^2) \\
= n^{-1/2} \phi(z) \Big[& \left[(a_2 g_x^2 - a_2' g_y^2)(1-p) + (a_1 g_x^2 - a_1' g_y^2) \right] [p/(1-p)]^{1/2} (2p^2 S_0^3)^{-1} (z^2) \\
& \quad - \left[2(a_2 g_x^2 - a_2' g_y^2) + (a_1 g_x^2 - a_1' g_y^2)/(1-p) \right] (2p S_0^3)^{-1} \Psi \sqrt{n} (z) \\
& \quad + \left[(a_2 g_x^2 - a_2' g_y^2) - S_0^2 (a_2 - a_2') \right] [p(1-p)]^{1/2} (2p^2 S_0^3)^{-1} \Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{4} m^{-1} \left(\frac{g_x^4 + g_y^4}{S_0^4} \right) \left[z^3 - 2\Psi \sqrt{n} \sqrt{p/(1-p)} (z^2) \right] \phi(z) \\
&- \frac{1}{4} m^{-1} (p^2 S_0^2)^{-1} \left\{ -2p^2 \frac{g_x^2 g_y^2}{S_0^2} + p^2 S_0^2 \left(\frac{g_x^4 + g_y^4}{S_0^4} \right) \Psi^2 np/(1-p) \right\} z\phi(z) \\
&+ \frac{1}{4} m^{-1} (p^2 S_0^2)^{-1} \left[-2p^2 \frac{g_x^2 g_y^2}{S_0^2} \right] \Psi \sqrt{n} \sqrt{p/(1-p)} (\phi(z)) \\
&+ \frac{1}{6} (m/n)^2 (g_x g_x'' + g_y g_y'') S_0^{-2} \left[z - \Psi \sqrt{n} \sqrt{p/(1-p)} \right] \phi(z) \\
&+ o(m^{-1} + (m/n)^2) \\
= n^{-1/2} \phi(z) \Big[& \left[(a_2 g_x^2 - a_2' g_y^2)(1-p) + (a_1 g_x^2 - a_1' g_y^2) \right] [p/(1-p)]^{1/2} (2p^2 S_0^3)^{-1} (z^2) \\
&\quad - \left[2(a_2 g_x^2 - a_2' g_y^2) + (a_1 g_x^2 - a_1' g_y^2)/(1-p) \right] (2pS_0^3)^{-1} \Psi \sqrt{n} (z) \\
&\quad + \left[(a_2 g_x^2 - a_2' g_y^2) - S_0^2 (a_2 - a_2') \right] [p(1-p)]^{1/2} (2p^2 S_0^3)^{-1} \Big] \\
&- \frac{1}{4} m^{-1} \left(\frac{g_x^4 + g_y^4}{S_0^4} \right) \left[z^3 - 2\gamma S_0^{-1} [p(1-p)]^{-1/2} z^2 \right] \phi(z) \\
&+ \frac{1}{4} m^{-1} \left\{ \frac{2g_x^2 g_y^2}{S_0^4} - \left(\frac{g_x^4 + g_y^4}{S_0^4} \right) \frac{\gamma^2}{S_0^2 p(1-p)} \right\} z\phi(z) \\
&- \frac{1}{4} m^{-1} \frac{2g_x^2 g_y^2}{S_0^5} [p(1-p)]^{-1/2} \gamma \phi(z) \\
&+ \frac{1}{6} (m/n)^2 \frac{g_x g_x'' + g_y g_y''}{S_0^2} \left[z - \gamma S_0^{-1} [p(1-p)]^{-1/2} \right] \phi(z) \\
&+ o(m^{-1} + (m/n)^2),
\end{aligned}
$$

using some other results that

$$
\begin{aligned}
\frac{a_1}{pg_x} &= -1 + O(n^{-1}) \implies a_1 = -pg_x + O(n^{-1}), a_1' = -pg_y + O(n^{-1}), \\
a_1^2 + (a_1')^2 &= p^2 g_x^2 + p^2 g_y^2 + O(n^{-1}) = p^2 S_0^2 + O(n^{-1}), \\
\frac{a_1^2 g_x^2 + (a_1')^2 g_y^2}{S_0^2} &- (a_1^2 + (a_1')^2) = \frac{p^2 g_x^2 g_x^2 + p^2 g_y^2 g_y^2 - p^2 S_0^4}{S_0^2} + O(n^{-1}) \\
&= -2p^2 \frac{g_x^2 g_y^2}{S_0^2} + O(n^{-1}), \\
\Psi &\equiv \gamma/(\sqrt{n} p S_0).
\end{aligned}
$$

From before, the inverse Fourier–Stieltjes transform of the part coming from L was

$$
\Phi(z) + n^{-1/2} \frac{1}{6} \frac{1+p}{\sqrt{p(1-p)}} \left(\frac{g_x^3 - g_y^3}{S_0^3} \right) (z^2 - 1)\phi(z) + O(n^{-1}),
$$

and from the centering,

$$
w_n \equiv \frac{\left((\lfloor np \rfloor + 1 - np) - 1 + \frac{1}{2}(1-p) \right) (g_x - g_y)}{S_0 \sqrt{p(1-p)}}.
$$

Thus, altogether, and with $C \equiv \gamma S_0^{-1} / \sqrt{p(1-p)}$,

$$
\begin{aligned}
P(T_{m,n} < z) &= P(K < z) - n^{-1/2} w_n \phi(z) \\
&= P(L < z) + P(K_{h.o.} < z) - n^{-1/2} w_n \phi(z) \\
&= \Phi(z) + \left[n^{-1/2} u_{1,\gamma}(z) + m^{-1} u_{2,\gamma}(z) + (m/n)^2 u_{3,\gamma}(z) \right] \phi(z) + o(m^{-1} + m^2/n^2),
\end{aligned}
$$

with

$$
\begin{aligned}
u_{1,\gamma}(z) &\equiv \frac{1}{6}\frac{1+p}{\sqrt{p(1-p)}}\left(\frac{g_x^3-g_y^3}{S_0^3}\right)(z^2-1) \\
&\quad + \left[(a_2g_x^2-a_2'g_y^2)(1-p)+(a_1g_x^2-a_1'g_y^2)\right][p/(1-p)]^{1/2}(2p^2S_0^3)^{-1}(z^2) \\
&\quad - \left[2(a_2g_x^2-a_2'g_y^2)+(a_1g_x^2-a_1'g_y^2)/(1-p)\right](2pS_0^3)^{-1}C[(1-p)/p]^{1/2}(z) \\
&\quad + \left[(a_2g_x^2-a_2'g_y^2)-S_0^2(a_2-a_2')\right][p(1-p)]^{1/2}(2p^2S_0^3)^{-1} \\
&\quad - \left((\lfloor np\rfloor+1-np)-1+\tfrac{1}{2}(1-p)\right)(g_x-g_y)\left[S_0\sqrt{p(1-p)}\right]^{-1}, \\
u_{2,\gamma}(z) &\equiv -\frac{1}{4}\left(\frac{g_x^4+g_y^4}{S_0^4}\right)z^3+\frac{1}{2}\frac{g_x^2g_y^2}{S_0^4}z-\frac{1}{2}\frac{g_x^2g_y^2}{S_0^4}C+\frac{1}{4}\left(\frac{g_x^4+g_y^4}{S_0^4}\right)(2Cz^2-C^2z), \\
u_{3,\gamma}(z) &\equiv \frac{g_xg_x''+g_yg_y''}{6S_0^2}(z-C).
\end{aligned}
$$

Appendix C. Corrected critical values

Although the expressions above are more complicated than the univariate case, the structure is similar. In the univariate case, our corrected critical values achieved $e_I \le \alpha$, where e_I contains the dominating components of type I error and α is the nominal size of the test. This was due to $u_{1,\gamma}$ being an even function of z under the null and $u_{3,\gamma} > 0$ under the null. Both are still true here: under the null, $C = 0$, so z only enters u_1 as z^2, and it was shown in the univariate case that $g_xg_x'' > 0$ (and equivalently, $g_yg_y'' > 0$).

