Solutions to Exercises 11.1

1. We have

\[ \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x e^{-ixw} \, dx \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x (\cos wx - i \sin wx) \, dx \]
\[ = -i \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x \sin wx \, dx \]
\[ = -2i \frac{1}{\sqrt{2\pi}} \int_{0}^{1} x \sin wx \, dx \]
\[ = -2i \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{w^2} \sin wx - \frac{x}{w} \cos wx \right]_{0}^{1} \]
\[ = -i \frac{2 \sin w - w \cos w}{w^2}. \]

5. Use integration by parts to evaluate the integrals:

\[ \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixw} \, dx \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|)(\cos wx - i \sin wx) \, dx \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|) \cos wx \, dx - i \sqrt{2\pi} \int_{-1}^{1} (1 - |x|) \sin wx \, dx \]
\[ = \frac{2}{\sqrt{2\pi}} \int_{0}^{1} (1 - x) \cos wx \, dx \]
\[ = \frac{2}{\sqrt{2\pi}} \left[ \frac{\sin wx}{w} \right]_{0}^{1} + \frac{2}{\sqrt{2\pi}w} \int_{0}^{1} \sin wx \, dx \]
\[ = -\sqrt{2 \pi} \frac{1 - \cos w}{w^2} \bigg|_{0}^{1} \]
\[ = \sqrt{2 \pi} \frac{1 - \cos w}{w^2}. \]
9. We have
\[ \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} \cos x \, e^{-ixw} \, dx \]
\[ = \frac{2}{\sqrt{2\pi}} \int_{0}^{\pi/2} \cos x \, \cos wx \, dx \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{0}^{\pi/2} \left[ \cos((w+1)x) + \cos((w-1)x) \right] \, dx \]
\[ = \frac{1}{\sqrt{2\pi}} \left[ \frac{\sin((w+1)x)}{w+1} + \frac{\sin((w-1)x)}{w-1} \right]_{0}^{\pi/2} \quad (w \neq \pm 1) \]
\[ = \frac{1}{\sqrt{2\pi}} \left[ \frac{\sin((w+1)\pi/2)}{w+1} + \frac{\sin((w-1)\pi/2)}{w-1} \right] \quad (w \neq \pm 1) \]
\[ = \frac{1}{\sqrt{2\pi}} \left[ \frac{\cos \frac{w\pi}{2}}{w+1} + \frac{\cos \frac{w\pi}{2}}{w-1} \right] \quad (w \neq \pm 1) \]
\[ = \sqrt{2\pi} \frac{\cos \frac{w\pi}{2}}{\pi - w^2} \quad (w \neq \pm 1). \]

Treat the case \( w = \pm 1 \) separately and you will find
\[ \hat{f}(\pm 1) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\pi/2} \cos x \cos x \, dx \]
\[ = \frac{\pi/2}{\sqrt{2\pi}} = \frac{\sqrt{2\pi}}{4}. \]

13. Apply the inverse Fourier transform to the transform of Exercise 9, then you will get the function back; that is,
\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\cos \frac{w\pi}{2}} \frac{1}{\pi - w^2} e^{ixw} \, dx = \begin{cases} \cos x & \text{if } |x| < \frac{\pi}{2}, \\ 0 & \text{if } |x| \geq \frac{\pi}{2}; \end{cases} \]
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \cos \frac{w\pi}{2} \cos wx \, dx = \begin{cases} \cos x & \text{if } |x| < \frac{\pi}{2}, \\ 0 & \text{if } |x| \geq \frac{\pi}{2}; \end{cases} \]
\[ \frac{2}{\pi} \int_{0}^{\infty} \cos \frac{w\pi}{2} \cos wx \, dx = \begin{cases} \cos x & \text{if } |x| < \frac{\pi}{2}, \\ 0 & \text{if } |x| \geq \frac{\pi}{2}; \end{cases} \]

In the last two steps, we used the fact that the integral of an odd function over a symmetric interval is 0 and that the integral of an even function over a symmetric interval is twice the integral over the positive half of the interval.
17. (a) Let $0 < \alpha < 1$. Applying the definition of the Fourier transform, we find, for $w > 0$,
\[
\mathcal{F} \left( \frac{1}{|x|^{\alpha}} \right) (w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{|x|^{\alpha}} e^{-iwx} \, dx
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{|x|^{\alpha}} \cos wx \, dx
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{|x|^{\alpha}} \cos wx \, dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{x^{\alpha}} \cos wx \, dx
\]
\[
= \sqrt{\frac{2}{\pi}} w^{\alpha-1} \int_{0}^{\infty} \frac{1}{t^{\alpha}} \cos t \, dt \quad (wx = t \Rightarrow x = t/w, \, dx = dt/w)
\]
\[
= \sqrt{\frac{2}{\pi}} w^{\alpha-1} \Gamma(1-\alpha) \sin \frac{\alpha \pi}{2}.
\]
If $w = 0$, both sides are infinite. If $w < 0$, the integral does not change, because $\cos wx$ is even. Hence the given formula follows.

(b) Take $\alpha = \frac{1}{2}$, then
\[
\mathcal{F} \left( \frac{1}{|x|^{1/2}} \right) (w) = \sqrt{\frac{2}{\pi}} w^{-1/2} \Gamma(1/2) \sin \frac{\pi}{4}
\]
\[
\frac{1}{w^{1/2}},
\]
where we have used $\Gamma(1/2) = \sqrt{\pi}$ (Exercise 25(a), Section 4.3).

