On quotients of real algebraic surfaces in $\mathbb{CP}^3$

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Abstract

We give explicitly the surgeries governing the changes of quotient manifolds of real algebraic surfaces in $\mathbb{CP}^3$. We also make a number of general observations regarding quotients of complex surfaces under antiholomorphic involutions.

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1. Introduction

A real algebraic surface in $\mathbb{CP}^3$ is by definition a set of the form $F^{-1}(0) \subset \mathbb{CP}^3$, where $F(x_0, x_1, x_2, x_3)$ is a homogeneous polynomial with real coefficients. Complex conjugation of the variables induces an antiholomorphic involution on real algebraic surfaces; the quotients $F^{-1}(0)/\text{conj}$ are the subject of this paper.

Fixing the degree $d$, real algebraic surfaces are parametrized by $\mathbb{RP}^N$, where $N$ depends on $d$. The singular surfaces comprise a hypersurface in $\mathbb{RP}^N$, defined by the zero locus of the discriminant. Nonsingular real algebraic surfaces consequently form a disconnected subset of $\mathbb{RP}^N$, and their quotients will generally differ on different components. In contrast, the surfaces $F^{-1}(0)$ themselves have a fixed diffeomorphism type. The main results in this paper describe surgeries that relate the quotients on the different components. Closely related is the classification of the real part $F^{-1}(0) \cap \mathbb{RP}^3$, which is unknown for $\deg F > 4$, despite the vintage work of Hilbert and the contemporary work of several Russian mathematicians.

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Donaldson [4] asked more generally whether quotients of Kähler surfaces under anti-holomorphic involutions can lead to new 4-dimensional manifolds. Attempting to answer this question by gauge-theoretic techniques was one of the motivations of the work [14].

This short paper is organized as follows: in Section 2 we describe explicitly all the quotients $F^{-1}(0)/\text{conj}$ when the degree $\deg F \leq 4$. For higher degree, we give the surgeries on the quotients when a nonsingular real algebraic surface is deformed to another one acrossing a singular surface with a double point. Then in Section 3, we indicate several analogous ways of constructing smooth 4-manifolds. We also prove that in general, these manifolds do not admit compatible Kähler structures, which explains why they are interesting to study.

2. Quotients and surgery

We first identify explicitly all quotients $F^{-1}(0)/\text{conj}$ when $\deg F \leq 4$. If $\deg F = 1$ then $(F^{-1}(0), \text{conj}) \cong (\mathbb{CP}^2, \text{conj})$; thus the only quotient $F^{-1}(0)/\text{conj}$ is diffeomorphic to $\mathbb{CP}^2/\text{conj} \cong S^4$ by [10] or [8]. For $\deg F = 2$, it is straightforward to verify that there are three diffeomorphic types for the quotients: $S^4$, $\overline{\mathbb{CP}^2}$ and $S^2 \times S^2/(x, y) \sim (x', y')$, where $x'$ is the antipode of $x$. For $\deg F = 3$, one can use the classification of real Del Pezzo surfaces to conclude that the quotient $F^{-1}(0)/\text{conj}$ is diffeomorphic to either $S^4$ or $m\overline{\mathbb{CP}^2}$ for $1 \leq m \leq 4$. Details are given in [13].

The case $\deg F = 4$ requires more modern techniques.

**Proposition 1.** For a nonsingular degree four polynomial $F(x_0, x_1, x_2, x_3)$ with real coefficients, the quotient $F^{-1}(0)/\text{conj}$ of $F^{-1}(0) \subset \mathbb{CP}^3$ is diffeomorphic to either an Enriques surface, $S^2 \times S^2$, or $\mathbb{CP}^2#\alpha\overline{\mathbb{CP}^2}$ with $0 \leq \alpha \leq 19$. All possibilities are realized.