Analogously, our new critical value $z_{\alpha,m}$ needs to cancel the u_2 terms that appear under the null. Using $u_{1,0}(-z) = u_{1,0}(z)$, $u_{2,0}(-z) = -u_{2,0}(z)$, and $u_{3,0}(-z) = -u_{3,0}(z)$, the type I error of a two-sided test is

$$
\begin{aligned}
P\left(|T_{m,n}| > z \mid H_0\right) &= P(T_{m,n} > z \mid H_0) + P(T_{m,n} < -z \mid H_0) \\
&= 1 - P(T_{m,n} < z \mid H_0) + P(T_{m,n} < -z \mid H_0) \\
&= 1 - \{\Phi(z) + [n^{-1/2}u_{1,0}(z) + m^{-1}u_{2,0}(z) + (m/n)^2u_{3,0}(z)]\phi(z)\} \\
&\quad + \Phi(-z) + [n^{-1/2}u_{1,0}(-z) + m^{-1}u_{2,0}(-z) + (m/n)^2u_{3,0}(-z)]\phi(-z) \\
&\quad + o(m^{-1} + m^2/n^2) \\
&= 2 - 2\Phi(z) + 0 - 2m^{-1}u_{2,0}(z)\phi(z) - 2(m/n)^2u_{3,0}(z)\phi(z) \\
&\quad + o(m^{-1} + m^2/n^2),
\end{aligned}
$$

and if $z_{\alpha,m} = z_{1-\alpha/2} + c/m$,

$$
\begin{aligned}
&= 2 - 2\Phi(z_{1-\alpha/2} + c/m) - 2m^{-1}\phi(z_{\alpha,m})\left[-\frac{1}{4}\left(\frac{g_x^4+g_y^4}{S_0^4}\right)z_{\alpha,m}^3 + \frac{1}{2}\frac{g_x^2g_y^2}{S_0^4}z_{\alpha,m}\right] \\
&\quad - 2(m/n)^2u_{3,0}(z_{\alpha,m})\phi(z_{\alpha,m}) + o(m^{-1} + m^2/n^2) \\
&= 2 - 2\Phi(z_{1-\alpha/2}) - 2\phi(z_{1-\alpha/2})(c/m) - O(m^{-2}) \\
&\quad - 2m^{-1}\phi(z_{1-\alpha/2})\left[-\frac{1}{4}\left(\frac{g_x^4+g_y^4}{S_0^4}\right)z_{1-\alpha/2}^3 + \frac{1}{2}\frac{g_x^2g_y^2}{S_0^4}z_{1-\alpha/2}\right] \\
&\quad - 2(m/n)^2u_{3,0}(z_{1-\alpha/2})\phi(z_{1-\alpha/2}) + o(m^{-1} + m^2/n^2) \\
&= \alpha - 2(m/n)^2u_{3,0}(z_{1-\alpha/2})\phi(z_{1-\alpha/2}) + o(m^{-1} + m^2/n^2) \\
&\quad - 2m^{-1}\phi(z_{1-\alpha/2})\left[c - \frac{1}{4}\left(\frac{g_x^4+g_y^4}{S_0^4}\right)z_{1-\alpha/2}^3 + \frac{1}{2}\frac{g_x^2g_y^2}{S_0^4}z_{1-\alpha/2}\right],
\end{aligned}
$$

and for a level α test we want this to be at most α. This is achieved by setting the m^{-1} term to zero, so that

$$
\begin{aligned}
c &= \frac{1}{4}\left(\frac{g_x^4+g_y^4}{S_0^4}\right) z_{1-\alpha/2}^3 - \frac{1}{2}\frac{g_x^2 g_y^2}{S_0^4} z_{1-\alpha/2},\\
z_{\alpha,m} &= z_{1-\alpha/2} + c/m\\
&= z_{1-\alpha/2} + m^{-1}[4S_0^4]^{-1}\left[(g_x^4+g_y^4)z_{1-\alpha/2}^3 - 2g_x^2g_y^2 z_{1-\alpha/2}\right].
\end{aligned}
$$

Appendix D. Fixed-m asymptotics

Define

$$
\begin{aligned}
&\theta \equiv S_0^{-4}(f_X^{-4}+\eta f_Y^{-4}), \quad \eta \equiv n_x/n_y,\\
&\delta \equiv f_X/f_Y,\\
&\lambda \equiv \left[1+\eta\delta^2\right]^{-1},\\
&\theta = \lambda^2 + (1-\lambda)^2, \quad 1-\theta = 2\lambda(1-\lambda).
\end{aligned}
$$

Here, we approximate the critical value derived from the fixed-m asymptotic distribution. (The constant sample size ratio η can be weakened to a limit of the sample size ratio, if the limit is approached at a fast enough rate.) Recall from earlier,

$$
\sqrt{n_x}(X_{nr}-\xi_p) - \sqrt{n_x/n_y}\sqrt{n_y}(Y_{nr}-\xi_p) \xrightarrow{d} N\left(0, p(1-p)(f_X^{-2}+\eta f_Y^{-2})\right),
$$

using the fact that the variance of the sum (or difference) of two independent normals is the sum of the variances, and with f_X shorthand for $f_X(F_X^{-1}(p))$ and similarly for f_Y. The pivot for the bivariate case is then

$$
\frac{\sqrt{n_x}(X_{nr}-Y_{nr})}{\sqrt{[f_X(\xi_p)]^{-2}+\eta[f_Y(\xi_p)]^{-2}}\sqrt{p(1-p)}} \xrightarrow{d} N(0,1),
$$

with the Studentized version using the sample estimates of f_X and f_Y by the same quantile spacing estimator as in the univariate case,

$$
T_{m,n} \equiv \frac{\sqrt{n_x}(X_{nr}-Y_{nr})}{\sqrt{(n_x/(2m))^2(X_{n,r+m}-X_{n,r-m})^2+\eta(n_y/(2m))^2(Y_{n,r+m}-Y_{n,r-m})^2}\sqrt{p(1-p)}}.
$$

The assumption has been that $X \perp\!\!\!\perp Y$, and $X_{n,r} \perp\!\!\!\perp (X_{n,r+m}-X_{n,r-m})$ asymptotically, per Siddiqui (1960). Siddiqui (1960) also gives

$$
\frac{\frac{n_x}{2m}(X_{n,r+m}-X_{n,r-m})}{1/f_X(\xi_p)} \xrightarrow{d} \mathcal{V}_{4m},
$$

where $\mathcal{V}_{4m} \sim \chi^2_{4m}/(4m)$. This will be true for Y also. Now,

$$
\begin{aligned}
\frac{S_{m,n}}{S_0} &= \left(\frac{[(n_x/(2m))(X_{n,r+m}-X_{n,r-m})]^2+\eta[(n_y/(2m))(Y_{n,r+m}-Y_{n,r-m})]^2}{f_X^{-2}+\eta f_Y^{-2}}\right)^{1/2}\\
&\xrightarrow{d} \left(\frac{\mathcal{V}_{4m,1}^2 f_X^{-2}+\eta\mathcal{V}_{4m,2}^2 f_Y^{-2}}{f_X^{-2}+\eta f_Y^{-2}}\right)^{1/2}\\
&= \left(\frac{f_X^{-2}+\eta f_Y^{-2}+(\mathcal{V}_{4m,1}^2-1)f_X^{-2}+(\mathcal{V}_{4m,2}^2-1)\eta f_Y^{-2}}{f_X^{-2}+\eta f_Y^{-2}}\right)^{1/2}\\
&= \left(1+\frac{(\mathcal{V}_{4m,1}^2-1)f_X^{-2}+\eta(\mathcal{V}_{4m,2}^2-1)f_Y^{-2}}{f_X^{-2}+\eta f_Y^{-2}}\right)^{1/2},
\end{aligned}
$$

and call a random variable with that distribution $\mathcal{U}$:

$$\mathcal{U} \sim (1+\epsilon)^{1/2}, \quad \epsilon \equiv \lambda(\mathcal{V}_{4m,1}^2 - 1) + (1-\lambda)(\mathcal{V}_{4m,2}^2 - 1)$$

Thus, under the null,

$$\begin{aligned} T_{m,n} &\equiv \frac{\sqrt{n_x}(X_{nr} - Y_{nr})}{\sqrt{(n_x/(2m))^2(X_{n,r+m} - X_{n,r-m})^2 + (n_y/(2m))^2(Y_{n,r+m} - Y_{n,r-m})^2}\sqrt{p(1-p)}} \\ &= \frac{\sqrt{n_x}(X_{nr} - \xi_p) - \sqrt{n_x/n_y}\sqrt{n_y}(Y_{nr} - \xi_p)}{\sqrt{[(n_x/(2m))(X_{n,r+m} - X_{n,r-m})]^2 + [(n_y/(2m))(Y_{n,r+m} - Y_{n,r-m})]^2}\sqrt{p(1-p)}} \\ &= \frac{\sqrt{n_x}(X_{nr} - Y_{nr})}{\sqrt{[f_X(\xi_p)]^{-2} + \eta[f_Y(\xi_p)]^{-2}}\sqrt{p(1-p)}} \frac{1}{S_{m,n}/S_0} \\ &\xrightarrow{d} \mathcal{Z}/\mathcal{U}. \end{aligned}$$

Analogous to the univariate case,

$$\begin{aligned} P(T_{m,\infty} < z) &= P(\mathcal{Z}/\mathcal{U} < z) = E[\Phi(z\mathcal{U})] = E[\Phi(z + z(\mathcal{U}-1))] \\ &= E\Big\{\Phi(z) + \Phi'(z)z(\mathcal{U}-1) + (1/2)\Phi''(z)[z(\mathcal{U}-1)]^2 + (1/6)\Phi'''(z)[z(\mathcal{U}-1)]^3 \\ &\quad + (1/24)\Phi''''(z)[z(\mathcal{U}-1)]^4\Big\} + O(E[(\mathcal{U}-1)^5]) \\ &= \Phi(z) + \phi(z)zE[\mathcal{U}-1] + (1/2)(-z\phi(z))z^2E[(\mathcal{U}-1)^2] + (1/6)(z^2-1)\phi(z)z^3E[(\mathcal{U}-1)^3] \\ &\quad + (1/24)(3z - z^3)\phi(z)z^4E[(\mathcal{U}-1)^4] + O(E[(\mathcal{U}-1)^5]). \end{aligned}$$

