12.1 3D-coordinate systems

The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$\text{dist}(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

An equation of a sphere with center at $C(h, k, l)$ and radius $r$

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

12.2 Vectors

The length of the vector $a = (a_1, a_2, a_3)$

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Properties

1. $a + b = b + a$
2. $a + (b + c) = (a + b) + c$
3. $a + 0 = a$
4. $a + (-a) = 0$
5. $\lambda(a + b) = \lambda a + \lambda b$
6. $(\lambda + \mu)(a) = \lambda a + \mu a$
7. $(\lambda \mu)a = \lambda(\mu a)$
8. $1a = a$

A unit vector in the direction of $a$

$$\mathbf{u} = \frac{1}{|a|} \mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$
12.3 The dot product

Given \( a = (a_1, a_2, a_3) \) and \( b = (b_1, b_2, b_3) \)
\[
a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3
\]

Properties

1. \( a \cdot a = |a|^2 \)
2. \( a \cdot b = b \cdot a \)
3. \( a \cdot (b + c) = a \cdot b + a \cdot c \)
4. \( \lambda a \cdot b = \lambda (a \cdot b) = a \cdot (\lambda b) \)
5. \( 0 \cdot a = 0 \)

Theorem
\[
a \cdot b = |a||b| \cos \theta
\]

Definition Direction numbers and direction cosines
\[
\begin{align*}
\cos \alpha &= \frac{a \cdot i}{|a||i|} = \frac{a_1}{|a|} \\
\cos \beta &= \frac{a \cdot j}{|a||j|} = \frac{a_2}{|a|} \\
\cos \gamma &= \frac{a \cdot k}{|a||k|} = \frac{a_3}{|a|}
\end{align*}
\]

Projections

Definition The scalar projection of \( b \) onto \( a \)
\[
\text{comp}_a b = |b| \cos \theta = \frac{a \cdot b}{|a|}
\]

Definition The vector projection of \( b \) onto \( a \)
\[
\text{proj}_a b = \text{comp}_a b \frac{a}{|a|} = \left( \frac{a \cdot b}{|a|^2} \right) a
\]
12.4 The cross product

\[ a \times b = \det \begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \]

**Theorem**

\[ a \times b \perp a, b \]

**Corollary**

\[ a \times b = 0 \iff a \parallel b \]

**Theorem**

\[ |a \times b| = |a||b|\sin \theta \]

**Warning**

\[ (a \times b) \times c \neq a \times (b \times c) \]

**Theorem** The area \( A \) of the parallelogram determined by the vectors \( a \) and \( b \) is given by

\[ A = |a \times b| \]

**Properties**

1. \( a \times b = -b \times a \)
2. \( \lambda(a \times b) = (\lambda a) \times b = a \times (\lambda b) \)
3. \( a \times (b + c) = a \times b + a \times c \)
4. \( (a + b) \times c = a \times c + b \times c \)
5. \( a \cdot (b \times c) = (a \times b) \cdot c \)
6. \( a \times (b \times c) = (a \cdot c)b - (a \cdot b)c \)

**Remark**

\[ a \cdot (b \times c) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \]

**Theorem** The volume \( V \) of the parallelepiped determined by the vectors \( a, b, \) and \( c \) is given by

\[ V = |a \cdot (b \times c)| \]
12.5 Equations of lines and planes

**Lines**

Vector equation

\[ \mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \]

Parametric equation

\[
\begin{align*}
x &= x_0 + at \\
y &= y_0 + bt \\
z &= z_0 + ct
\end{align*}
\]

Symmetric equation

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}
\]

The line passing through the points \( P_0(x_0, y_0, z_0) \) and \( P_1(x_1, y_1, z_1) \)

\[
\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}
\]

The line segment from \( r_0 \) to \( r_1 \)

\[ r = (1 - t)r_0 + tr_1, \quad 0 \leq t \leq 1 \]

**Planes**

Vector equation

\[ (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0 \]

\[ \mathbf{n} = (a, b, c) \] is a normal vector

Linear equation

\[ ax + by + cz + d = 0 \] (\( \ast \))

Distance from the point \( P(X, Y, Z) \) to the plane given by equation (\( \ast \))

\[
\text{dist} = \frac{|aX + bY + cZ + d|}{\sqrt{a^2 + b^2 + c^2}}
\]
13.1 Vector functions and space curves

13.2 Derivatives and integrals of vector functions

Suppose that
\[ \mathbf{r}(t) = (f(t), g(t), h(t)) \]

**Definition**
\[ \mathbf{r}'(t) = (f'(t), g'(t), h'(t)) \]

**Properties**
1. \((u + v)' = u' + v'\)
2. \((cu)' = cu'\)
3. \((f(t)u)' = f'u + fu'\)
4. \((u \cdot v)' = u' \cdot v + u \cdot v'\)
5. \((u \times v)' = u' \times v + u \times v'\)
6. \((u (f(t)))' = f'(t)u' (f(t))\)