21. We have
\[
\mathcal{F} \left( \frac{\sqrt{2} x}{\sqrt{\pi} (1 + x^2)^2} \right) (w) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{(1 + x^2)^2} e^{-iwx} \, dx
\]
\[
= \frac{-i}{\pi} \int_{-\infty}^{\infty} \frac{x \sin wx}{(1 + x^2)^2} \, dx.
\]
Note that the Fourier transform is odd in the sense that $\hat{f}(-w) = -\hat{f}(w)$. For the sake of the residue method, we take $w < 0$ and write the Fourier transform as
\[
\mathcal{F} \left( \frac{\sqrt{2} x}{\sqrt{\pi} (1 + x^2)^2} \right) (w) = \frac{1}{\pi} \int_{\gamma_R} \frac{z}{(1 + z^2)^2} e^{-iwz} \, dz,
\]
\[
= \frac{1}{\pi} \int_{\gamma_R} \frac{z}{(1 + z^2)^2} e^{iWz} \, dz \quad (W = -w, W > 0),
\]
where $\gamma_R$ is the contour in Figure 8, with $R$ large (in fact, it is enough to take $R > 1$). To justify this equality, we note that the contour integral is equal to $2\pi i$ times the sum of the residues inside the contour. Inside the contour we have only one residue at $z = i$ and so the integral is constant for all $R > 1$. Letting $R \to \infty$, the integral on the semi-circle converges to 0 (by Jordan’s lemma) and the integral on the real line becomes
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{(1 + x^2)^2} e^{iWx} \, dx,
\]
which is the desired integral. So let us compute the contour integral, \( I_R \), using residues. Let

\[
F(z) = \frac{z}{(1 + z^2)^2} e^{iWz}
\]

, then \( F \) has one pole of order 2 at \( z = i \) inside the contour \( \gamma_R \). The residue at \( z = i \) is equal to

\[
\text{Res} (F, i) = \frac{d}{dz} \left. \frac{z}{(z + i)^2} \right|_{z=i} e^{iWz} = \frac{d}{dz} \left. \frac{z e^{iWz}}{(z + i)^2} \right|_{z=i}
\]

\[
= \frac{d}{dz} \left. \frac{z}{(z + i)^2} \right|_{z=i} e^{iWz} = \frac{(e^{iWz} + iW e^{iWz} z)(z + i)^2 - 2(z + i) z e^{iWz}}{(z + i)^4} \bigg|_{z=i}
\]

\[
= e^{-W} W \frac{4}{4} = -w e^{-|w|}.
\]

Thus the contour integral is equal to

\[
-i \pi w e^{-|w|}
\]

and the Fourier transform for \( w < 0 \) is equal to

\[
\frac{1}{\pi} I_R = -i w e^{-|w|} \frac{1}{2}.
\]

The formula works for \( w \geq 0 \) since it defines an odd function.
Solutions to Exercises 11.2

1. We have $\mathcal{F}(e^{-x^2}) = \frac{1}{\sqrt{2\pi}} e^{-w^2/4}$. Applying Theorem 1(ii) (with $n = 2$), we obtain

$$\mathcal{F}(x^2e^{-x^2}) = -\frac{d^2}{dw^2} \left[ \frac{1}{\sqrt{2}} e^{-w^2/4} \right] = -\frac{1}{\sqrt{2}} \frac{d}{dw} \left[ \frac{-w}{2} e^{-w^2/4} \right] = \frac{e^{-w^2/4}}{4\sqrt{2}} \left[ 2 - w^2 \right].$$

5. We have $\mathcal{F}(e^{-|x|}) = \sqrt{\frac{2}{\pi}} \frac{1}{1+w^2}$. So

$$\mathcal{F}(e^{-|x|} + 6xe^{-|x|}) = \sqrt{\frac{2}{\pi}} \left( \frac{1}{1+w^2} + 6i \frac{d}{dw} \left[ \frac{1}{1+w^2} \right] \right) = \sqrt{\frac{2}{\pi}} \left( \frac{1}{1+w^2} + \frac{-2w}{(1+w^2)^2} \right) = \sqrt{\frac{2}{\pi}} \frac{1 - 12iw + w^2}{(1+w^2)^2}.$$

9. Let

$$g(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise}, \end{cases}$$

and note that $f(x) = xg(x)$. Now $\mathcal{F}(g(x)) = \frac{i}{\sqrt{2\pi}} \frac{e^{-iw} - 1}{w}$ (Exercise 3, Section 11.1); so, by Theorem 1,

$$\mathcal{F}(f(x)) = \mathcal{F}(xg(x)) = \frac{d}{dw} \left[ \frac{i}{\sqrt{2\pi}} \frac{e^{-iw} - 1}{w} \right] = -\frac{1}{\sqrt{2\pi}} \frac{-iwe^{-iw} - e^{-iw} + 1}{w^2} = \frac{1}{\sqrt{2\pi}} \frac{(1+ iw) \cos w + (w - i) \sin w}{w^2}.$$

13. We have $\mathcal{F}(e^{-x^2}) = \frac{1}{\sqrt{2}} e^{-w^2/4}$. Using Exercise 1, we obtain

$$\mathcal{F}((1-x^2)e^{-x^2}) = \mathcal{F}(e^{-x^2}) - \mathcal{F}(x^2e^{-x^2}) = \frac{1}{\sqrt{2}} e^{-w^2/4} - e^{-w^2/4} \frac{2 - w^2}{4\sqrt{2}}.$$

$$= \frac{e^{-w^2/4}}{2\sqrt{2}} \left[ 1 + \frac{w^2}{2} \right].$$
17. Let \( f(x) = e^{-x^2} \) and \( g(x) = xe^{-x^2} \). We have \( \mathcal{F}(e^{-x^2}) = \frac{1}{\sqrt{2}} e^{-w^2/4} \) and
\[
\mathcal{F}(xe^{-x^2}) = \frac{i}{\sqrt{2}} \frac{d}{dw} e^{-w^2/4} = -\frac{i}{2\sqrt{2}} we^{-w^2/4}.
\]
So
\[
\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g) = \frac{-i}{4} we^{-w^2/2} = \frac{1}{4} \frac{d}{dw} \left[ e^{-w^2/2} \right] = \frac{1}{4} \mathcal{F}(xe^{-x^2/2}).
\]
Hence
\[
f * g(x) = \frac{1}{4} xe^{-x^2/2}.
\]

