1. Location:

   Conservation Auditorium  
   (Anheuser-Busch Natural Resources Building)

   Time:
   8:00pm–10:00pm, Wednesday, May 11, 2016.

2. Exam papers will be handed out as in the three Tests.

3. Your student I.D. card will be required.

4. You can bring a 3x5 note-card with you, on which you may write anything you want. This card should be handed in to me at the same time you hand in your exam paper. No other papers will be allowed. In addition, no electronic devices such as calculators and cell phones will be allowed.

5. The office hours before the final exam are:
   9:00am–11:00am, May 6
   9:00am–11:30am, May 11

   You may also email me your questions. If you do, copy down the original problem in details and state your questions clearly.

6. It is a comprehensive final examination. You are responsible for the following materials (inclusive):
   (1) §12.1—12.5;
   (2) §13.1—13.4;
   (3) §14.1—14.8 (excluding §14.2);
   (4) §15.1—15.10 (excluding §15.6);
   (5) §16.1—16.9.
QUIZ 10—MATHEMATICS 2300

Your last name: ____________________________

Your first name: ____________________________

1. Let $C$ be the rectangle with vertices $(0, 0), (2, 0), (2, 3), (0, 3)$ oriented counter-clock-wise. Use Green’s Theorem to compute the line integral:

$$\int_C y\, dx + x\, dy.$$

**SOLUTION:** Let $R$ be the rectangular region. Then, $C = (\partial R)^+$. So by Green’s Theorem,

$$\int_C y\, dx + x\, dy = \int \int_R (1 - 1)\, dx\, dy = 0.$$

---

2. Let $\mathbf{F}(x, y, z) = (yz, xz, xy + 2z)$, and $C$ be the line segment from $(1, 0, -2)$ to $(4, 6, 3)$.

(i) Show that $\mathbf{F}$ is conservative;

(ii) Find scalar functions $f(x, y, z)$ such that $\mathbf{F} = \nabla f$;

(iii) Compute the integral of $\mathbf{F}$ along the path $C$.

**SOLUTION:**

(i) We check directly that $\nabla \times \mathbf{F} = \mathbf{0}$. So $\mathbf{F}$ is conservative.

(ii) We have

$$f = \int_0^x F_1(t, 0, 0)\, dt + \int_0^y F_2(x, t, 0)\, dt + \int_0^z F_3(x, y, t)\, dt + C$$

$$= \int_0^z (xy + 2t)\, dt + C$$

$$= xyz + z^2 + C.$$  

(iii) By the Fundamental Theorem of Line Integrals, we have

$$\int_C \mathbf{F} \cdot ds = \int_C \nabla f \cdot ds = f(4, 6, 3) - f(1, 0, -2) = 77.$$
1. Let \( f(x, y, z) = x + y + z \), and \( \mathbf{r} : [0, 2] \rightarrow \mathbb{R}^3 \) be the line segment defined by \( \mathbf{r}(t) = (t, 2t, 2t) \). Compute the integral of \( f \) along the path \( \mathbf{r} \).

**SOLUTION:**
Note that \( \mathbf{r}'(t) = (1, 2, 2) \). So \( \|\mathbf{r}'(t)\| = 3 \). We have
\[
\int_{\mathbf{r}} f \, ds = \int_{a}^{b} f(x(t), y(t), z(t)) \|\mathbf{r}'(t)\| \, dt = \int_{0}^{2} (t + 2t + 2t) \cdot 3 \, dt = 15 \int_{0}^{2} t \, dt = 15 \left( \frac{t^2}{2} \right) \bigg|_{0}^{2} = 30.
\]

2. Let \( \mathbf{F}(x, y, z) = (x, y, z) \), and \( \mathbf{r} : [0, 2] \rightarrow \mathbb{R}^3 \) be the line segment defined by \( \mathbf{r}(t) = (t, 2t, 2t) \). Compute the integral of \( \mathbf{F} \) along the path \( \mathbf{r} \).