**Definition**
\[ \int_a^b \mathbf{r}(t) \, dt = \left( \int_a^b f(t) \, dt, \int_a^b g(t) \, dt, \int_a^b h(t) \, dt \right) \]
### 13.3 Arc length and curvature

**Definition**

**Arc length**

\[ L = \int_a^b |\mathbf{r}'(t)| \, dt \]

**Definition**

**Arc length function**

\[ s(t) = \int_a^t |\mathbf{r}'(t)| \, dt = \int_a^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \]

**Corollary**

\[ \frac{ds}{dt} = |\mathbf{r}'(t)| \]

**The (T, N, B)–frame**

**Definition**

The unit tangent vector

\[ \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \]

**Definition**

The principal normal vector

\[ \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \]

**Definition**

The binormal vector

\[ \mathbf{B}(t) = \mathbf{T} \times \mathbf{N} \]

**Definition** The plane determined by \( \mathbf{B} \) and \( \mathbf{N} \) is called the normal plane of a curve at a point.

**Definition** The plane determined by \( \mathbf{T} \) and \( \mathbf{N} \) is called the osculating plane of a curve at a point.
Curvature

**Definition** The curvature of a curve

\[ \kappa = \frac{dT}{ds} \]

where \( T \) is the unit tangent vector

**Lemma**

\[ \kappa = \frac{|T'(t)|}{|r'(t)|} \]

**Theorem**

\[ \kappa = \frac{|r' \times r''|}{|r'|^3} \]

The Frenet-Serret formulas

The \((T, N, B)\) principle frame describes the second-order (local) behavior of a curve

\[ \frac{dT}{ds} = \kappa N \quad (\kappa \text{-curvature}) \]
\[ \frac{dB}{ds} = -\tau N \quad (\tau \text{-torsion}) \]
\[ \frac{dN}{ds} = -\kappa T + \tau B \]

\[
\frac{d}{ds} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}
\]
13.4 Motion in Space. Velocity and acceleration

**Definition** The speed

\[ v = |v| = |r'| \]

**Theorem** Let \((T, N, B)\) be the principle frame. Then

\[ \vec{v} = v'T \]

**Remark** The normal and binormal components of velocity are always zero

**Definition** The acceleration

\[ a = v' = r'' \]

**Theorem** Let \((T, N, B)\) be the principle frame. Then

\[ \vec{a} = a_T T + a_N N \]

with

\[ a_T = v' = \frac{r' \cdot r''}{|r'|} \]  
(tangential component)

\[ a_N = \kappa v^2 = \frac{|r' \times r''|}{|r'|} \]  
(normal component)

**Remark** The binormal component of the acceleration is always zero
14.1 Functions of several variables

14.3 Partial derivatives

**Definition** Partial derivatives

\[
\begin{align*}
    f_x(x, y) &= \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \\
    f_y(x, y) &= \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}
\end{align*}
\]

**Geometric interpretation**

\( \frac{\partial f}{\partial x} (a, b) \) is the slope of the tangent line to the graph of \( f \) in the plane \( y = b \)

\( \frac{\partial f}{\partial y} (a, b) \) is the slope of the tangent line to the graph of \( f \) in the plane \( x = a \)

**Higher partial derivatives** \( f_{xx}, f_{yy}, f_{xy}, f_{yx} \) are introduced accordingly

**Theorem** If \( f_{xy} \) and \( f_{yx} \) are continuous, then

\[
f_{xy} = f_{yx}
\]
14.4 Tangent plane and linear approximation

The tangent plane equation

\[ z - z_0 = f_x(x - x_0) + f_y(y - y_0) \]

**Definition** The linearization of \( f(x, y) \) at \((a, b)\)

\[ z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \]

**Definition** A function \( f(x, y) \) is called differentiable if

\[ \left| f(a + \Delta x, b + \Delta y) - \left[ f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y \right] \right| = \epsilon_1|\Delta x| + \epsilon_2|\Delta y| \]

where \( \epsilon_{1,2} \to 0 \) as \( \Delta x, \Delta y \to 0 \)

**Theorem** If \( f_x \) and \( f_y \) are continuous, then \( f \) is differentiable
14.5 The chain rule

**Theorem** If \( z = f(x, y) \) and \( x = x(t) \) and \( y = y(t) \), then

\[
\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}
\]

**Consequence** If \( z = f(x, y) \) and \( x = x(s, t) \) and \( y = y(s, t) \), then

\[
\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}
\]

and

\[
\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
\]

More generally, if \( u = u(x_1, x_2, \ldots, x_n) \) and

\[
x_j = x_j(t_1, t_2, \ldots, t_m), \quad j = 1, 2, \ldots, n,
\]

then

\[
\frac{\partial u}{\partial t_j} = \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \frac{\partial x_k}{\partial t_j}, \quad j = 1, 2, \ldots, n.
\]

**Implicit Differentiation**

**The Implicit Function Theorem** If

\[
F(a, b, c) = 0,
\]