21. Let \( f(x) = \mathcal{U}_a - \mathcal{U}_b(x) \) and \( g(x) = \mathcal{U}_b - \mathcal{U}_b(x) \), where \( 0 < a \leq b \). We have
\[
\mathcal{F}(f) = \sqrt{\frac{2}{\pi}} \frac{\sin aw}{w} \quad \text{and} \quad \mathcal{F}(g) = \sqrt{\frac{2}{\pi}} \frac{\sin bw}{w};
\]
so
\[
\mathcal{F}(f * g) = \frac{2}{\pi} \frac{\sin(aw) \sin(bw)}{w^2}.
\]
Instead of inverting the Fourier transform to find \( f * g \), we will compute \( f * g \) by using the method of Example 10. (This is an interesting Fourier transform that is not in the table of transforms at the end of the book.) We have
\[
f'(x) = \delta_{-a}(x) - \delta_a(x);
g'(x) = \delta_{-b}(x) - \delta_b(x);
\]
\[
\frac{d^2}{dx^2}(f * g)(x) = \frac{d}{dx} f * \frac{d}{dx} g(x)
= (\delta_{-a}(x) - \delta_a(x)) * (\delta_{-b}(x) - \delta_b(x))
= \frac{1}{\sqrt{2\pi}} (-\delta_{b-a}(x) + \delta_{-(a+b)}(x) + \delta_{a+b}(x) - \delta_{a-b}(x)).
\]
So \( f * g \) is the second antiderivative of this sum of Dirac deltas, that vanishes at infinity; that is, \( f * g(x) = 0 \) for large \( |x| \) (in fact, for \( |x| > a+b \)). The reason for the last assertion is that both \( f \) and \( g \) have bounded support, so \( f * g \) will have bounded support. To compute the antiderivatives it is best to do it on a graph, as we did in Example 10. We can also proceed as follows. Antidifferentiate once and use (17), then
\[
\frac{d}{dx}(f * g)(x) = \frac{1}{\sqrt{2\pi}} \left( -\mathcal{U}_{b-a}(x) + \mathcal{U}_{-(a+b)}(x) + \mathcal{U}_{a+b}(x) - \mathcal{U}_{a-b}(x) \right).
\]
An antiderivative of \( \mathcal{U}_a \) is the function \((x-a)\mathcal{U}_a\) or
\[
\frac{d}{dx}(x-a)\mathcal{U}_a = \mathcal{U}_a(x).
\]
To see this, just draw the graph of \((x-a)\mathcal{U}_a\); it is continuous and equal to 0 if \( x < a \) and \( x - a \) if \( x > a \). Thus its derivative is 0 if \( x < a \) and 1 if \( x > a \); thus the derivative formula is true. So and antiderivative of \( f * g \) is
\[
\frac{1}{\sqrt{2\pi}} \left( - (x-b+a)\mathcal{U}_{b-a}(x) + (x+b+a)\mathcal{U}_{-(a+b)}(x) + (x-b-a)\mathcal{U}_{a+b}(x) - (x+b-a)\mathcal{U}_{a-b}(x) \right).
\]
If this function vanishes for large $|x|$, then it would be the desired antiderivative. Let us check: Take $x > a + b$, then
\[
\frac{1}{\sqrt{2\pi}} \left( -(x-b+a)\mathcal{U}_{b-a}(x) + (x+b+a)\mathcal{U}_{-(a+b)}(x) + (x-b-a)\mathcal{U}_{a+b}(x) - (x+b-a)\mathcal{U}_{a-b}(x) \right)
\]
\[
= \frac{1}{\sqrt{2\pi}} \left( -(a-b) + (x+b+a) + (x-b-a) - (x+b-a) \right) = 0,
\]
as desired. Similarly, if $x < -a - b$, then
\[
\frac{1}{\sqrt{2\pi}} \left( -(x-b+a)\mathcal{U}_{b-a}(x) + (x+b+a)\mathcal{U}_{-(a+b)}(x) + (x-b-a)\mathcal{U}_{a+b}(x) - (x+b-a)\mathcal{U}_{a-b}(x) \right)
\]
\[
= \frac{1}{\sqrt{2\pi}} \left( -0 + 0 + 0 - 0 \right) = 0.
\]
This proves that
\[
f \ast g(x) = \frac{1}{\sqrt{2\pi}} \left( (x+b+a)\mathcal{U}_{-(a+b)}(x) - (x-b-a)\mathcal{U}_{a-b}(x) - (x-b+a)\mathcal{U}_{b-a}(x) + (x-b-a)\mathcal{U}_{a+b}(x) \right).
\]
More explicitly, we have
\[
f \ast g(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} 
0 & \text{if } |x| > a + b; \\
(x+b+a) - (x+b-a) & \text{if } -a-b < x < -b+a; \\
(x+b+a) - (x-b-a) - (x-b+a) & \text{if } -b+a < x < b-a; \\
-2a & \text{if } -b-a < x < b-a; \\
-x+b+a & \text{if } -a-b < x < -b+a.
\end{cases}
\]
so, after simplifying,
\[
f \ast g(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} 
0 & \text{if } |x| > a + b; \\
x+b+a & \text{if } -a-b < x < -b+a; \\
2a & \text{if } -b+a < x < b-a; \\
x-b+a & \text{if } -a-b < x < -b+a.
\end{cases}
\]
The graph of $f \ast g$ is a nice tent.

25. (a) Use the definition of convolutions:
\[
e^{i\alpha x} \ast f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{i\alpha (x-y)} dy
\]
\[
= e^{i\alpha x} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{-i\alpha y} dy = e^{i\alpha x} \hat{f}(\alpha).
\]
(b) Using (a) and $\cos \alpha x = \frac{e^{i\alpha x} + e^{-i\alpha x}}{2}$, we find
\[
\cos(\alpha x) \ast f(x) = \frac{1}{2} \left( e^{i\alpha x} \ast f(x) + e^{-i\alpha x} \ast f(x) \right)
\]
\[
= \frac{1}{2} \left( e^{i\alpha x} \hat{f}(\alpha) + e^{-i\alpha x} \hat{f}(-\alpha) \right).
Specializing to $\alpha = 1$ and $f(x) = e^{-|x|}$, $\hat{f}(w) = \sqrt{\frac{2}{\pi}} \frac{1}{1+w^2}$, we find

\[
\cos x \ast e^{-|x|} = \frac{1}{2} \left( e^{ix} \sqrt{\frac{2}{\pi}} \frac{1}{1+1} + e^{-ix} \sqrt{\frac{2}{\pi}} \frac{1}{1+1} \right)
\]

\[
= \frac{1}{2} \left( e^{ix} \sqrt{\frac{2}{\pi}} \frac{1}{1+1} + e^{-ix} \sqrt{\frac{2}{\pi}} \frac{1}{1+1} \right) = \frac{1}{\sqrt{2\pi}} \cos x.
\]