Below, we will need the moments of ϵ, which depend on moments of $\mathcal{V}_{4m}^2 - 1$. Recall that $\mathcal{V}_{4m} \sim \chi_{4m}^2/(4m)$, and the noncentral moments of χ_{4m}^2 are $4m$, $4m(4m+2)$, $4m(4m+2)(4m+4)$, $4m(4m+2)(4m+4)(4m+6)$, $4m(4m+2)(4m+4)(4m+6)(4m+8)$, etc.

$$\begin{aligned} E[\mathcal{V}_{4m}^2 - 1] &= \frac{4m(4m+2)}{(4m)^2} - 1 = (1/2)m^{-1}, \\ E[(\mathcal{V}_{4m}^2 - 1)^2] &= E(\mathcal{V}_{4m}^4) - 2E[\mathcal{V}_{4m}^2] + 1 \\ &= (256m^4)^{-1}4m(4m+2)(4m+4)(4m+6) - 2[1 + (1/2)m^{-1}] + 1 \\ &= 1 + 3m^{-1} + (11/4)m^{-2} + (3/4)m^{-3} - m^{-1} - 1 \\ &= 2m^{-1} + (11/4)m^{-2} + (3/4)m^{-3} \\ &= 2m^{-1} + O(m^{-2}), \\ E[(\mathcal{V}_{4m}^2 - 1)^3] &= E[\mathcal{V}_{4m}^6] - 3E[\mathcal{V}_{4m}^4] + 3E[\mathcal{V}_{4m}^2] - 1 \\ &= (4096m^6)^{-1}4m(4m+2)(4m+4)(4m+6)(4m+8)(4m+10) \\ &\quad - 3\left[1 + 3m^{-1} + (11/4)m^{-2} + (3/4)m^{-3}\right] + 3[1 + (1/2)m^{-1}] - 1 \\ &= \frac{4096m^6 + 30720m^5 + 87040m^4 + 115200m^3 + 70144m^2 + 15360m}{4096m^6} \\ &\quad - 3 + 3 - 1 - 9m^{-1} + (3/2)m^{-1} - (33/4)m^{-2} - (9/4)m^{-3} \\ &= (15/2)m^{-1} + (85/4)m^{-2} + (225/8)m^{-3} + (137/8)m^{-4} + (15/4)m^{-5} \\ &\quad - 9m^{-1} + (3/2)m^{-1} - (33/4)m^{-2} - (9/4)m^{-3} \\ &= 13m^{-2} + (207/8)m^{-3} + (137/8)m^{-4} + (15/4)m^{-5} \\ &= O(m^{-2}), \end{aligned}$$

$$
\begin{aligned}
E[(\mathcal{V}_{4m}^2-1)^4] &= E[\mathcal{V}_{4m}^8]-4E[\mathcal{V}_{4m}^6]+6E[\mathcal{V}_{4m}^4]-4E[\mathcal{V}_{4m}^2]+1\\
&= (4m)^{-8}4m(4m+2)(4m+4)(4m+6)(4m+8)(4m+10)(4m+12)(4m+14)\\
&\quad -4\left[1+(15/2)m^{-1}+(85/4)m^{-2}+(225/8)m^{-3}+(137/8)m^{-4}+(15/4)m^{-5}\right]\\
&\quad +6\left[1+3m^{-1}+(11/4)m^{-2}+(3/4)m^{-3}\right]\\
&\quad -4\left[1+(1/2)m^{-1}\right]+1\\
&= 14m^{-1}+(161/2)m^{-2}+O(m^{-3})\\
&\quad -30m^{-1}-85m^{-2}+18m^{-1}+(33/2)m^{-2}-2m^{-1}+O(m^{-3})\\
&= m^{-1}[0]+m^{-2}[12]+O(m^{-3}),\\
E[(\mathcal{V}_{4m}^2-1)^5] &= E[\mathcal{V}_{4m}^{10}]-5E[\mathcal{V}_{4m}^8]+10E[\mathcal{V}_{4m}^6]-10E[\mathcal{V}_{4m}^4]+5E[\mathcal{V}_{4m}^2]-1\\
&= (4m)^{-10}4m(4m+2)(4m+4)(4m+6)(4m+8)(4m+10)\\
&\qquad \times(4m+12)(4m+14)(4m+16)(4m+18)\\
&\quad -5[1+14m^{-1}+(161/2)m^{-2}]\\
&\quad +10\left[1+(15/2)m^{-1}+(85/4)m^{-2}\right]\\
&\quad -10\left[1+3m^{-1}+(11/4)m^{-2}\right]\\
&\quad +5\left[1+(1/2)m^{-1}\right]-1\\
&\quad +O(m^{-3})\\
&= (45/2)m^{-1}+(435/2)m^{-2}\\
&\quad +m^{-1}\left[-70+75-30+(5/2)\right]+m^{-2}\left[(-805/2)+(425/2)-(55/2)\right]\\
&= m^{-1}[0]+m^{-2}[0]+O(m^{-3}).
\end{aligned}
$$

The moments of ϵ are

$$
\begin{aligned}
E(\epsilon) &= \frac{f_X^{-2}}{f_X^{-2}+f_Y^{-2}}E(\mathcal{V}_{4m,1}^2-1)+\frac{f_Y^{-2}}{f_X^{-2}+f_Y^{-2}}E(\mathcal{V}_{4m,2}^2-1)\\
&= \frac{f_X^{-2}+f_Y^{-2}}{f_X^{-2}+f_Y^{-2}}\left(\frac{4m(4m+2)}{(4m)^2}-1\right)\\
&= (1/2)m^{-1},\\
E(\epsilon^2) &= E\left[\lambda^2(\mathcal{V}_{4m,1}^2-1)+(1-\lambda)^2(\mathcal{V}_{4m,2}^2-1)+2\lambda(1-\lambda)(\mathcal{V}_{4m,1}^2-1)(\mathcal{V}_{4m,2}^2-1)\right]\\
&= \left[2m^{-1}+(11/4)m^{-2}+(3/4)m^{-3}\right]\theta+(1-\theta)(1/4)m^{-2}\\
&= 2m^{-1}\theta+(1/4)m^{-2}(11\theta+1-\theta)+(3/4)m^{-3}\theta,\\
E(\epsilon^3) &= E\left\{[\lambda^3+(1-\lambda)^3](\mathcal{V}_{4m}^2-1)^3+3[\lambda^2(1-\lambda)+\lambda(1-\lambda)^2](\mathcal{V}_{4m,1}^2-1)^2(\mathcal{V}_{4m,2}^2-1)\right\}\\
&= [1-3\lambda(1-\lambda)]E[(\mathcal{V}_{4m}^2-1)^3]+3\lambda(1-\lambda)E[(\mathcal{V}_{4m}^2-1)^2]E[\mathcal{V}_{4m}^2-1]\\
&= [\theta-(1-\theta)/2][13m^{-2}+(207/8)m^{-3}+(137/8)m^{-4}+(15/4)m^{-5}]\\
&\quad +(3/2)(1-\theta)[2m^{-1}+(11/4)m^{-2}+(3/4)m^{-3}][(1/2)m^{-1}]\\
&= [(3/2)\theta-(1/2)][13m^{-2}+(207/8)m^{-3}+(137/8)m^{-4}+(15/4)m^{-5}]\\
&\quad +(3/2)(1-\theta)[m^{-2}+(11/8)m^{-3}+(3/8)m^{-4}]\\
&= m^{-2}[(36/2)\theta-10/2]+m^{-3}[(588/16)\theta-(174/16)]+m^{-4}[(402/16)\theta-(128/16)]\\
&\quad +m^{-5}[(45/8)\theta-(15/8)]\\
&= m^{-2}[18\theta-5]+m^{-3}[(147/4)\theta-(87/8)]+m^{-4}[(201/8)\theta-8]+m^{-5}[(45/8)\theta-(15/8)]\\
&= O(m^{-2}),
\end{aligned}
$$

$$
\begin{aligned}
E(\epsilon^4) &= E\Big\{[\lambda^4+(1-\lambda)^4](\mathcal{V}_{4m}^2-1)^4+4[\lambda^3(1-\lambda)+\lambda(1-\lambda)^3](\mathcal{V}_{4m,1}^2-1)^3(\mathcal{V}_{4m,2}^2-1) \\
&\qquad +6\lambda^2(1-\lambda)^2(\mathcal{V}_{4m,1}^2-1)^2(\mathcal{V}_{4m,2}^2-1)^2\Big\} \\
&= [\lambda^4+(1-\lambda)^4]12m^{-2}+4[\lambda^3(1-\lambda)+\lambda(1-\lambda)^3](1/2)m^{-1}13m^{-2}+6\lambda^2(1-\lambda)^2 4m^{-2}+O(m^{-3}) \\
&= 12m^{-2}\left[\lambda^4+(1-\lambda)^4+2\lambda^2(1-\lambda)^2\right]+O(m^{-3}) \\
&= 12m^{-2}\theta^2+O(m^{-3}), \\
E(\epsilon^5) &= O\left(E\Big\{(\mathcal{V}_{4m}^2-1)^5+(\mathcal{V}_{4m,1}^2-1)^4(\mathcal{V}_{4m,2}^2-1)+(\mathcal{V}_{4m,1}^2-1)^3(\mathcal{V}_{4m,2}^2-1)^2\Big\}\right)=O(m^{-3}).
\end{aligned}
$$