**SOLUTION:**
Again, \( \mathbf{r}'(t) = (1, 2, 2) \). We have
\[
\int_{\mathbf{r}} \mathbf{F} \cdot ds = \int_{a}^{b} \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) \, dt = \int_{0}^{2} (1, 2t, 2t) \cdot (1, 2, 2) \, dt = \int_{0}^{2} 9t \, dt = \frac{9}{2} t^2 \bigg|_{0}^{2} = 18.
\]
QUIZ 8—MATHEMATICS 2300

Your last name: ______________________________________________________________________

Your first name: ____________________________________________________________________

1. Compute \( \iiint_{B^*} \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz \) where \( B^* \) is the solid region:

\[
4 \leq x^2 + y^2 + z^2 \leq 9.
\]

SOLUTION:

Use spherical coordinates. The solid region \( R \) in the \( \rho \theta \phi \)-space is:

\[
2 \leq \rho \leq 3, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.
\]

Note that \( J = -\rho^2 \sin \phi \) and \( \sqrt{x^2 + y^2 + z^2} = \rho \). So we have

\[
\iiint_{B^*} \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz = \iiint_{B} \rho \cdot \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
\]

\[
= \int_{2}^{3} \int_{0}^{2\pi} \int_{0}^{\pi} \rho^3 \sin \phi \, d\phi \, d\theta \, d\rho
\]

\[
= 2 \int_{2}^{3} \int_{0}^{2\pi} \rho^3 \, d\theta \, d\rho
\]

\[
= 4\pi \int_{2}^{3} \rho^3 \, d\rho
\]

\[
= 65\pi.
\]
1. Compute the double integral \( \iint_{R^*} \sqrt{x^2 + y^2} \, dx \, dy \) where \( R^* \) is the region:

\[
x^2 + y^2 \leq 9, \quad x \geq 0, \quad y \geq 0.
\]

**SOLUTION:**

Use polar coordinates. We have \( R = [0, 3] \times [0, \pi/2] \). Since \( x^2 + y^2 = r^2 \), we have

\[
\iint_{R^*} \sqrt{x^2 + y^2} \, dx \, dy = \iint_{R} \sqrt{r^2} \cdot r \, dr \, d\theta = \frac{9\pi}{2}.
\]

2. Compute the double integral \( \iint_{R} f \) where \( f(x, y) = x^2 + y^2 \) and \( R = [0, 1] \times [0, 6] \).

**SOLUTION:**

\[
\iint_{R} f = \int_{0}^{1} \left[ \int_{0}^{6} (x^2 + y^2) \, dy \right] \, dx \\
= \int_{0}^{1} \left[ x^2y + y^3/3 \right]_{0}^{6} \, dx \\
= \int_{0}^{1} (6x^2 + 72) \, dx \\
= (2x^3 + 72x)\bigg|_{0}^{1} \\
= 74.
\]
1. Using 2nd derivatives test, find all the local extremal values of
\[ f(x, y) = x^2 + y^2 - 2016. \]

SOLUTION:

Step 1. Find all the critical points.
We have \( \frac{\partial f}{\partial x} = 2x \) and \( \frac{\partial f}{\partial y} = 2y \). So \( \frac{\partial f}{\partial x} = 0 \) and \( \frac{\partial f}{\partial y} = 0 \) forces \( x = 0 \) and \( y = 0 \). This says that the only critical point of \( f \) is the origin \((0, 0)\).

Step 2. Compute the Hessian.
We have \( \frac{\partial^2 f}{\partial x^2} = 2 \), \( \frac{\partial^2 f}{\partial y^2} = 2 \), and \( \frac{\partial^2 f}{\partial x \partial y} = 0 \). So \( H_f = 4 \).

Step 3. Use the 2nd derivatives test.
We have \( H_f(0, 0) = 4 > 0 \) and \( \frac{\partial^2 f}{\partial x^2}(0, 0) = 2 > 0 \). So \( f \) has a local minimum at \((0, 0)\), and the local minimum is \( f(0, 0) = -2016 \).
1. Compute the gradient of \( f(x, y, z) = x^2 \cos(8y^3z) \).

\textbf{SOLUTION:}
\[
\nabla f = (2x \cos(8y^3z), -24x^2y^2z \sin(8y^3z), -8x^2y^3 \sin(8y^3z)).
\]
1. Let \( f(x, y, z) = x^3z^5 + x^2\sin(8y^3z) \). Compute \( \partial f / \partial y \).

**SOLUTION:**

\[ \partial f / \partial y = 24x^2y^2z\cos(8y^3z). \]
1. Find the tangent vector $\mathbf{r}'(t)$ of the following curve:

$$\mathbf{r}(t) = (3t, e^{5t}, \cos(2t)), \quad t \in [-\pi, \pi].$$

**SOLUTION:**

We have $\mathbf{r}'(t) = (3, 5e^{5t}, -2\sin(2t))$. 
1. Let $X = (1, 0, 0)$ and $Y = (2, 1, 0)$. Compute $X \times Y$.

**SOLUTION:**
We have $X \times Y = k$. 
1. Write down the equation of the sphere with radius 2 and center \( C = (1, 0, 0) \).

**SOLUTION:**

The equation is:

\[
(x - 1)^2 + y^2 + z^2 = 2^2.
\]