33. Apply (23):

\[
\mathcal{F}(U_2(x) - U_4(x)) = \mathcal{F}(U_2(x)) - \mathcal{F}(U_4(x))
\]

\[
= \frac{-i}{\sqrt{2\pi w}} \left( e^{-2iw} - e^{-4iw} \right)
\]

\[
= \frac{-i}{\sqrt{2\pi w}} \left( e^{-2iw} - e^{-4iw} \right).
\]
Solutions to Exercises 11.3

1.\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = \frac{1}{1 + x^2}, \quad \frac{\partial u}{\partial t}(x, 0) = 0.
\]

Follow the solution of Example 1. Fix \( t \) and Fourier transform the problem with respect to the variable \( x \):

\[
\frac{d^2}{dt^2} \hat{u}(w, t) = -w^2 \hat{u}(w, t), \quad \hat{u}(w, 0) = \mathcal{F}\left(\frac{1}{1 + x^2}\right) = \sqrt{\frac{\pi}{2}} e^{-|w|}, \quad \frac{d}{dt} \hat{u}(w, 0) = 0.
\]

Solve the second order differential equation in \( \hat{u}(w, t) \):

\[
\hat{u}(w, t) = A(w) \cos wt + B(w) \sin wt.
\]

Using \( \frac{d}{dt} \hat{u}(w, 0) = 0 \), we get

\[
-A(w)w \sin wt + B(w)w \cos wt \bigg|_{t=0} = 0 \Rightarrow B(w)w = 0 \Rightarrow B(w) = 0.
\]

Hence

\[
\hat{u}(w, t) = A(w) \cos wt.
\]

Using \( \hat{u}(w, 0) = \sqrt{\frac{\pi}{2}} e^{-|w|} \), we see that \( A(w) = \sqrt{\frac{\pi}{2}} e^{-|w|} \) and so

\[
\hat{u}(w, t) = \sqrt{\frac{\pi}{2}} e^{-|w|} \cos wt.
\]

Taking inverse Fourier transforms, we get

\[
u(x, t) = \int_{-\infty}^{\infty} e^{-|w|} \cos wt e^{ixw} dw.
\]

5.\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = \sqrt{\frac{2}{\pi}} \sin \frac{x}{2}, \quad \frac{\partial u}{\partial t}(x, 0) = 0.
\]

Fix \( t \) and Fourier transform the problem with respect to the variable \( x \):

\[
\frac{d^2}{dt^2} \hat{u}(w, t) = -c^2 w^2 \hat{u}(w, t), \quad \hat{u}(w, 0) = \mathcal{F}\left(\sqrt{\frac{2}{\pi}} \sin \frac{x}{2}\right)(w) = \hat{f}(w) = \begin{cases} 1 & \text{if } |w| < 1 \\ 0 & \text{if } |w| > 1 \end{cases}, \quad \frac{d}{dt} \hat{u}(w, 0) = 0.
\]
Solve the second order differential equation in \( \hat{u}(w, t) \):

\[
\hat{u}(w, t) = A(w) \cos cwt + B(w) \sin cwt.
\]

Using \( \frac{d}{dt} \hat{u}(w, 0) = 0 \), we get

\[
\hat{u}(w, t) = A(w) \cos cwt.
\]

Using \( \hat{u}(w, 0) = \hat{f}(w) \), we see that

\[
\hat{u}(w, t) = \hat{f}(w) \cos wt.
\]

Taking inverse Fourier transforms, we get

\[
u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) \cos cwt e^{ixw} dw = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos cwt e^{ixw} dw.
\]

9.

\[ t^2 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0, \]

\[ u(x, 0) = 3 \cos x. \]

The solution of this problem is very much like the solution of Exercise 7. However, there is a difficulty in computing the Fourier transform of \( \cos x \), because \( \cos x \) is not integrable on the real line. One can make sense of the Fourier transform by treating \( \cos x \) as a generalized function, but there is no need for this in this solution, since we do not need the exact formula of the Fourier transform, as you will see shortly.

Let \( f(x) = 3 \cos x \) and Fourier transform the problem with respect to the variable \( x \):

\[
t^2 i \hat{w} \hat{u}(w, t) - \frac{d}{dt} \hat{u}(w, t) = 0,
\]

\[
\hat{u}(w, 0) = \mathcal{F}(3 \cos x)(w) = \hat{f}(w).
\]

Solve the first order differential equation in \( \hat{u}(w, t) \):

\[
\hat{u}(w, t) = A(w)e^{-\frac{t^3}{3}}.
\]

Using the transformed initial condition, we get

\[
\hat{u}(w, t) = \hat{f}(w)e^{-\frac{t^3}{3}}.
\]

Taking inverse Fourier transforms, we get

\[
u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-\frac{t^3}{3}} e^{ixw} dw
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx + \frac{t^3}{3}} dw
\]

\[
= f(x + \frac{t^3}{3}) = 3 \cos(x + \frac{t^3}{3}).
\]

13.

\[
\frac{\partial u}{\partial t} = t \frac{\partial^2 u}{\partial x^2},
\]

\[ u(x, 0) = f(x). \]
Fix $t$ and Fourier transform the problem with respect to the variable $x$:

$$\frac{d}{dt} \hat{u}(w, t) + tw^2 \hat{u}(w, t) = 0,$$

$$\hat{u}(w, 0) = \hat{f}(w).$$

Solve the first order differential equation in $\hat{u}(w, t)$:

$$\hat{u}(w, t) = A(w) e^{-\frac{t^2 w^2}{2}}.$$

Use the initial condition: $A(w) = \hat{f}(w)$. Hence

$$\hat{u}(w, t) = \hat{f}(w) e^{-\frac{t^2 w^2}{2}}.$$

Taking inverse Fourier transforms, we get

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-\frac{t^2 w^2}{2}} e^{iwx} dw.$$

21. (a) To verify that

$$u(x, t) = \frac{1}{2} \left[ f(x - ct) + f(x + ct) \right] + \frac{1}{2c} \int_{x - ct}^{x + ct} g(s) ds$$

is a solution of the boundary value problem of Example 1 is straightforward. You just have to plug the solution into the equation and the initial and boundary conditions and see that the equations are verified. The details are sketched in Section 3.4, following Example 1 of that section.