To calculate the expectations involving $\mathcal{U}$ above,

$$
\begin{aligned}
E[\mathcal{U}-1] &= E[(1+\epsilon)^{1/2}-1] \\
&= E[1]+E[\epsilon/2]-E[\epsilon^2/8]+(1/6)E[\epsilon^3(3/8)]-(1/24)E[\epsilon^4(15/16)]+O(E(\epsilon^5))-1 \\
&= (1/2)(1/2)m^{-1}-(1/8)(\theta 2m^{-1}+(1/4)m^{-2}(10\theta+1)+(3/4)m^{-3}\theta) \\
&\quad +(1/16)m^{-2}[18\theta-5]-(5/128)12m^{-2}\theta^2+O(m^{-3}) \\
&= m^{-1}(1/4)(1-\theta)+m^{-2}[(-10/32)\theta-(1/32)+(18/16)\theta-(5/16)-(15/32)\theta^2]+O(m^{-3}) \\
&= m^{-1}(1/4)(1-\theta)+m^{-2}[(13/16)\theta-(11/32)-(15/32)\theta^2]+O(m^{-3}) \\
&= m^{-1}(1/4)(1-\theta)+O(m^{-2}), \\
E[(\mathcal{U}-1)^2] &= E[\mathcal{U}^2]-2E[\mathcal{U}]+1 \\
&= E[1+\epsilon]-2\left\{1+m^{-1}(1/4)(1-\theta)+m^{-2}[(13/16)\theta-(11/32)-(15/32)\theta^2]+O(m^{-3})\right\}+1 \\
&= 1+(1/2)m^{-1}-2-m^{-1}(1/2)(1-\theta)-m^{-2}[(13/8)\theta-(11/16)-(15/16)\theta^2]+1+O(m^{-3}) \\
&= m^{-1}(1/2)\theta+m^{-2}[(11/16)-(13/8)\theta+(15/16)\theta^2]+O(m^{-3}), \\
E[(\mathcal{U}-1)^3] &= E[\mathcal{U}^3]-3E[\mathcal{U}^2]+3E[\mathcal{U}]-1 \\
&= E[(1+\epsilon)(1+\epsilon)^{1/2}] \\
&\quad -3(1+(1/2)m^{-1}) \\
&\quad +3\left[1+m^{-1}(1/4)(1-\theta)+m^{-2}[(13/16)\theta-(11/32)-(15/32)\theta^2]+O(m^{-3})\right]-1 \\
&= E[(1+\epsilon)^{1/2}]+E[\epsilon(1+\epsilon)^{1/2}] \\
&\quad -3(1+(1/2)m^{-1}) \\
&\quad +3\left[1+m^{-1}(1/4)(1-\theta)+m^{-2}[(13/16)\theta-(11/32)-(15/32)\theta^2]+O(m^{-3})\right]-1 \\
&= \left\{1+m^{-1}(1/4)(1-\theta)+m^{-2}[(13/16)\theta-(11/32)-(15/32)\theta^2]+O(m^{-3})\right\} \\
&\quad +E[\epsilon]+E[\epsilon^2/2]-E[\epsilon^3/8]+(1/6)E[\epsilon^4(3/8)]+O(E[\epsilon^5]) \\
&\quad -3(1+(1/2)m^{-1}) \\
&\quad +3\left[1+m^{-1}(1/4)(1-\theta)+m^{-2}[(13/16)\theta-(11/32)-(15/32)\theta^2]\right]-1+O(m^{-3}) \\
&= (1/2)m^{-1}+(1/2)[2\theta m^{-1}+(1/4)m^{-2}(10\theta+1)]-(1/8)m^{-2}[18\theta-5]+(1/16)12m^{-2}\theta^2 \\
&\quad +m^{-1}[(1/4)(1-\theta)-(3/2)+(3/4)(1-\theta)] \\
&\quad +m^{-2}[(13/16)\theta-(11/32)+(39/16)\theta-(33/32)-4(15/32)\theta^2]+O(m^{-3}) \\
&= m^{-2}[(10/8)\theta+(1/8)-(18/8)\theta+(5/8)+(52/16)\theta-(44/32)-(9/8)\theta^2]+O(m^{-3}) \\
&= m^{-2}[(9/4)\theta-(5/8)-(9/8)\theta^2]+O(m^{-3}), \\
E[(\mathcal{U}-1)^4] &= E[\mathcal{U}^4-4\mathcal{U}^3+6\mathcal{U}^2-4\mathcal{U}+1] \\
&= E[(1+\epsilon)^2]-4E[\mathcal{U}^3]+6E[1+\epsilon]-4E[\mathcal{U}]+1
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + 2m^{-1}\theta + (1/4)m^{-2}(11\theta + 1 - \theta) + O(m^{-3}) + 2(1/2)m^{-1} \\
&\quad - 4\Big\{1 + m^{-1}(1/4)(1-\theta) + m^{-2}[(13/16)\theta - (11/32) - (15/32)\theta^2] \\
&\qquad + (1/2)m^{-1} + (1/2)[2\theta m^{-1} + (1/4)m^{-2}(10\theta + 1)] \\
&\qquad - (1/8)m^{-2}[18\theta - 5] + (3/4)m^{-2}\theta^2 + O(m^{-3})\Big\} \\
&\quad + 6[1 + (1/2)m^{-1}] \\
&\quad - 4\left\{1 + m^{-1}(1/4)(1-\theta) + m^{-2}[(13/16)\theta - (11/32) - (15/32)\theta^2] + O(m^{-3})\right\} + 1 \\
&= m^{-1}\left\{2\theta + 1 - 1 + \theta - 2 - 4\theta + 3 - 1 + \theta\right\} \\
&\quad + m^{-2}\Big\{(10/4)\theta + (1/4) - (13/4)\theta + (11/8) - 5\theta - (1/2) + 9\theta - (5/2) \\
&\qquad - (13/4)\theta + (11/8) + \theta^2[(60/32) - 3 + (60/32)]\Big\} + O(m^{-3}) \\
&= m^{-2}\theta^2(3/4) + O(m^{-3}),
\end{aligned}
$$