\( F_z(a, b, c) \neq 0 \), and \( F_x, F_y, F_z \) are continuous in a neighborhood of \((a, b, c)\), then the equation

\[
F(x, y, z) = 0
\]

determines a function \( z(x, y) \) in a neighborhood of the point \((a, b, c)\) such that

\[
F(x, y, z(x, y)) = 0.
\]

In this case,

\[
z_x = -\frac{F_x}{F_z}
\]

and

\[
z_y = -\frac{F_y}{F_z}
\]
14.6 Directional derivatives and the gradient vector

Definition The directional derivative of \( f \) at \((x_0, y_0)\) in the direction of a unit vector \( u = (a, b) \)

\[
D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}
\]

Theorem

\[
D_u f = f_x a + f_y b, \quad u = (a, b), \quad a^2 + b^2 = 1.
\]

Definition The gradient \( \nabla f \) of a function \( f \) is a vector

\[
\nabla f = f_x \mathbf{i} + f_y \mathbf{j}
\]

Hence

\[
D_u f = \nabla f \cdot u
\]

Theorem

\[
\max_{\|u\|=1} D_u f(X) = |\nabla f(X)|
\]

Tangent planes to level surfaces

The tangent plane equation to the level surface \( F = k \) passing through the point \( P(x_0, y_0, z_0) \)

\[
F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0
\]

The normal line equation to the level surface

\[
\frac{x - x_0}{F_x} = \frac{y - y_0}{F_y} = \frac{z - z_0}{F_z} = 0
\]

Remark

1. \( \nabla f \) gives the direction of fastest increase
2. \( \nabla f \perp \) to the level surface of \( f \)
14.7 Maximum and minimum values

**Definition** We say that a function \( f \) has a local maximum at \( P \) if

\[
 f(X) \leq f(P) \quad \text{in a neighborhood of} \quad P
\]

**First derivative test**

**Theorem** If a differentiable function has a local minimum or local maximum at \( P \), then

\[
 \nabla f(P) = 0
\]

**Definition** \( P = (a, b) \) is called a critical point of \( f \) if

\[
 f_x(a, b) = f_y(a, b) = 0
\]

or one of these derivatives does not exist.

**Second derivative test**

Suppose that \( f_x(P) = f_y(P) = 0 \) (\( P \) is a critical point). Set

\[
 D = \det \begin{pmatrix}
 f_{xx} & f_{xy} \\
 f_{yx} & f_{yy}
 \end{pmatrix} = f_{xx}f_{yy} - (f_{xy})^2
\]

1. \( D > 0 \) and \( f_{xx} > 0 \) - loc min
2. \( D > 0 \) and \( f_{xx} < 0 \) - loc max
3. \( D < 0 \) - saddle point

**Extreme Value Theorem** If \( \mathcal{D} \) is a closed bounded set in \( \mathbb{R}^2 \) and \( f : \mathcal{D} \to \mathbb{R} \) is continuous, then \( f \) attains an absolute minimum and absolute maximum at some point \( P \in \mathcal{D} \)

**Instructions**

Step 1. Find the values of \( f \) at the critical points in \( \mathcal{D} \)
Step 2. Do the same on the boundary \( \partial\mathcal{D} \)
Step 3. Take max/min of those obtained from Steps 1 and 2
14.8 Lagrange Multipliers

**Theorem** If \( f(x, y, z) \) attains its extreme value at a point \( P = (x_0, y_0, z_0) \) on the level curve of the function \( g(x, y, z) \),

\[ g(x, y, z) = k, \quad (k \text{ is fixed}), \]

and hence

\[ g(x_0, y_0, z_0) = k, \]

then

\[ \nabla f(x_0, y_0, z_0) \parallel \nabla g(x_0, y_0, z_0) \]

That is, if \( \nabla g(x_0, y_0, z_0) \neq 0 \), then there exists a \( \lambda \) such that

\[ \nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \]

**Method of Lagrange multipliers**

1. Solve for \( x, y, z \) and \( \lambda \) the following system of **four** equations

\[
\begin{aligned}
\nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\
g(x, y, z) &= k
\end{aligned}
\]

2. Evaluate the values of \( f \) at the points \( (x, y, z) \) that result from Step 1 and take \( \max/min \) of those

More generally, to optimize the objective function \( f(x, y, z) \) subject to two constraints

\[
\begin{aligned}
g(x, y, z) &= k \\
h(x, y, z) &= c
\end{aligned}
\]

solve for \( x, y, z, \lambda \) and \( \mu \) the following system of **five** equations

\[
\begin{aligned}
\nabla f(x, y, z) &= \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \\
g(x, y, z) &= k \\
h(x, y, z) &= c
\end{aligned}
\]
15.1 Double Integrals

A double integral
\[ \iint_{D} f(x,y) dA = \lim \{ \text{Riemann sums} \} \]

15.2 Iterated Integrals

Fubini’s Theorem If \( D = [a, b] \times [c, d] \), then
\[
\iint_{D} f(x,y) dA = \int_{a}^{b} \left[ \int_{c}^{d} f(x,y) dy \right] dx = \int_{c}^{d} \left[ \int_{a}^{b} f(x,y) dx \right] dy
\]