(b) In Example 1, we derived the solution as an inverse Fourier transform:

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) \cos cwt + \frac{1}{cw} \hat{g}(w) \sin cwt e^{iwx} dw.$$
because the first integral is simply the inverse Fourier transform of \( \hat{f} \) evaluated at \( x + ct \), and the second integral is the inverse Fourier transform of \( \hat{f} \) evaluated at \( x - ct \). This proves (1). To prove (2), we note that the left side of (2) is an inverse Fourier transform. So (2) will follow if we can show that

\[
\mathcal{F} \left\{ \int_{x-ct}^{x+ct} g(s) \, ds \right\} = \frac{2}{w} \hat{g}(w) \sin cwt. \tag{3}
\]

Let \( G \) denote an antiderivative of \( g \). Then (3) is equivalent to

\[
\mathcal{F} \left( G(x + ct) - G(x - ct) \right) (w) = \frac{2}{w} \hat{G}'(w) \sin cwt. \tag{4}
\]

Since \( \hat{G}' = iw\hat{G} \), the last equation is equivalent to

\[
\mathcal{F} \left( G(x + ct) \right) (w) - \mathcal{F} \left( G(x - ct) \right) (w) = 2i\hat{G}(w) \sin cwt. \tag{4}
\]

Using Exercise 19, Sec. 7.2, we have

\[
\mathcal{F} \left( G(x + ct) \right) (w) - \mathcal{F} \left( G(x - ct) \right) (w) = e^{ictw} \mathcal{F}(G)(w) - e^{-ictw} \mathcal{F}(G)(w) = \mathcal{F}(G)(w) (e^{ictw} - e^{-ictw}) = 2it \hat{G}(w) \sin cwt,
\]

where we have applied the formula

\[
\sin ctw = \frac{e^{ictw} - e^{-ictw}}{2i}.
\]

This proves (4) and completes the solution.
Solutions to Exercises 11.4

1. Repeat the solution of Example 1 making some adjustments: \( c = \frac{1}{2}, \ g_t(x) = \frac{\sqrt{\pi}}{2} e^{-\frac{x^2}{4}} \),

\[
\begin{align*}
\hat{u}(x, t) &= f \ast g_t(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) \sqrt{\frac{2}{\pi}} e^{-\frac{(x-s)^2}{4}} \, ds \\
&= \frac{20}{\sqrt{\pi}} \int_{-1}^{1} e^{-\frac{(x-s)^2}{4}} \, ds \quad (v = \frac{x-s}{\sqrt{t}}, \ dv = -\frac{1}{\sqrt{t}} \, ds) \\
&= \frac{20}{\sqrt{\pi}} \int_{\frac{x-1}{\sqrt{t}}}^{\frac{x+1}{\sqrt{t}}} e^{-v^2} \, dv \\
&= 10 \left( \text{erf} \left( \frac{x+1}{\sqrt{t}} \right) - \text{erf} \left( \frac{x-1}{\sqrt{t}} \right) \right). 
\end{align*}
\]

9. Fourier transform the problem:

\[
\frac{du}{dt} \hat{u}(w, t) = -e^{-t} w^2 \hat{u}(w, t), \quad \hat{u}(w, 0) = \hat{f}(w).
\]

Solve for \( \hat{u}(w, t) \):

\[
\hat{u}(w, t) = \hat{f}(w) e^{-w^2(1-e^{-t})}.
\]

Inverse Fourier transform and note that

\[
u(x, t) = f \ast \mathcal{F}^{-1} \left( e^{-w^2(1-e^{-t})} \right).
\]

With the help of Theorem 5, Sec. 7.2 (take \( a = 1 - e^{-t} \)), we find

\[
\mathcal{F}^{-1} \left( e^{-w^2(1-e^{-t})} \right) = \frac{1}{\sqrt{2\sqrt{1-e^{-t}}}} e^{-\frac{w^2}{4(1-e^{-t})}}.
\]

Thus

\[
u(x, t) = \frac{1}{\sqrt{2\sqrt{1-e^{-t}}}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4(1-e^{-t})}} \, ds.
\]

13. If in Exercise 9 we take

\[
f(x) = \begin{cases} 
100 & \text{if } |x| < 1, \\
0 & \text{otherwise},
\end{cases}
\]

then the solution becomes

\[
u(x, t) = \frac{50}{\sqrt{\pi} \sqrt{1-e^{-t}}} \int_{-1}^{1} e^{-\frac{(x-s)^2}{4(1-e^{-t})}} \, ds.
\]

Let \( z = \frac{x-s}{2\sqrt{1-e^{-t}}}, \ dz = \frac{-ds}{2\sqrt{1-e^{-t}}} \). Then

\[
u(x, t) = \frac{50}{\sqrt{\pi} \sqrt{1-e^{-t}}} 2\sqrt{1-e^{-t}} \int_{\frac{x-1}{2\sqrt{1-e^{-t}}}}^{\frac{x+1}{2\sqrt{1-e^{-t}}}} e^{-z^2} \, dz
\]

\[
= \frac{100}{\sqrt{\pi}} \int_{\frac{x-1}{2\sqrt{1-e^{-t}}}}^{\frac{x+1}{2\sqrt{1-e^{-t}}}} e^{-z^2} \, dz
\]

\[
= 50 \left[ \text{erf} \left( \frac{x+1}{2\sqrt{1-e^{-t}}} \right) - \text{erf} \left( \frac{x-1}{2\sqrt{1-e^{-t}}} \right) \right].
\]
As \( t \) increases, the expression \( \text{erf} \left( \frac{x + 1}{2} \right) - \text{erf} \left( \frac{x - 1}{2} \right) \) approaches very quickly \( \text{erf} \left( \frac{x + 1}{2} \right) - \text{erf} \left( \frac{x - 1}{2} \right) \), which tells us that the temperature approaches the limiting distribution

\[
50 \left[ \text{erf} \left( \frac{x + 1}{2} \right) - \text{erf} \left( \frac{x - 1}{2} \right) \right].
\]

You can verify this assertion using graphs.