$$
\begin{aligned}
E[(\mathcal{U}-1)^5] &= E[\mathcal{U}^5 - 5\mathcal{U}^4 + 10\mathcal{U}^3 - 10\mathcal{U}^2 + 5\mathcal{U} - 1] \\
&= E[(1+\epsilon)(1+\epsilon)^{3/2}] \\
&\quad - 5\Big\{1 + 2m^{-1}\theta + (1/4)m^{-2}(10\theta + 1) + 2(1/2)m^{-1}\Big\} \\
&\quad + 10\Big\{1 + m^{-1}(1/4)(1-\theta) + m^{-2}[(13/16)\theta - (11/32) - (15/32)\theta^2] \\
&\qquad + (1/2)m^{-1} + (1/2)[2\theta m^{-1} + (1/4)m^{-2}(10\theta + 1)] \\
&\qquad - (1/8)m^{-2}[18\theta - 5] + (3/4)m^{-2}\theta^2\Big\} \\
&\quad - 10\Big\{1 + (1/2)m^{-1}\Big\} \\
&\quad + 5\Big\{1 + m^{-1}(1/4)(1-\theta) + m^{-2}[(13/16)\theta - (11/32) - (15/32)\theta^2]\Big\} - 1 + O(m^{-3}) \\
&= E[\mathcal{U}^3] + E[(\epsilon^2 + \epsilon)(1+\epsilon)^{1/2}] + E[-5\mathcal{U}^4 + 10\mathcal{U}^3 - 10\mathcal{U}^2 + 5\mathcal{U} - 1] \\
&= E\Big\{[\epsilon^2 + \epsilon^3/2 - \epsilon^4/8] + [\epsilon + \epsilon^2/2 - \epsilon^3/8 + \epsilon^4/16]\Big\} \\
&\quad + m^{-1}\Big\{ - 10\theta - 5 + (11/4)(1-\theta) + (11/2) + 11\theta - 5 + (5/4)(1-\theta)\Big\} \\
&\quad + m^{-2}\Big\{(-50/4)\theta - (5/4) + (143/16)\theta - (121/32) - (165/32)\theta^2 + (11/2)(10/4)\theta + (11/8) \\
&\qquad - (198/8)\theta + (55/8) + (33/4)\theta^2 + (65/16)\theta - (55/32) - (75/32)\theta^2\Big\} + O(m^{-3}) \\
&= E[\epsilon] + (3/2)E[\epsilon^2] + (3/8)E[\epsilon^3] - E[\epsilon^4/16] \\
&\quad + m^{-1}\Big\{ - 10\theta - (11/4)\theta + 11\theta - \theta(5/4) - 5 + (11/2) + (11/4) - 5 + (5/4)\Big\} \\
&\quad + m^{-2}\Big\{ - (5/4) - (121/32) + (11/8) + (55/8) - (55/32) \\
&\qquad - (165/32)\theta^2 + (33/4)\theta^2 - (75/32)\theta^2 \\
&\qquad - (50/4)\theta + (143/16)\theta + (55/4)\theta - (198/8)\theta + (65/16)\theta\Big\} + O(m^{-3}) \\
&= (1/2)m^{-1} + (3/2)[2m^{-1}\theta + (10/4)m^{-2}\theta + (1/4)m^{-2}] + (3/8)m^{-2}[18\theta - 5] - (1/16)12m^{-2}\theta^2 \\
&\quad + m^{-1}\Big\{\theta - (16/4)\theta - 10 + (38/4)\Big\} \\
&\quad + m^{-2}\Big\{ - (40/32) - (121/32) + (44/32) + (220/32) - (55/32)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - (165/32)\theta^2 + (264/32)\theta^2 - (75/32)\theta^2 \\
&\quad - (200/16)\theta + (143/16)\theta + (220/16)\theta - (396/16)\theta + (65/16)\theta\Big\} + O(m^{-3}) \\
&= m^{-1}\Big\{ - 3\theta - (1/2) + (1/2) + 3\theta\Big\} \\
&\quad + m^{-2}\Big\{(48/32) + (3/8) - (15/8) \\
&\quad + (24/32)\theta^2 - (12/16)\theta^2 \\
&\quad - (168/16)\theta + (30/8)\theta + (54/8)\theta\Big\} + O(m^{-3}) \\
&= m^{-1}[0] + m^{-2}[0] + O(m^{-3}).
\end{aligned}
$$

Summarizing and then plugging in,

$$
\begin{aligned}
E[\mathcal{U} - 1] &= m^{-1}(1/4)(1-\theta) + m^{-2}[(13/16)\theta - (11/32) - (15/32)\theta^2] + O(m^{-3}) \\
E[(\mathcal{U} - 1)^2] &= m^{-1}(1/2)\theta + m^{-2}[(11/16) - (13/8)\theta + (15/16)\theta^2] + O(m^{-3}), \\
E[(\mathcal{U} - 1)^3] &= m^{-2}[(9/4)\theta - (5/8) - (9/8)\theta^2] + O(m^{-3}), \\
E[(\mathcal{U} - 1)^4] &= m^{-2}\theta^2(3/4) + O(m^{-3}), \\
E[(\mathcal{U} - 1)^5] &= O(m^{-3}), \\
P(T_{m,\infty} < z) &= \Phi(z) + \phi(z)zE[\mathcal{U}-1] + (1/2)(-z\phi(z))z^2E[(\mathcal{U}-1)^2] + (1/6)(z^2-1)\phi(z)z^3E[(\mathcal{U}-1)^3] \\
&\quad + (1/24)(3z - z^3)\phi(z)z^4E[(\mathcal{U}-1)^4] + O(E[(\mathcal{U}-1)^5]) \\
&= \Phi(z) + \phi(z)z\left\{m^{-1}(1/4)(1-\theta) + m^{-2}[(13/16)\theta - (11/32) - (15/32)\theta^2]\right\} \\
&\quad - (1/2)z^3\phi(z)\left\{m^{-1}(1/2)\theta + m^{-2}[(11/16) - (13/8)\theta + (15/16)\theta^2]\right\} \\
&\quad + (1/6)\phi(z)(z^5 - z^3)\left\{m^{-2}[(9/4)\theta - (5/8) - (9/8)\theta^2]\right\} \\
&\quad + (1/24)(3z^5 - z^7)\phi(z)\left\{m^{-2}\theta^2(3/4)\right\} + O(m^{-3}) \\
&= \Phi(z) + (1/4)\phi(z)m^{-1}\left\{z(1-\theta) - z^3\theta\right\} \\
&\quad + \phi(z)m^{-2}\Big\{(13/16)z\theta - (11/32)z - (15/32)z\theta^2 - z^3(11/32) + z^3(13/16)\theta - z^3(15/32)\theta^2 \\
&\qquad + (z^5 - z^3)[(3/8)\theta - (5/48) - (9/48)\theta^2] \\
&\qquad + (1/32)(3z^5 - z^7)\theta^2\Big\} + O(m^{-3}) \\
&= \Phi(z) + (1/4)\phi(z)m^{-1}\left\{z(1-\theta) - z^3\theta\right\} \\
&\quad + (1/16)\phi(z)m^{-2} \\
&\qquad \times \Big\{z[13\theta - (11/2) - (15/2)\theta^2] + z^3[7\theta - (23/6) - (9/2)\theta^2] \\
&\qquad\quad + z^5[6\theta - (5/3) - (3/2)\theta^2] - (1/2)z^7\theta^2\Big\} \\
&\quad + O(m^{-3}) \\
&= \Phi(z) - (1/4)\phi(z)m^{-1}[z^3\theta - z(1-\theta)] + O(m^{-2}).
\end{aligned}
$$

Note that when Y is just a constant (the univariate special case), $\theta = 1$, and this matches the univariate fixed-m distribution of $\Phi(z) - \phi(z)(1/4)m^{-1}z^3 + O(m^{-2})$.

Above, the corrected critical value was

$$z_{\alpha,m} = z_{1-\alpha/2} + m^{-1}[4S_0^4]^{-1}\left[(g_x^4 + g_y^4)z_{1-\alpha/2}^3 - 2g_x^2g_y^2z_{1-\alpha/2}\right],$$

which is the same critical value suggested by the above approximation of the fixed-m limiting distribution. In the bivariate case here, same as in the univariate case, the fixed-m distribution picks up the Edgeworth term associated with the variance of the quantile spacing estimator. Once again, the associated critical value reduces the dominant components of type I error, e_I, below the nominal test size, α, for all "common" distributions (including normal, t, Fréchet, uniform, χ^2_k, exponential, and others).

For the third-order corrected critical value, Let $z = z_{1-\alpha} + c_1/m + c_2/m^2$ as in the univariate case. Up to $O(m^{-3})$ terms,

$$\begin{aligned}
P(T_{m,\infty} < z) &= \Phi(z_{1-\alpha}) + (c_1/m + c_2/m^2)\phi(z_{1-\alpha}) + (1/2)(c_1/m + c_2/m^2)^2\phi'(z_{1-\alpha}) \\
&\quad + (1/4)\phi(z_{1-\alpha} + c_1/m)m^{-1}\left\{(z_{1-\alpha} + c_1/m)(1-\theta) - (z_{1-\alpha} + c_1/m)^3\theta\right\} \\
&\quad + (1/16)\phi(z_{1-\alpha})m^{-2}\Big\{z_{1-\alpha}[13\theta - (11/2) - (15/2)\theta^2] + z^3_{1-\alpha}[7\theta - (23/6) - (9/2)\theta^2] \\
&\qquad\qquad + z^5_{1-\alpha}[6\theta - (5/3) - (3/2)\theta^2] - z^7_{1-\alpha}(1/2)\theta^2\Big\} + O(m^{-3}) \\
&= 1 - \alpha + m^{-1}c_1\phi(z_{1-\alpha}) + m^{-2}\left[c_2\phi(z_{1-\alpha}) - (1/2)c_1^2 z_{1-\alpha}\phi(z_{1-\alpha})\right] \\
&\quad + (1/4)m^{-1}\left[\phi(z_{1-\alpha}) - c_1 m^{-1} z_{1-\alpha}\phi(z_{1-\alpha})\right] \\
&\qquad \times \left[z_{1-\alpha}(1-\theta) + m^{-1}c_1(1-\theta) - \theta z^3_{1-\alpha} - 3m^{-1}c_1\theta z^2_{1-\alpha}\right] \\
&\quad + (1/16)\phi(z_{1-\alpha})m^{-2}\Big\{z_{1-\alpha}[13\theta - (11/2) - (15/2)\theta^2] + z^3_{1-\alpha}[7\theta - (23/6) - (9/2)\theta^2] \\
&\qquad\qquad + z^5_{1-\alpha}[6\theta - (5/3) - (3/2)\theta^2] - z^7_{1-\alpha}(1/2)\theta^2\Big\} + O(m^{-3}) \\
&= 1 - \alpha \\
&\quad + m^{-1}\phi(z_{1-\alpha})\Big\{c_1 + (1/4)\left[z_{1-\alpha}(1-\theta) - \theta z^3_{1-\alpha}\right]\Big\} \\
&\quad + m^{-2}\phi(z_{1-\alpha})\Big\{c_2 - (1/2)c_1^2 z_{1-\alpha} - (1/4)c_1 z_{1-\alpha}\left[z_{1-\alpha}(1-\theta) - \theta z^3_{1-\alpha}\right] \\
&\qquad + (1/4)\left[c_1(1-\theta) - 3c_1\theta z^2_{1-\alpha}\right] \\
&\qquad + (1/16)z_{1-\alpha}[13\theta - (11/2) - (15/2)\theta^2] \\
&\qquad + (1/16)z^3_{1-\alpha}[7\theta - (23/6) - (9/2)\theta^2] \\
&\qquad + (1/16)z^5_{1-\alpha}[6\theta - (5/3) - (3/2)\theta^2] - z^7_{1-\alpha}(1/32)\theta^2\Big\} \\
&\quad + O(m^{-3}).
\end{aligned}$$