15.3 Double integrals over general regions

Type I: If \( D = \{ a \leq x \leq b, g_1(x) \leq y \leq g_2(x) \} \), then
\[
\iint_{D} f(x,y) dA = \int_{a}^{b} \left[ \int_{g_1(x)}^{g_2(x)} f(x,y) dy \right] dx
\]

Type II: If \( D = \{ c \leq y \leq d, h_1(y) \leq x \leq h_2(y) \} \), then
\[
\iint_{D} f(x,y) dA = \int_{c}^{d} \left[ \int_{h_1(y)}^{h_2(y)} f(x,y) dx \right] dy
\]
Properties

1. \[\iint_D (f + g) dA = \iint_D f dA + \iint_D g dA\]
2. \[\iint_D c f dA = c \iint_D f dA\]
3. \[f \geq g \Rightarrow \iint_D f dA \geq \iint_D g dA\]
4. \[\iint_D dA = \text{Area of } D = A(D)\]
5. \[m \leq f \leq M \Rightarrow mA(D) \leq \iint_D f dA \leq MA(D)\]
6. if \(f \geq 0\), then
   \[\iint_D f dA = \text{Volume (of the solid under the graph of} \ f)\]
7. if \(D_1\) and \(D_2\) do not intersect \((D_1 \cap D_2 = \emptyset)\), then
   \[\iint_{D_1 \cup D_2} f dA = \iint_{D_1} f dA + \iint_{D_2} f dA\]

15.4 Double integrals in polar coordinates

Polar coordinates

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  r &= \sqrt{x^2 + y^2} \\
  \theta &= \arctan \frac{y}{x}
\end{align*}
\]

Type I: If \(D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}\), then

\[\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \left[ \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr \right] d\theta\]

Type II: If \(D = \{(r, \theta) \mid r_1 \leq r \leq r_2, g_1(r) \leq \theta \leq g_2(r)\}\), then

\[\iint_D f(x, y) dA = \int_{r_1}^{r_2} \left[ \int_{g_1(r)}^{g_2(r)} f(r \cos \theta, r \sin \theta) d\theta \right] r dr\]
15.5 Applications of double integrals

2D-density $\rho \sim \frac{\Delta \text{(mass)}}{\Delta \text{(area)}}$

$$m = \iint_D \rho \, dA \quad \text{total mass}$$

The moment

$$M_x = \iint_D y \rho(x, y) \, dA \quad \text{(about the } x\text{-axis)}$$

$$M_y = \iint_D x \rho(x, y) \, dA \quad \text{(about the } y\text{-axis)}$$

The coordinate $(\overline{x}, \overline{y})$ of the center of mass

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x \rho(x, y) \, dA$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y \rho(x, y) \, dA$$

where

$$m = \iint_D \rho \, dA$$

The moment of inertia

$$I_x = \iint_D y^2 \rho(x, y) \, dA \quad \text{(about the } x\text{-axis)}$$

$$I_y = \iint_D x^2 \rho(x, y) \, dA \quad \text{(about the } y\text{-axis)}$$

$$I_0 = \iint_D (x^2 + y^2) \rho(x, y) \, dA \quad \text{(about the origin)}$$
A triple integral
\[ \iiint_E f(x, y, z) dV = \lim \{ \text{Riemann sums} \} \]

**Iterated Integrals**

**Fubini’s Theorem** If \( E = [a, b] \times [c, d] \times [r, s] \), then
\[ \iiint_E f(x, y, z) dV = \int_a^b \left[ \int_c^d \left[ \int_r^s f(x, y, z) dz \right] dy \right] dx = \ldots \quad (6 \text{ equalities}) \]

**Triple integrals over general regions**

If
\[ E = \{(x, y, z) \mid (x, y) \in \mathcal{D}, \; u_1(x, y) \leq z \leq u_2(x, y)\}, \]
then
\[ \iiint_E f(x, y, z) dV = \iint_{\mathcal{D}} \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA \]

In particular, if
\[ \mathcal{D} = \{(x, y) \mid a \leq x \leq b, \; g_1(x) \leq y \leq g_2(x)\} \]
that is,
\[ E = \{a \leq x \leq b, \; g_1(x) \leq y \leq g_2(x), \; u_1(x, y) \leq z \leq u_2(x, y)\}, \]
then
\[ \iiint_E f(x, y, z) dV = \int_a^b \left[ \int_{g_1(x)}^{g_2(x)} \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dy \right] dx \]
15.8 Triple integrals in cylindrical coordinates

The cylindrical coordinate system

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\]

\[0 \leq r < \infty, \quad 0 \leq \theta < 2\pi, \quad -\infty < z < \infty\]

The volume element

\[dV = r \, dr \, d\theta \, dz\]

If

\[E = \{(x, y, z) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta), u_1(x, y) \leq z \leq u_2(x, y)\},\]

then

\[
\iiint_E f(x, y, z) \, dV = \int_\alpha^\beta \left[ \int_{h_1(\theta)}^{h_2(\theta)} \left[ \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) \, dz \right] \, dr \right] \, d\theta
\]
15.9 Triple integrals in spherical coordinates