17. (a) If

\[
f(x) = \begin{cases} 
T_0 & \text{if } a < x < b, \\
0 & \text{otherwise},
\end{cases}
\]

then

\[
u(x, t) = \frac{T_0}{2c \sqrt{\pi t}} \int_a^b e^{-\frac{(x-s)^2}{4c^2t}} \, ds.
\]

(b) Let \( z = \frac{x-s}{2c \sqrt{t}} \), \( dz = \frac{ds}{2c \sqrt{t}} \). Then

\[
u(x, t) = \frac{T_0}{2c \sqrt{\pi t}} \int_{-\infty}^{\frac{x-a}{2c \sqrt{t}}} e^{-z^2} \, dz
\]

\[
= \frac{T_0}{2} \left[ \text{erf} \left( \frac{x-a}{2c \sqrt{t}} \right) - \text{erf} \left( \frac{x-b}{2c \sqrt{t}} \right) \right].
\]

25. Let \( u_2(x, t) \) denote the solution of the heat problem with initial temperature distribution \( f(x) = e^{-(x-1)^2} \). Let \( u(x, t) \) denote the solution of the problem with initial distribution \( e^{-x^2} \). Then, by Exercise 23, \( u_2(x, t) = u(x-1, t) \)

By (4), we have

\[
u(x, t) = \frac{1}{c \sqrt{2t}} e^{-x^2/(4c^2t)} * e^{-x^2}.
\]

We will apply Exercise 24 with \( a = \frac{1}{4c^2t} \) and \( b = 1 \). We have

\[
a \cdot b = \frac{1}{a+b} = \frac{1}{4c^2t} \times \frac{1}{\frac{1}{4c^2t} + 1}
\]

\[
= \frac{1}{1 + 4c^2t} = \frac{1}{\sqrt{2(\frac{1}{4c^2t} + 1)}}
\]

\[
\sqrt{2(a+b)} = \frac{c \sqrt{2t}}{\sqrt{4c^2t + 1}}
\]

So

\[
u(x, t) = \frac{1}{c \sqrt{2t}} e^{-x^2/(4c^2t)} * e^{-x^2}
\]

\[
= \frac{1}{c \sqrt{2t}} \cdot \frac{c \sqrt{2t}}{\sqrt{4c^2t + 1}} e^{-\frac{1+4c^2t}{1+4c^2t}},
\]

\[
= \frac{1}{\sqrt{4c^2t + 1}} e^{-\frac{1+4c^2t}{1+4c^2t}},
\]

and hence

\[
u_2(x, t) = \frac{1}{\sqrt{4c^2t + 1}} e^{-\frac{(x-1)^2}{1+4c^2t}}.
\]
29. Parts (a)-(c) are obvious from the definition of \( g_t(x) \).

(d) The total area under the graph of \( g_t(x) \) and above the \( x \)-axis is

\[
\int_{-\infty}^{\infty} g_t(x) \, dx = \frac{1}{c \sqrt{2t}} \int_{-\infty}^{\infty} e^{-x^2/(4c^2t)} \, dx
\]

\[
= \frac{2c \sqrt{\pi}}{c \sqrt{2t}} \int_{-\infty}^{\infty} e^{-z^2} \, dz \quad (z = \frac{x}{2c \sqrt{t}}, \, dx = 2c \sqrt{t} \, dz)
\]

\[
\sqrt{2} \int_{-\infty}^{\infty} e^{-z^2} \, dz = \sqrt{2\pi},
\]

by (4), Sec. 7.2.

(e) To find the Fourier transform of \( g_t(x) \), apply (5), Sec. 7.2, with

\[
a = \frac{1}{4c^2 t}, \quad \frac{1}{\sqrt{2a}} = 2c \sqrt{2t}, \quad \frac{1}{4a} = c^2 t.
\]

We get

\[
\hat{g}_t(w) = \frac{1}{c \sqrt{2t}} \mathcal{F} \left( e^{-x^2/(4c^2t)} \right) \, dx
\]

\[
= \frac{1}{c \sqrt{2t}} \times 2c \sqrt{2t} e^{-c^2 t \omega^2}
\]

\[
= e^{-c^2 t \omega^2}.
\]

(f) If \( f \) is an integrable and piecewise smooth function, then at its points of continuity, we have

\[
\lim_{t \to 0} g_t * f(x) = f(x).
\]

This is a true fact that can be proved by using properties of Gauss’s kernel. If we interpret \( f(x) \) as an initial temperature distribution in a heat problem, then the solution of this heat problem is given by

\[
u(x, t) = g_t * f(x).
\]

If \( t \to 0 \), the temperature \( u(x, t) \) should approach the initial temperature distribution \( f(x) \). Thus

\[
\lim_{t \to 0} g_t * f(x) = f(x).
\]

Alternatively, we can use part (e) and argue as follows. Since

\[
\lim_{t \to 0} \mathcal{F}(g_t)(\omega) = \lim_{t \to 0} e^{-c^2 t \omega^2} = 1,
\]

So

\[
\lim_{t \to 0} \mathcal{F}(g_t * f) = \lim_{t \to 0} \mathcal{F}(g_t) \mathcal{F}(f) = \mathcal{F}(f).
\]

You would expect that the limit of the Fourier transform be the transform of the limit function. So taking inverse Fourier transforms, we get \( \lim_{t \to 0} g_t * f(x) = f(x) \). (Neither one of the arguments that we gave is rigorous.)
Solutions to Exercises 11.5

1. To solve the Dirichlet problem in the upper half-plane with the given boundary function, we use formula (5). The solution is given by

\[
  u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2 + y^2} \, ds
  = \frac{50y}{\pi} \int_{-1}^{1} \frac{1}{(x-s)^2 + y^2} \, ds
  = \frac{50}{\pi} \left\{ \tan^{-1} \left( \frac{1 + x}{y} \right) + \tan^{-1} \left( \frac{1 - x}{y} \right) \right\},
\]

where we have used Example 1 to evaluate the definite integral.