Zeroing the m^{-1} term,

$$\begin{aligned}
0 &= c_1 + (1/4)[z_{1-\alpha}(1-\theta) - \theta z^3_{1-\alpha}], \\
c_1 &= (1/4)[\theta z^3_{1-\alpha} - z_{1-\alpha}(1-\theta)],
\end{aligned}$$

same as before (as it should be).

Plugging in and zeroing the m^{-2} term,

$$\begin{aligned}
c_2 &= (1/2)z_{1-\alpha}(1/16)[\theta z^3_{1-\alpha} - z_{1-\alpha}(1-\theta)]^2 + (1/4)(1/4)[\theta z^3_{1-\alpha} - z_{1-\alpha}(1-\theta)][z^2_{1-\alpha}(1-\theta) - \theta z^4_{1-\alpha}] \\
&\quad - (1/4)(1/4)[\theta z^3_{1-\alpha} - z_{1-\alpha}(1-\theta)](1-\theta) + (3/4)(1/4)[\theta z^3_{1-\alpha} - z_{1-\alpha}(1-\theta)]\theta z^2_{1-\alpha} \\
&\quad - (1/16)z_{1-\alpha}[13\theta - (11/2) - (15/2)\theta^2] - (1/16)z^3_{1-\alpha}[7\theta - (23/6) - (9/2)\theta^2] \\
&\quad - (1/16)z^5_{1-\alpha}[6\theta - (5/3) - (3/2)\theta^2] + z^7_{1-\alpha}(1/32)\theta^2 \\
&= (1/32)[\theta^2 z^7_{1-\alpha} + (1-\theta)^2 z^3_{1-\alpha} - 2\theta(1-\theta)z^5_{1-\alpha}] \\
&\quad + (1/16)[\theta(1-\theta)z^5_{1-\alpha} + \theta(1-\theta)z^5_{1-\alpha} - (1-\theta)^2 z^3_{1-\alpha} - \theta^2 z^7_{1-\alpha}] \\
&\quad - (1/16)[\theta(1-\theta)z^3_{1-\alpha} - (1-\theta)^2 z_{1-\alpha}] \\
&\quad + (1/16)[3\theta^2 z^5_{1-\alpha} - 3\theta(1-\theta)z^3_{1-\alpha}] \\
&\quad - (1/16)z_{1-\alpha}[13\theta - (11/2) - (15/2)\theta^2] - (1/16)z^3_{1-\alpha}[7\theta - (23/6) - (9/2)\theta^2] \\
&\quad - (1/16)z^5_{1-\alpha}[6\theta - (5/3) - (3/2)\theta^2] + z^7_{1-\alpha}(1/32)\theta^2
\end{aligned}$$

$$
\begin{aligned}
&= (1/32)\Big\{z_{1-\alpha}\left[2(1-\theta)^2 - 26\theta + 11 + 15\theta^2\right] \\
&\qquad + z_{1-\alpha}^3\left[(1-\theta)^2 - 2(1-\theta)^2 - 2\theta(1-\theta) - 6\theta(1-\theta) - 14\theta + (23/3) + 9\theta^2\right] \\
&\qquad + z_{1-\alpha}^5\left[-2\theta(1-\theta) + 4\theta(1-\theta) + 6\theta^2 - 12\theta + (10/3) + 3\theta^2\right] \\
&\qquad + z_{1-\alpha}^7\left[\theta^2 - 2\theta^2 + \theta^2\right]\Big\} \\
&= (1/32)\Big\{z_{1-\alpha}\left[17\theta^2 - 30\theta + 13\right] + z_{1-\alpha}^3\left[16\theta^2 - 20\theta + (20/3)\right] \\
&\qquad + z_{1-\alpha}^5\left[7\theta^2 - 10\theta + (10/3)\right]\Big\}.
\end{aligned}
$$

With $\theta = 1$, this matches the one-sample c_1 and c_2, as it should.

Appendix E. Type I error

As shown above, the type I error when using critical value $z_{\alpha,m}$ is

$$\alpha - 2(m/n)^2 u_{3,0}(z_{1-\alpha/2})\phi(z_{1-\alpha/2}) + o(m^{-1} + m^2/n^2),$$

where the dominant term $e_I = \alpha - 2(m/n)^2 u_{3,0}(z_{1-\alpha/2})\phi(z_{1-\alpha/2}) \le \alpha$ for all reasonably common distributions, as discussed above.

Appendix F. Type II error

F.1. **With ideal corrected critical value.** As in the univariate case, I calculate type II error against the alternative hypothesis that yields 50% power under the first-order asymptotic distribution; i.e., $0.5 = P(|T_{m,n}| < z_{1-\alpha/2}) \doteq G_{C^2}(z_{1-\alpha/2}^2)$ using the same notation as before. For $\alpha = 0.05$, $C = \pm 1.96$. Using $z_{\alpha,m}$, under this alternative hypothesis,

$$
\begin{aligned}
P(|T_{m,n}| < z_{\alpha,m}) &= P(T_{m,n} < z_{\alpha,m}) - P(T_{m,n} < -z_{\alpha,m}) \\
&= \Phi(z_{\alpha,m} + C) \\
&\quad + \phi(z_{\alpha,m} + C)\Big[n^{-1/2}u_{1,\gamma}(z_{\alpha,m} + C) + m^{-1}u_{2,\gamma}(z_{\alpha,m} + C) \\
&\qquad\qquad + (m/n)^2 u_{3,\gamma}(z_{\alpha,m} + C)\Big] + o(m^{-1} + (m/n)^2) \\
&\quad - \Phi(-z_{\alpha,m} + C) \\
&\quad - \phi(-z_{\alpha,m} + C)\Big[n^{-1/2}u_{1,\gamma}(-z_{\alpha,m} + C) + m^{-1}u_{2,\gamma}(-z_{\alpha,m} + C) \\
&\qquad\qquad + (m^2/n^2)u_{3,\gamma}(-z_{\alpha,m} + C)\Big] + o(m^{-1} + (m/n)^2).
\end{aligned}
$$

I can write $O(n^{-1/2})$ for the $n^{-1/2}$ terms since they do not depend on m, and thus they will not affect the optimization problem to select m.

If the alternatives $+C$ and $-C$ each have 0.5 probability, the average power can be calculated. Noticing that $\phi(-x) = \phi(x)$, this would yield

$$
\begin{aligned}
P(|T_{m,n}| < z_{\alpha,m}) = \frac{1}{2}\Big\{&\Phi(z_{\alpha,m} + C) - \Phi(-z_{\alpha,m} + C) + \Phi(z_{\alpha,m} - C) - \Phi(-z_{\alpha,m} - C) \\
&+ \phi(z_{\alpha,m} + C)\big[m^{-1}(u_{2,\gamma}(z_{\alpha,m} + C) - u_{2,\gamma}(-z_{\alpha,m} - C)) \\
&\qquad + (m/n)^2(u_{3,\gamma}(z_{\alpha,m} + C) - u_{3,\gamma}(-z_{\alpha,m} - C))\big] \\
&+ \phi(z_{\alpha,m} - C)\Big[m^{-1}(u_{2,\gamma}(z_{\alpha,m} - C) - u_{2,\gamma}(-z_{\alpha,m} + C)) \\
&\qquad + \frac{m^2}{n^2}(u_{3,\gamma}(z_{\alpha,m} - C) - u_{3,\gamma}(-z_{\alpha,m} + C))\Big]
\end{aligned}
$$

$$+ O(n^{-1/2}) + o(m^{-1} + (m/n)^2)\Bigg\}, \tag{19}$$

noting that wherever C enters any of the $u_{i,\gamma}$ functions as $-C$, as in $u_{2,\gamma}(-z_{\alpha,m} - C)$, likewise the γ in the definition of $u_{i,\gamma}$ is negative, specifically $-C\sqrt{p(1-p)}/f(\xi_p)$ as given before.