The spherical coordinate system

\[
\begin{align*}
    x &= r \sin \varphi \cos \theta \\
    y &= r \sin \varphi \sin \theta \\
    z &= r \cos \varphi
\end{align*}
\]

\[0 \leq r < \infty, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \varphi < \pi\]

The volume element

\[dV = r^2 \sin \varphi \, dr \, d\theta \, d\varphi\]

If

\[E = \{ (x, y, z) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq \varphi \leq h_2(\theta), u_1(\theta, \varphi) \leq r \leq u_2(\theta, \varphi) \},\]

then

\[
\iiint_E f(x, y, z) \, dV = \int_{\alpha}^{\beta} \left[ \int_{h_1(\theta)}^{h_2(\theta)} \left[ \int_{u_1(\theta, \varphi)}^{u_2(\theta, \varphi)} f(r \sin \varphi \cos \theta, r \sin \varphi \sin \theta, r \cos \varphi) \, r^2 \, dr \right] \sin \varphi \, d\varphi \right] \, d\theta
\]
Suppose that
\[ T : S \rightarrow R \]
is a one-to-one \( C^1 \)-transformation of the domain \( S \) onto the domain \( R \) so that
\[
\begin{align*}
  x &= x(u, v) \\
  y &= y(u, v)
\end{align*}
\]
\((u, v) \in S.\)

Then
\[
\iint_R f(x, y) \, dx \, dy = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv
\]
where
\[
\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}, \quad \text{the Jacobian determinant}
\]
16.1 Vector fields

**Definition** A vector field on $\mathbb{R}^n$ is a function that assigns to each point an $n$-dimensional vector.

**Definition** A vector field $\mathbf{F}$ is called a *conservative* field if it is a gradient of some function $f : \mathbb{R}^3 \to \mathbb{R}$, that is,

$$\mathbf{F} = \nabla f.$$

16.2 Line integrals

Suppose $C$ is a smooth curve and $f : C \to \mathbb{R}$. If $C$ is given by parametric equations

$$x = x(t) \quad \text{and} \quad y = y(t), \quad a \leq t \leq b,$$

then the line integral of $f$ along $C$ with respect to arc length is given by

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}dt$$

**Remark** The value of the integral in the right hand side does not depend on the choice of parametrization of the curve $C$.

Let $P(x, y)dx + Q(x, y)dy$ be a 1-form. Then the line integral of the 1-form along $C$ is given by

$$\int_C P(x, y)dx + Q(x, y)dy = \int_a^b \left[ P(x(t), y(t))\dot{x}(t) + Q(x(t), y(t))\dot{y}(t) \right] dt$$
Line integrals in space

Line integral of a scalar field

If \( C \) is a curve parametrized by the vector-function \( \mathbf{r}(t), a \leq t \leq b \), then the line integral over a scalar field \( f \) along \( C, f : C \to \mathbb{R} \), with respect to arc length can be evaluated as

\[
\int_C f \, ds = \int_a^b f(\mathbf{r}(t)) \left| \mathbf{r}'(t) \right| \, dt
\]

Remark. Line integrals of scalar fields are independent of the parametrization \( \mathbf{r}(t) \).

Line integral of a 1-form

Similarly, the line integral of the 1-form \( P \, dx + Q \, dy + R \, dz \) along \( C \) can be computed as

\[
\int_C (P \, dx + Q \, dy + R \, dz) = \int_a^b \left( P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) \, dt
\]

Line integral of a vector field

The line integral of a vector field \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \) along a curve \( C \)

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P \, dx + Q \, dy + R \, dz
\]

Since \( \mathbf{r}' = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \) and \( \mathbf{T} = \frac{\mathbf{r}'}{\left| \mathbf{r}' \right|} \)

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
\]

Remark. Line integrals of vector fields are independent of the parametrization \( \mathbf{r}(t) \) in absolute value, but they do depend on its orientation.
16.3 The fundamental theorem for line integrals

**Theorem** Let $C : r(t), a \leq t \leq b$, be a curve and $f$ a differentiable function whose gradient field is continuous. Then

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

**Remark** (Path-independence) If $F = \nabla f$, then

$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$

where the curves $C_1$ and $C_2$ have the same initial and terminal points, respectively.

**Theorem** $\int_C F \cdot dr$ is independent of path in $D$ iff

$$\int_C F \cdot dr = 0$$

for any closed path (loop) in $D$.