5. Appealing to (4) in Section 7.5, with \( y = y_1, y_2, y_1 + y_2 \), we find

\[
  \mathcal{F}(P_{y_1})(w) = e^{-y_1|w|}, \quad \mathcal{F}(P_{y_2})(w) = e^{-y_2|w|}, \quad \mathcal{F}(P_{y_1+y_2})(w) = e^{-(y_1+y_2)|w|}.
\]

Hence

\[
  \mathcal{F}(P_{y_1})(w) \cdot \mathcal{F}(P_{y_2})(w) = e^{-y_1|w|}e^{-y_2|w|} = e^{-(y_1+y_2)|w|} = \mathcal{F}(P_{y_1+y_2})(w).
\]

But

\[
  \mathcal{F}(P_{y_1})(w) \cdot \mathcal{F}(P_{y_2})(w) = \mathcal{F}(P_{y_1} * P_{y_2})(w),
\]

Hence

\[
  \mathcal{F}(P_{y_1+y_2})(w) = \mathcal{F}(P_{y_1} * P_{y_2})(w);
\]

and so \( P_{y_1+y_2} = P_{y_1} * P_{y_2} \).
Solutions to Exercises 11.6

1. The even extension of \( f(x) \) is

\[
f_e(x) = \begin{cases} 
1 & \text{if } -1 < x < 1, \\
0 & \text{otherwise}.
\end{cases}
\]

The Fourier transform of \( f_e(x) \) is computed in Example 1, Sec. 7.2 (with \( a = 1 \)). We have, for \( w \geq 0 \),

\[
\mathcal{F}_c(f)(w) = \mathcal{F}(f_e)(w) = \sqrt{\frac{2}{\pi}} \sin \frac{w}{w}.
\]

To write \( f \) as an inverse Fourier cosine transform, we appeal to (6). We have, for \( x > 0 \),

\[
f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathcal{F}_c(f)(w) \cos wx \, dw,
\]

or

\[
\frac{2}{\pi} \int_0^\infty \frac{\sin w}{w} \cos wx \, dw = \begin{cases} 
1 & \text{if } 0 < x < 1, \\
0 & \text{if } x > 1, \\
\frac{1}{2} & \text{if } x = 1.
\end{cases}
\]

Note that at the point \( x = 1 \), a point of discontinuity of \( f \), the inverse Fourier transform is equal to \((f(x^+) + f(x^-))/2\).

5. The even extension of \( f(x) \) is

\[
f_e(x) = \begin{cases} 
\cos x & \text{if } -2\pi < x < 2\pi, \\
0 & \text{otherwise}.
\end{cases}
\]

Let’s compute the Fourier cosine transform using definition (5), Sec. 7.6:

\[
\mathcal{F}_c(f)(w) = \sqrt{\frac{2}{\pi}} \int_0^{2\pi} \cos x \cos wx \, dx
\]

\[
= \sqrt{\frac{2}{\pi}} \int_0^{2\pi} \frac{1}{2} \cos(w+1)x + \cos(w-1)x \, dx
\]

\[
= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left[ \frac{\sin(w+1)x}{w+1} + \frac{\sin(w-1)x}{w-1} \right]_0^{2\pi} \quad (w \neq 1)
\]

\[
= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left[ \frac{\sin(2w+1)x}{w+1} + \frac{\sin(2w-1)x}{w-1} \right] \quad (w \neq 1)
\]

\[
= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left[ \frac{\sin(2\pi w + \pi)x}{w+1} + \frac{\sin(2\pi w)x}{w-1} \right] \quad (w \neq 1)
\]

\[
= \sqrt{\frac{2}{\pi}} \sin 2\pi w \frac{w}{w^2 - 1} \quad (w \neq 1).
\]

Also, by l’Hospital’s rule, we have

\[
\lim_{w \to 0} \sqrt{\frac{2}{\pi}} \sin 2\pi w \frac{w}{w^2 - 1} = \sqrt{2\pi},
\]

which is the value of the cosine transform at \( w = 1 \).

To write \( f \) as an inverse Fourier cosine transform, we appeal to (6). We have, for \( x > 0 \),

\[
\frac{2}{\pi} \int_0^\infty \frac{w}{w^2 - 1} \sin 2\pi w \cos wx \, dw = \begin{cases} 
\cos x & \text{if } 0 < x < 2\pi, \\
0 & \text{if } x > 2\pi.
\end{cases}
\]
For \( x = 2\pi \), the integral converges to \( 1/2 \). So

\[
\frac{2}{\pi} \int_0^\infty \frac{w}{w^2 - 1} \sin 2\pi w \cos 2\pi w \, dw = \frac{1}{2}.
\]

9. Applying the definition of the transform and using Exercise 17, Sec. 2.6 to evaluate the integral,

\[
\mathcal{F}_s(e^{-2x})(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \sin wx \, dx
\]
\[
= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-2x}}{4 + w^2} \left[-w \cos wx - 2 \sin wx\right]_{x=0}^\infty
\]
\[
= \sqrt{\frac{2}{\pi}} \frac{w}{4 + w^2}.
\]

The inverse sine transform becomes

\[
f(x) = \frac{2}{\pi} \int_0^\infty \frac{w}{4 + w^2} \sin wx \, dw.
\]

13. We have \( f_c(x) = \frac{1}{1+x^2} \). So

\[
\mathcal{F}_c\left(\frac{1}{1+x^2}\right) = \mathcal{F}\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{\pi}{2}} e^{-w} \quad (w > 0),
\]

by Exercise 11, Sec. 7.2.

17. We have \( f_c(x) = \frac{\cos x}{1+x^2} \). So

\[
\mathcal{F}_c\left(\frac{\cos x}{1+x^2}\right) = \mathcal{F}\left(\frac{\cos x}{1+x^2}\right) = \sqrt{\frac{\pi}{2}} \left(e^{-|w|-1} + e^{-(w+1)}\right) \quad (w > 0),
\]

by Exercises 11 and 20(b), Sec. 7.2.

21. From the definition of the inverse transform, we have \( \mathcal{F}_c f = \mathcal{F}_c^{-1} f \). So \( \mathcal{F}_c \mathcal{F}_c f = \mathcal{F}_c \mathcal{F}_c^{-1} f = f \).