For the first-order terms,

$$\begin{aligned}
\Phi(z_{\alpha,m}&+C) - \Phi(-z_{\alpha,m} + C)\\
&= \Phi(C + z_{1-\alpha/2} + m^{-1}[4S_0^4]^{-1}\left[(g_x^4 + g_y^4)z_{1-\alpha/2}^3 - 2g_x^2 g_y^2 z_{1-\alpha/2}\right])\\
&\quad - \Phi(C - z_{1-\alpha/2} - m^{-1}[4S_0^4]^{-1}\left[(g_x^4 + g_y^4)z_{1-\alpha/2}^3 - 2g_x^2 g_y^2 z_{1-\alpha/2}\right])\\
&= \Phi(C + z_{1-\alpha/2})\\
&\qquad + \phi(z_{1-\alpha/2} + C)m^{-1}[4S_0^4]^{-1}\left[(g_x^4 + g_y^4)z_{1-\alpha/2}^3 - 2g_x^2 g_y^2 z_{1-\alpha/2}\right] + O(m^{-2})\\
&\quad - \Phi(C - z_{1-\alpha/2})\\
&\qquad - \phi(C - z_{1-\alpha/2})(-1)m^{-1}[4S_0^4]^{-1}\left[(g_x^4 + g_y^4)z_{1-\alpha/2}^3 - 2g_x^2 g_y^2 z_{1-\alpha/2}\right] + O(m^{-2})\\
&= [\Phi(z_{1-\alpha/2} + C) - \Phi(-z_{1-\alpha/2} + C)]\\
&\quad + m^{-1}[4S_0^4]^{-1}\left[(g_x^4 + g_y^4)z_{1-\alpha/2}^3 - 2g_x^2 g_y^2 z_{1-\alpha/2}\right](\phi(z_{1-\alpha/2} + C) + \phi(C - z_{1-\alpha/2}))\\
&= 0.5 + m^{-1}[4S_0^4]^{-1}\left[(g_x^4 + g_y^4)z_{1-\alpha/2}^3 - 2g_x^2 g_y^2 z_{1-\alpha/2}\right][\phi(z_{1-\alpha/2} + C) + \phi(C - z_{1-\alpha/2})]\\
&= 0.5 + \tfrac{1}{4}m^{-1}\left[\theta z_{1-\alpha/2}^3 - (1-\theta)z_{1-\alpha/2}\right][\phi(z_{1-\alpha/2} + C) + \phi(z_{1-\alpha/2} - C)]
\end{aligned}$$

since above C was chosen to solve $0.5 = \Phi(z_{1-\alpha/2} + C) - \Phi(-z_{1-\alpha/2} + C)$. Since $\Phi(x) = 1 - \Phi(-x)$, then $\Phi(z + C) - \Phi(-z + C) = 1 - \Phi(-z - C) - (1 - \Phi(z - C)) = \Phi(z - C) - \Phi(-z - C)$, so

$$\begin{aligned}
\Phi(z_{\alpha,m}&-C) - \Phi(-z_{\alpha,m} - C) = \Phi(z_{\alpha,m} + C) - \Phi(-z_{\alpha,m} + C)\\
&= 0.5 + \tfrac{1}{4}m^{-1}\left[\theta z_{1-\alpha/2}^3 - (1-\theta)z_{1-\alpha/2}\right][\phi(z_{1-\alpha/2} + C) + \phi(z_{1-\alpha/2} - C)].
\end{aligned}$$

Within the m^{-1} and $(m/n)^2$ terms, anything $o(1)$ will end up in the remainder (not in e_{II}). So for the corrected $z_{\alpha,m}$, $z_{\alpha,m} = z_{1-\alpha/2} + O(m^{-1})$, $z_{\alpha,m}^2 = z_{1-\alpha/2}^2 + O(m^{-1})$, and $\phi(z_{\alpha,m} + C) = \phi(z_{1-\alpha/2} + C) + \phi'(z_{1-\alpha/2} + C)O(m^{-1}) + \ldots = \phi(z_{1-\alpha/2} + C) + O(m^{-1})$.

For the m^{-1} terms, since $\phi(x) = \phi(-x)$ and $C \equiv \gamma f(\xi_p)/\sqrt{p(1-p)}$, and letting $d_1 \equiv z + C$, $d_2 \equiv z - C$, and again $\theta \equiv S_0^{-4}(f_X^{-4} + f_Y^{-4})$,

$$\begin{aligned}
u_{2,\gamma}(d_1) - u_{2,\gamma}(-d_1) &= 2u_{2,\gamma}(d_1),\\
u_{2,\gamma}(d_1) &= -\tfrac{1}{4}\theta(z + C)^3 + \tfrac{1}{2}(1-\theta)(z + C) - \tfrac{1}{2}(1-\theta)C\\
&\quad + \tfrac{1}{4}\theta(2C(z + C)^2 - C^2(z + C))\\
&= \tfrac{1}{4}\theta(z + C)(-(z + C)^2 + 2C(z + C) - C^2) + \tfrac{1}{2}(1-\theta)(z + C - C)\\
&= \tfrac{1}{4}\theta(z + C)[-z^2 - 2Cz - C^2 + 2Cz + 2C^2 - C^2] + \tfrac{1}{2}(1-\theta)z\\
&= \tfrac{1}{4}\theta(z + C)(-z^2) + \tfrac{1}{2}(1-\theta)z,\\
u_{2,\gamma}(-d_2) &= -\tfrac{1}{4}\theta(-z + C)^3 + \tfrac{1}{2}(1 - \theta(-z + C) - \tfrac{1}{2}(1-\theta)C\\
&\quad + \tfrac{1}{4}\theta(2C(-z + C)^2 - C^2(-z + C))\\
&= \tfrac{1}{4}\theta(-z + C)[-(-z + C)^2 + 2C(-z + C) - C^2] + \tfrac{1}{2}(1-\theta)(-z + C - C)\\
&= \tfrac{1}{4}\theta(-z + C)(-z^2) - \tfrac{1}{2}(1-\theta)z,\\
u_{2,\gamma}(d_2) &= \tfrac{1}{4}\theta(-z + C)z^2 + \tfrac{1}{2}(1-\theta)z,
\end{aligned}$$

$$u_{3,\gamma}(d_1) - u_{3,\gamma}(-d_1) = \frac{g_x g_x'' + g_y g_y''}{6S_0^2}[d_1 - C - (-d_1 - (-C))] = \frac{g_x g_x'' + g_y g_y''}{6S_0^2} 2z,$$
$$u_{3,\gamma}(d_2) - u_{3,\gamma}(-d_2) = \frac{g_x g_x'' + g_y g_y''}{6S_0^2}[d_2 - (-C) - (-d_2 - C)] = \frac{g_x g_x'' + g_y g_y''}{6S_0^2} 2z,$$

and thus altogether,

$$\begin{aligned}
P(|T_{m,n}| < z_{\alpha,m}) &= 0.5 \\
&\quad + \tfrac{1}{4}m^{-1}\big\{\phi(z_{1-\alpha/2} + C)[\theta z_{1-\alpha/2}^3 - (1-\theta)z_{1-\alpha/2} \\
&\qquad\qquad - \theta(z_{1-\alpha/2} + C)z_{1-\alpha/2}^2 + 2(1-\theta)z_{1-\alpha/2}] \\
&\qquad + \phi(z_{1-\alpha/2} - C)[\theta z_{1-\alpha/2}^3 - (1-\theta)z_{1-\alpha/2} \\
&\qquad\qquad + \theta(-z_{1-\alpha/2} + C)z_{1-\alpha/2}^2 + 2(1-\theta)z_{1-\alpha/2}]\big\} \\
&\quad + (m/n)^2 \frac{g_x g_x'' + g_y g_y''}{6S_0^2} z_{1-\alpha/2}\big\{\phi(z_{1-\alpha/2} + C) + \phi(z_{1-\alpha/2} - C)\big\} \\
&\quad + O(n^{-1/2}) + o(m^{-1} + m^2/n^2) \\
&= 0.5 \\
&\quad + \tfrac{1}{4}m^{-1}\big\{\phi(z_{1-\alpha/2} + C)[-\theta C z_{1-\alpha/2}^2 + (1-\theta)z_{1-\alpha/2}] \\
&\qquad + \phi(z_{1-\alpha/2} - C)[\theta C z_{1-\alpha/2}^2 + (1-\theta)z_{1-\alpha/2}]\big\} \\
&\quad + (m/n)^2 \frac{g_x g_x'' + g_y g_y''}{6S_0^2} z_{1-\alpha/2}\big\{\phi(z_{1-\alpha/2} + C) + \phi(z_{1-\alpha/2} - C)\big\} \\
&\quad + O(n^{-1/2}) + o(m^{-1} + m^2/n^2),
\end{aligned}$$

where as $\theta \to 1$ (and either $g_x \to 0$ or $g_y \to 0$) this approaches the univariate expression. The terms depending on m are