**Theorem** Let $D$ be an open connected region. Assume that $\int_C F \cdot dr$ is path-independent in $D$.

Then $F$ is conservative. That is

$$F = \nabla f \quad \text{for some } f.$$ 

In this case,

$$f(P) = \int_{P'}^P F \cdot dr$$

**Theorem** If $F = Pi + Qj$ is a conservative vector field and $P$ and $Q$ have continuous first order partial derivatives, then

$$P_y = Q_x.$$

**Theorem** If

$$F = Pi + Qj$$

is a vector field in an open **simply connected** region $D$, $P$ and $Q$ have continuous first order partial derivatives in $D$ and

$$P_y = Q_x, \quad (x, y) \in D,$$

then $F$ is conservative.
16.4 Green’s Theorem

**Green’s Theorem** Let \( C = \partial D \) be a positively oriented simple curve and \( P \) and \( Q \) have continuous first order partial derivatives in \( D \). Then

\[
\oint_{C=\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA
\]

In particular,

\[
\text{Area}(D) = \frac{1}{2} \oint_{\partial D} x\,dy - y\,dx
\]
16.5 Curl and divergence

**Definition** Let \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \) be a vector field. Then the vector field’s rate of rotation is given by

\[
\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} 
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial_x & \partial_y & \partial_z \\
P & Q & R 
\end{vmatrix}
= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
\]

**Theorem** If \( f : \mathbb{R}^3 \to \mathbb{R} \) is a \( C^1 \)-function, then

\[
\text{curl}(\nabla f) = 0 \quad \quad (\nabla \times (\nabla f) = 0)
\]

**Definition** Let \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k} \) be a vector field. Then the magnitude of the vector field’s source (or sink) at a given point is given by

\[
\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

**Theorem** If \( \mathbf{F} \) is a \( C^1 \)-vector field, then

\[
\text{div } (\text{curl } \mathbf{F}) = 0 \quad \quad (\nabla \cdot (\nabla \times \mathbf{F}) = 0)
\]
Vector forms of Green’s Theorem

**Theorem** Let $D$ be a planar domain such that its boundary $\partial D$ is a piecewise-smooth simple positively oriented curve. Suppose that

$$\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

is a $C^1$-vector field in $D$.

Then

(i) the line integral of the tangential component of $\mathbf{F}$ along the boundary $\partial D$ coincides with the double integral of the vertical component of curl $\mathbf{F}$ over $D$

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{T} \, ds = \iint_D (\text{curl} \, \mathbf{F}) \cdot \mathbf{k} \, dA$$

(ii) the line integral of the normal component of $\mathbf{F}$ coincides with the double integral of the divergence of $\mathbf{F}$ over $D$

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \text{div} \, \mathbf{F} \, dA$$
16.6 Parametric surfaces

**Definition** A vector function \( \mathbb{R}^2 \supset D \ni (u, v) \mapsto r(u, v) \in \mathbb{R}^3 \),

\[
r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k
\]

is called a parametric surface.

**Definition** A surface \( S \) is said to be smooth if

\[
r_u \times r_v \neq 0.
\]

**Remark** The cross product \( r_u \times r_v \) (of the tangent vectors \( r_u \) and \( r_v \)) is a normal vector to the tangent plane to the surface \( S \) at the point \( r \in S \).

**Theorem** If a smooth parametric surface \( S \) given by

\[
r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k, \quad (u, v) \in D,
\]
is one-to-one, then the surface area of \( S \) is

\[
\text{Area}(S) = \iint_D |r_u \times r_v| dA
\]

In particular, if \( S \) is the graph of a function \( z = f(x, y) \),

\[
f : D = \text{Dom}(f) \to \mathbb{R},
\]
then

\[
\text{Area}(S) = \iint_D \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} dA
\]
A surface integral of $f$ over the surface $S$

$$\int\int_S f \, dS = \lim \{ \text{Riemann sums} \}$$

If $S$ is a parametric surface, parametrized by the vector-function $\mathbf{r} : D \rightarrow \mathbb{R}^3$, then

$$\int\int_S f(x, y, z) \, dS = \int\int_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

In particular, if $S$ is the graph of a function $z = g(x, y)$,

$$g : D \rightarrow \mathbb{R},$$

then

$$\int\int_S f(x, y, z) \, dS = \int\int_D f(x, y, g(x, y)) \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dA$$

**Oriented surfaces**

A surface $S$ is said to be orientable if a two-dimensional figure cannot be moved around the surface and back to where it started so that it looks like its own mirror image. In this case it is possible to choose a unit normal vector $\mathbf{n}$ at any point of $S$ so that $\mathbf{n}$ varies continuously over $S$. 
Surface integrals of vector fields

**Definition** If \( \mathbf{F} \) is a continuous vector field on an oriented surface \( S \) with unit normal vector \( \mathbf{n} \), then the flux of \( \mathbf{F} \) across \( S \) is

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS
\]

**Theorem** If \( S \) is a parametric surface \((\mathbf{r} : D \to \mathbb{R}^3)\) with the orientation of the unit normal vector

\[
\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}
\]

then

\[
\iint_S \mathbf{F} \cdot dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA
\]
**16.8 Stokes’ Theorem**

**Theorem** (Stokes’ Theorem) The circulation of a vector field \( \mathbf{F} \) around the boundary \( \partial S \) of an oriented surface \( S \) coincides with the flux of curl \( \mathbf{F} \) across the surface

\[
\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S}
\]

(compare this formula with its planar version)

\[
\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl} \mathbf{F}) \cdot \mathbf{k} \, dA
\]