Similarly, \( \mathcal{F}_s \mathcal{F}_s f = \mathcal{F}_s \mathcal{F}_s^{-1} f = f \).
Solutions to Exercises 11.7

1. Fourier sine transform with respect to $x$:

\[
\frac{d}{dt} \hat{u}_s(w, t) = -w^2 \hat{u}_s(w, t) + \sqrt{\frac{2}{\pi}} w \hat{u}_s(w, t) = 0
\]

Solve the first-order differential equation in $\hat{u}_s(w, t)$ and get

\[
\hat{u}_s(w, t) = A(w) e^{-w^2 t}.
\]

Fourier sine transform the initial condition

\[
\hat{u}_s(w, 0) = A(w) = \mathcal{F}_s(f(x))(w) = T_0 \sqrt{\frac{2}{\pi}} \frac{1 - \cos bw}{w}.
\]

Hence

\[
\hat{u}_s(w, t) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos bw}{w} e^{-w^2 t}.
\]

Taking inverse Fourier sine transform:

\[
u(x, t) = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos bw}{w} e^{-w^2 t} \sin(wx) \, dw.
\]

5. If you Fourier cosine the equations (1) and (2), using the Neumann type condition

\[
\frac{\partial u}{\partial x}(0, t) = 0,
\]

you will get

\[
\frac{d}{dt} \hat{u}_c(w, t) = c^2 \left[ -w^2 \hat{u}_c(w, t) - \sqrt{\frac{2}{\pi}} \frac{d}{dx} \hat{u}_c(w, t) \right] = 0
\]

Solve the first-order differential equation in $\hat{u}_c(w, t)$ and get

\[
\hat{u}_c(w, t) = A(w) e^{-c^2 w^2 t}.
\]

Fourier cosine transform the initial condition

\[
\hat{u}_c(w, 0) = A(w) = \mathcal{F}_c(f)(w).
\]

Hence

\[
\hat{u}_s(w, t) = \mathcal{F}_c(f)(w) e^{-c^2 w^2 t}.
\]

Taking inverse Fourier cosine transform:

\[
u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \mathcal{F}_c(f)(w)e^{-c^2 w^2 t} \cos wx \, dw.
\]
9. (a) Taking the sine transform of the heat equation (1) and using \( u(0, t) = T_0 \) for \( t > 0 \), we get
\[
\frac{d}{dt} \hat{u_s}(w, t) = c^2 \left[ -w^2 \hat{u_s}(w, t) + \sqrt{\frac{2}{\pi}} w u(0, t) \right];
\]
or
\[
\frac{d}{dt} \hat{u_s}(w, t) + c^2 \omega^2 \hat{u_s}(w, t) = c^2 \sqrt{\frac{2}{\pi}} w T_0.
\]
Taking the Fourier sine transform of the boundary condition \( u(x, 0) = 0 \) for \( x > 0 \), we get \( \hat{u_s}(w, 0) = 0 \).

(b) A particular solution of the differential equation can be guessed easily: \( \hat{u_s}(w, t) = \sqrt{\frac{2}{\pi}} \frac{T_0}{w} \). The general solution of the homogeneous differential equation:
\[
\frac{d}{dt} \hat{u_s}(w, t) + c^2 \omega^2 \hat{u_s}(w, t) = 0
\]
is \( \hat{u_s}(w, t) = A(w) e^{-c^2 w^2 t} \). So the general solution of the nonhomogeneous differential equation is
\[
\hat{u_s}(w, t) = A(w) e^{-c^2 w^2 t} \sqrt{\frac{2}{\pi}} \frac{T_0}{w}.
\]
Using \( \hat{u_s}(w, 0) = A(w) \sqrt{\frac{2}{\pi}} \frac{T_0}{w} = 0 \), we find \( A(w) = -\sqrt{\frac{2}{\pi}} \frac{T_0}{w} \). So
\[
\hat{u_s}(w, t) = \sqrt{\frac{2}{\pi}} \frac{T_0}{w} - \sqrt{\frac{2}{\pi}} \frac{T_0}{w} e^{-c^2 w^2 t}.
\]

Taking inverse sine transforms, we find
\[
u(x, t) = \frac{2}{\pi} \int_0^\infty \left( \frac{T_0}{w} - \frac{T_0}{w} e^{-c^2 w^2 t} \right) \sin wx \, dw
\]
\[
= \text{sgn}(x) = 1
\]
\[
= T_0 \frac{2}{\pi} \int_0^\infty \frac{\sin wx}{w} \, dw - 2T_0 \int_0^\infty \frac{\sin wx}{w} e^{-c^2 w^2 t} \, dw
\]
\[
= T_0 - \frac{2T_0}{\pi} \int_0^\infty \frac{\sin wx}{w} e^{-c^2 w^2 t} \, dw
\]

13. Proceed as in Exercise 11 using the Fourier sine transform instead of the cosine transform and the condition \( u(x 0) = 0 \) instead of \( u_y(x, 0) = 0 \). This yields
\[
\frac{d^2}{dx^2} \hat{u_s}(x, w) - w^2 \hat{u_s}(x, w) + \sqrt{\frac{2}{\pi}} u(x, 0) = 0
\]
\[
\frac{d^2}{dx^2} \hat{u_s}(x, w) = w^2 \hat{u_s}(x, w).
\]
The general solution is
\[
\hat{u_s}(x, w) = A(w) \cosh wx + B(w) \sinh wx.
\]
Using
\[
\hat{u_s}(0, w) = 0 \quad \text{and} \quad \hat{u_s}(1, w) = \mathcal{F}_s(e^{-y}) = \sqrt{\frac{2}{\pi}} \frac{w}{1 + w^2},
\]
we get
\[
A(w) = 0 \quad \text{and} \quad B(w) = \sqrt{\frac{2}{\pi}} \frac{w}{1 + w^2} \cdot \frac{1}{\sinh w}.
\]
Hence

\[ \hat{u}_s(x, w) = \sqrt{\frac{2}{\pi}} \frac{w}{\sqrt{1 + w^2}} \frac{\sinh wx}{\sinh w}. \]

Taking inverse sine transforms:

\[ u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{w}{\sqrt{1 + w^2}} \frac{\sinh wx}{\sinh w} \sin wy \, dw. \]