$$\begin{aligned}
&\tfrac{1}{4}m^{-1}\big\{\phi(z_{1-\alpha/2} + C)[-\theta C z_{1-\alpha/2}^2 + (1-\theta)z_{1-\alpha/2}] + \phi(z_{1-\alpha/2} - C)[\theta C z_{1-\alpha/2}^2 + (1-\theta)z_{1-\alpha/2}]\big\} \\
&+ (m/n)^2 \left\{\frac{g_x g_x'' + g_y g_y''}{6S_0^2} z_{1-\alpha/2}[\phi(z_{1-\alpha/2} + C) + \phi(z_{1-\alpha/2} - C)]\right\} \\
&+ o(m^{-1} + m^2/n^2) \\
&\quad = \tfrac{1}{4}m^{-1}\{A\} + (m/n)^2\{B\},
\end{aligned}$$

and the FOC is

$$\begin{aligned}
0 &= -\tfrac{1}{4}m^{-2}\{A\} + 2(m/n^2)\{B\}, \\
m_K &= \sqrt[3]{n^2 A/(8B)} \\
&= \Bigg\{ n^2\big\{\phi(z_{1-\alpha/2} + C)[-\theta C z_{1-\alpha/2}^2 + (1-\theta)z_{1-\alpha/2}] \\
&\qquad + \phi(z_{1-\alpha/2} - C)[\theta C z_{1-\alpha/2}^2 + (1-\theta)z_{1-\alpha/2}]\big\} \\
&\qquad / \left[8\frac{g_x g_x'' + g_y g_y''}{6S_0^2} z_{1-\alpha/2}[\phi(z_{1-\alpha/2} + C) + \phi(z_{1-\alpha/2} - C)]\right] \Bigg\}^{1/3} \\
&= n^{2/3}(3/4)^{1/3} z_{1-\alpha/2}^{-1/3} z_{1-\alpha/2}^{1/3} \left(\frac{S_0^2}{g_x g_x'' + g_y g_y''}\right)^{1/3} \\
&\quad \times \Bigg\{ \big\{\phi(z_{1-\alpha/2} + C)[-\theta C z_{1-\alpha/2} + 1 - \theta] + \phi(z_{1-\alpha/2} - C)[\theta C z_{1-\alpha/2} + 1 - \theta]\big\}
\end{aligned}$$

$$/\left[\phi(z_{1-\alpha/2}+C)+\phi(z_{1-\alpha/2}-C)\right]\Big\}^{1/3}$$

$$
\begin{aligned}
&= n^{2/3}(3/4)^{1/3}\left(\frac{S_0^2}{g_x g_x'' + g_y g_y''}\right)^{1/3} \\
&\quad \times \left\{1-\theta+\theta C z_{1-\alpha/2}\frac{\phi(z_{1-\alpha/2}-C)-\phi(z_{1-\alpha/2}+C)}{\phi(z_{1-\alpha/2}-C)+\phi(z_{1-\alpha/2}+C)}\right\}^{1/3} \\
&= n^{2/3}(3/4)^{1/3}\left(\frac{S_0^2}{g_x g_x'' + g_y g_y''}\right)^{1/3} \\
&\quad \times \left\{\frac{2g_x^2 g_y^2}{S_0^4}+\frac{g_x^4+g_y^4}{S_0^4}C z_{1-\alpha/2}\frac{\phi(z_{1-\alpha/2}-C)-\phi(z_{1-\alpha/2}+C)}{\phi(z_{1-\alpha/2}-C)+\phi(z_{1-\alpha/2}+C)}\right\}^{1/3}.
\end{aligned}
$$

There are a few options now for the unknown g: Gaussian plug-in, estimate from data, shrinking an estimate from the data toward the Gaussian value, or some combination thereof.

F.2. **With univariate corrected critical value (conservative type I error).** This subsection is the same as F.1 but with the univariate $z_{\alpha,m}$. The general expression (19) is the same starting point.

For the first-order terms, it's the same as the univariate case,

$$
\begin{aligned}
\Phi(z_{\alpha,m}+C)-\Phi(-z_{\alpha,m}+C) &= \Phi(z_{\alpha,m}-C)-\Phi(-z_{\alpha,m}-C) \\
&= 0.5+m^{-1}\frac{1}{4}z_{1-\alpha/2}^3[\phi(z_{1-\alpha/2}+C)+\phi(C-z_{1-\alpha/2})].
\end{aligned}
$$

Within the m^{-1} and $(m/n)^2$ terms, anything $o(1)$ will end up in the remainder (not in e_{II}). Since only the $z_{1-\alpha/2}$ part of $z_{\alpha,m}$ remains, these terms are the same as in Section F.1. Altogether,

$$
\begin{aligned}
P(|T_{m,n}|<z_{\alpha,m}) &= 0.5 \\
&\quad + \tfrac{1}{4}m^{-1}\big\{\phi(z_{1-\alpha/2}+C)[z_{1-\alpha/2}^3-\theta(z_{1-\alpha/2}+C)z_{1-\alpha/2}^2+2(1-\theta)z_{1-\alpha/2}] \\
&\qquad + \phi(z_{1-\alpha/2}-C)[z_{1-\alpha/2}^3+\theta(-z_{1-\alpha/2}+C)z_{1-\alpha/2}^2+2(1-\theta)z_{1-\alpha/2}]\big\} \\
&\quad + (m/n)^2\frac{g_x g_x''+g_y g_y''}{6S_0^2}z_{1-\alpha/2}\big\{\phi(z_{1-\alpha/2}+C)+\phi(z_{1-\alpha/2}-C)\big\} \\
&\quad + O(n^{-1/2})+o(m^{-1}+m^2/n^2) \\
&= 0.5 \\
&\quad + \tfrac{1}{4}m^{-1}z_{1-\alpha/2}\big\{\phi(z_{1-\alpha/2}+C)[(1-\theta)(z_{1-\alpha/2}^2+2)-\theta C z_{1-\alpha/2}] \\
&\qquad + \phi(z_{1-\alpha/2}-C)[(1-\theta)(z_{1-\alpha/2}^2+2)+\theta C z_{1-\alpha/2}]\big\} \\
&\quad + (m/n)^2\frac{g_x g_x''+g_y g_y''}{6S_0^2}z_{1-\alpha/2}\big\{\phi(z_{1-\alpha/2}+C)+\phi(z_{1-\alpha/2}-C)\big\} \\
&\quad + O(n^{-1/2})+o(m^{-1}+m^2/n^2) \\
&= \tfrac{1}{4}m^{-1}z_{1-\alpha/2}A+(m/n)^2 z_{1-\alpha/2}B+1/2+O(n^{-1/2})+o(m^{-1}+m^2/n^2).
\end{aligned}
$$

To find the m that minimizes the type II error, the FOC is

$$
\begin{aligned}
0 &= -\tfrac{1}{4}m^{-2}A+2mn^{-2}B, \\
m_K &= \sqrt[3]{An^2/(8B)} = n^{2/3}\sqrt[3]{A/(8B)} \\
&= n^{2/3}(3/4)^{1/3}\left(\frac{S_0^2}{g_x g_x''+g_y g_y''}\right)^{1/3}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Bigg\{ \big\{ \phi(z_{1-\alpha/2} + C)[(1-\theta)(z_{1-\alpha/2}^2 + 2) - \theta C z_{1-\alpha/2}] \\
&\qquad + \phi(z_{1-\alpha/2} - C)[(1-\theta)(z_{1-\alpha/2}^2 + 2) + \theta C z_{1-\alpha/2}] \big\} \\
&\qquad / \left[\phi(z_{1-\alpha/2} + C) + \phi(z_{1-\alpha/2} - C) \right] \Bigg\}^{1/3} \\
&= n^{2/3} (3/4)^{1/3} \left(\frac{S_0^2}{g_x g_x'' + g_y g_y''} \right)^{1/3} \\
&\times \left\{ (1-\theta)(z_{1-\alpha/2}^2 + 2) + \theta C z_{1-\alpha/2} \frac{\phi(z_{1-\alpha/2} - C) - \phi(z_{1-\alpha/2} + C)}{\phi(z_{1-\alpha/2} - C) + \phi(z_{1-\alpha/2} + C)} \right\}^{1/3} \\
&= n^{2/3} (3/4)^{1/3} \left(\frac{S_0^2}{g_x g_x'' + g_y g_y''} \right)^{1/3} \\
&\times \left\{ \frac{2 g_x^2 g_y^2}{S_0^4} (z_{1-\alpha/2}^2 + 2) + \frac{g_x^4 + g_y^4}{S_0^4} C z_{1-\alpha/2} \frac{\phi(z_{1-\alpha/2} - C) - \phi(z_{1-\alpha/2} + C)}{\phi(z_{1-\alpha/2} - C) + \phi(z_{1-\alpha/2} + C)} \right\}^{1/3} .
\end{aligned}
$$

Again, estimates from the data, assumptions, or both can be plugged into this expression to calculate m.

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