**Corollary** If curl \( \mathbf{F} = 0 \) on all of \( \mathbb{R}^3 \), then \( \mathbf{F} \) is conservative.
16.9 Divergence Theorem

**Theorem** (The Divergence Theorem)

\[
\iint_{S = \partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_{E} \text{div} \mathbf{F} \, dV
\]

(compare with its 2-dimensional analog)

\[
\oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_{D} \text{div} \mathbf{F} \, dA
\]

So, the divergence \( \text{div} \mathbf{F}(P) \) of a vector field \( \mathbf{F} \) at a point \( P \) is the limit of the net flow of \( \mathbf{F} \) across the boundary \( \partial E \) of a three dimensional region \( E \) divided by the volume of \( E \) as \( E \) shrinks to the point \( P \)

\[
\text{div} \mathbf{F}(P) = \lim_{E \to \{P\}} \frac{1}{\text{Volume}(E)} \iint_{\partial E} \mathbf{F} \, d\mathbf{S}
\]
SCHOLIUM (a la Feynman)

1. The Nabla. The differential operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ can be considered as the three components of a vector operator $\nabla$

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$$

---

2. The Fundamental Theorem of Calculus for Line Integrals. The difference of the values of a scalar field $f$ at two points is equal to the line integral of the tangential component of the gradient of that field along any curve at all between the first and second points

$$f(2) - f(1) = \int_{(1)}^{(2)} \nabla f \cdot ds$$
3. **The Stokes Theorem.** The line integral of the tangential component of a vector field \( \mathbf{F} \) around a closed loop \( c = \partial S \) is equal to the surface integral of the normal component of the curl of that field over any surface \( S \) which is bounded by the loop

\[
\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}
\]

If the surface \( S \) has two boundary components \( \partial S_1 \) and \( \partial S_2 \) oriented (for an external observer) as shown below\(^2\):

![Diagram](image)

but, as a bonus, we get

\[
\oint_{\partial S_2} \mathbf{F} \cdot d\mathbf{s} - \oint_{\partial S_1} \mathbf{F} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}
\]

\(^2\)If you walk in the counter-clockwise direction around \( \partial S_2 \) with your head pointing the direction of the outward-pointing \( \mathbf{n} \), then the surface will always be on your left.
4. **The Divergence Theorem.** The surface integral of the normal component of a vector field $\mathbf{F}$, the flux of $\mathbf{F}$, over a closed surface $S = \partial V$ is equal to the integral of the divergence of the vector field over the volume $V$ interior to the surface

$$\int_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \int_{V} \nabla \cdot \mathbf{F} \, dV$$

If the region $V$ lies between two closed surfaces $\partial V_2$ and $\partial V_1$

we obtain

$$\int_{\partial V_2} \mathbf{F} \cdot d\mathbf{S} - \int_{\partial V_1} \mathbf{F} \cdot d\mathbf{S} = \int_{V} \nabla \cdot \mathbf{F} \, dV$$

\[\textit{The End}\]

\[\^3\partial V_1\text{ is negatively oriented!}\]
Calculus via Differential forms

Line integral of a 1-form

Recall that the line integral of the 1-form \( Pdx + Qdy + Rdz \) along \( C \),

\[
\int_C Pdx + Qdy + Rdz = \lim \{ \text{Riemann sums} \},
\]
can be computed as

\[
\int_C Pdx + Qdy + Rdz = \int_a^b \left( P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt.
\]

Since clearly

\[
r' = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k
\]

and

\[
T = \frac{r'}{|r'|}, \quad \text{the unit tangent vector},
\]

we obtain

\[
\int_C Pdx + Qdy + Rdz = \int_C \mathbf{F} \cdot T \, ds, \quad \text{the circulation}
\]

Surface integral of a 2-form

Similarly, the surface integral of the 2-form \( Pdydz + Qdzdx + Rdxdy \) over \( S \),

\[
\int_S Pdydz + Qdzdx + Rdxdy = \lim \{ \text{Riemann sums} \},
\]
can be computed as

\[
\int_S Pdydz + Qdzdx + Rdxdy = \int_S \left[ P \cos \alpha + Q \cos \beta + R \cos \gamma \right] dS
\]

\[
= \int_S \mathbf{F} \cdot \mathbf{n} \, dS \quad \text{(the flux of } \mathbf{F} \text{ across } S)\]

where \( \alpha, \beta \) and \( \gamma \) are the direction angles of the unit normal vector

\[
\mathbf{n} = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}
\]

If \( S \) is a (orientable) parametric surface such that

\[
\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}
\]
the flux can be computed as
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \left[ P \cos \alpha + Q \cos \beta + R \cos \gamma \right] \, dS
\]
\[
= \iint_D \left[ P \frac{\partial(y,z)}{\partial(u,v)} + Q \frac{\partial(z,x)}{\partial(u,v)} + R \frac{\partial(x,y)}{\partial(u,v)} \right] \, dudv
\]

**Proof.** Indeed,
\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_D \mathbf{F} \cdot \mathbf{r}_u \times \mathbf{r}_v \, dudv
\]
but
\[
\mathbf{r}_u \times \mathbf{r}_v = \frac{\partial(y,z)}{\partial(u,v)} \mathbf{i} + \frac{\partial(z,x)}{\partial(u,v)} \mathbf{j} + \frac{\partial(x,y)}{\partial(u,v)} \mathbf{k}
\]

**Scholium**

\[
\mathbf{F} = Pi + Qj + Rk
\]

\[
\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_D \det \begin{pmatrix} P & Q & R \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix} \, dudv
\]
Infinitesimal Calculus

Functions (0-forms)

\[ f \]

Line forms (1-forms)

\[ ds = |r'| dt \]
\[ ds = r' dt = dr \]
\[ ds = T \cdot ds \]

Surface forms (2-forms)

\[ dS = |r_u \times r_v| du dv \]
\[ dS = r_u \times r_v \; du dv \]
\[ dS = n \cdot dS \]

The principal volume form (3-forms)

\[ dV = dx dy dz \]
**Infinitesimal Calculus** (continued)

**The infinitesimal Circulation**

\[ \mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot \mathbf{r}' dt = \mathbf{F} \cdot \frac{\mathbf{r}'}{|\mathbf{r}'|} |\mathbf{r}'| dt = \mathbf{F} \cdot \mathbf{T} ds \]

where

\[ \mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} \]

is the unit tangent vector to the oriented curve \( c \)

**The infinitesimal Flux**

\[ \mathbf{F} \cdot dS = \mathbf{F} \cdot \mathbf{r}_u \times \mathbf{r}_v \, dudv = \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v| dudv \]

where

\[ \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \]

is the unit normal vector to the oriented surface \( S \)
The General Stokes Formula

Formal (noncommutative) manipulations with forms respect the following rules

\[ d^2 = 0 \]
\[ da \wedge db = -db \wedge da \]
\[ d(ab) = bda + adb \]
\[ df = f_x dx + f_y dy + f_z dz \]

**Supertheorem.** [The General Stokes formula]

\[
\int_{\partial \Omega} \omega = \int_{\Omega} d\omega
\]

(\(\partial \Omega\) is the oriented boundary of \(\Omega\) induced by the orientation on \(\Omega\))

As the result, by the General Stokes Formula, we get
**Theorem 1.** [The Fundamental Theorem of Calculus for line integrals]

\[ df = \nabla f \cdot ds \]

\[ \int_C f \cdot ds = \int_C df = \int_C f = f_{\text{final}} - f_{\text{initial}} \]

**Proof.**

\[ df = f_x dx + f_y dy + f_z dz \]
\[ = \nabla f \cdot dr \]

\[ \square \]
Theorem 2. [The Stokes Theorem]

\[
d(F \cdot ds) = \nabla \times F \cdot dS \]

\[
\iint_S \nabla \times F \cdot dS = \iint_S d(F \cdot ds) = \oint_{\partial S} F \cdot ds \tag{Stokes} \]

Proof.

\[
d(F \cdot ds) = d(Pdx + Qdy + Rdz) \\
= (P_x dx + P_y dy + P_z dz) \wedge dx \\
+ (Q_x dx + Q_y dy + Q_z dz) \wedge dy \\
+ (R_x dx + R_y dy + R_z dz) \wedge dz \\
= (R_y - Q_z) dy \wedge dz + (P_z - R_y) dz \wedge dx + (Q_x - P_y) dx \wedge dy \\
= \nabla \times F \cdot dS \]

\[
\square
\]
**Theorem 3.** [The Divergence Theorem]

\[
\iiint_V \nabla \cdot \mathbf{F} \, dV = \iint_{\partial V} \mathbf{F} \cdot d\mathbf{S}
\]

\[
\iiint_{\partial V = S} \mathbf{F} \cdot d\mathbf{S} = \iint_{V} \nabla \cdot \mathbf{F} \, dV
\]

**Proof.**

\[
d(\mathbf{F} \cdot d\mathbf{S}) = d(P \, dy \wedge dz + Q \, dz \wedge dx + R \, dx \wedge dy)
\]

\[
= (P_x \, dx + P_y \, dy + P_z \, dz) \wedge dy \wedge dz
\]

\[
+ (Q_x \, dx + Q_y \, dy + Q_z \, dz) \wedge dz \wedge dx
\]

\[
+ (R_x \, dx + R_y \, dy + R_z \, dz) \wedge dx \wedge dy
\]

\[
= (P_x + Q_y + R_z) \, dx \wedge dy \wedge dz
\]

\[
= \nabla \cdot \mathbf{F} \, dV
\]

\[
\square
\]

**Summary Table**

<table>
<thead>
<tr>
<th>ω</th>
<th>dω</th>
<th>Ω</th>
<th>∂Ω</th>
<th>dimension</th>
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<td>curve</td>
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<td>F · dS</td>
<td>\nabla \cdot F , dV</td>
<td>solid</td>
<td>surface</td>
<td>3</td>
</tr>
</tbody>
</table>

\[
dim(\partial \Omega) = dim(\Omega) - 1
\]