**Proof.** It is observed in Donaldson [4] that the quotient of a K3 surface $X$ under any antiholomorphic involution $\sigma$ is always diffeomorphic to a rational or Enriques surface. Briefly the proof goes like this: S.-T. Yau’s solution to the Calabi conjecture yields a hyper Kähler structure $I, J, K$ on $X$, with respect to one of which $\sigma$ becomes holomorphic. Hence the quotient $X/\sigma$ has an induced complex structure. A use of Castelnuovo’s criteria implies that if $\sigma$ is not free, then $X/\sigma$ must be a rational surface, thus diffeomorphic to either $S^2 \times S^2$, or $\mathbb{CP}^2#\alpha\overline{\mathbb{CP}^2}$ for some $\alpha$. (Indeed, since $X/\sigma$ is easily seen to be simply connected, its irregularity $q = 0$. To see the second plurigenus $P_2 = 0$, notice that the square of the canonical bundle is isomorphic to a bundle corresponding to a negative divisor by the proof of the assertion below.) If the involution is free, the quotient is of course diffeomorphic to an Enriques surface.

We can certainly apply this observation to $X = F^{-1}(0)$, since $F^{-1}(0)$ is a K3 surface. To see precisely which $\alpha$’s are realized in the quotients, we only need to find all the possible values of the Betti number $b_2$ of the quotients, and in the case $b_2 = 2$, also whether the quotient is spin or not (giving quotients $S^2 \times S^2$ or $\mathbb{CP}^2#\overline{\mathbb{CP}^2}$). For clarification, let $\Sigma = X \cap \mathbb{RP}^3$ and $Y = X/\text{conj}$. Notice that $X \rightarrow Y$ is a double cover branched over $\Sigma$, the Euler characteristic satisfies $\chi(X) = 2\chi(Y) - \chi(\Sigma)$, so $b_2(Y) = \frac{1}{2}\chi(\Sigma) + 10$. From
Kharlamov's classification of $\Sigma$ in Silhol [11, p. 189], we have $-18 \leq \chi(\Sigma) \leq 20$; thus $1 \leq b_2(Y) \leq 20$. This in turn yields that $Y$ is diffeomorphic to $\mathbb{C}P^2 \# \alpha \mathbb{C}P^2$ with $0 \leq \alpha \leq 19, \alpha \neq 1$, and possibly $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ when $b_2(Y) = 2$. The last two possible quotients correspond to $F$'s with real parts $\Sigma = T_{n_1} \bigsqcup \cdots \bigsqcup S^2$ and $\Sigma = T_n$ in Silhol [11, p. 189], where $T_n$ denotes a genus $n$ surface. Thus the proof is completed by the following claim:

**Assertion.** For all real degree 4 polynomials $F$ such that $\Sigma = T_{n_1} \bigsqcup \cdots \bigsqcup S^2$, the quotients $Y$ are always $S^2 \times S^2$. For $F$ with $\Sigma = T_n$, both $S^2 \times S^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ appear as quotients.

**Proof of Assertion.** First we show that the $[\Sigma] = 0$ or $\neq 0$ in $H_2(X, \mathbb{Z}_2)$ determines that the quotient is $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. From Donaldson's observation cited above, we can view conj as a holomorphic involution on $X$ if we change the complex structure on $X$. (We are only concerned with diffeomorphism types of the quotient $Y$ here.) So $Y$ inherits a complex structure. Let $L$ be a holomorphic line bundle on $Y$ with $L \otimes L$ corresponding to the divisor $\Sigma \subset Y$. Then the holomorphic branched covering $p : X \to Y$ yields $K_X = p^*(K_Y \otimes L)$, where $K_X, K_Y$ are canonical bundles. But $X$ is $K3$: $K_X = 1$, and $p^*$ is injective. So $L = K_Y^{-1}$, namely $\text{PD}^*[\Sigma] = 2c_1(Y)$ on $Y$. Now, if $[\Sigma] = 0 \in H_2(X, \mathbb{Z}_2)$ then there is a double covering of $X$ branched over $\Sigma$, therefore a four-fold covering of $Y$ over $\Sigma$. Thus $4 \mid \text{PD}^*[\Sigma]$ on $Y$, so $4 \mid 2c_1(Y)$, which forces $Y = S^2 \times S^2$. If, on the other hand, $[\Sigma] \neq 0 \in H_2(X, \mathbb{Z}_2)$, then $c_1(Y)$ is not divisible by 4 and $Y$ has to be $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Then from Viro [12], $\Sigma = T_{n_1} \bigsqcup \cdots \bigsqcup S^2$ is always realized by $F$ with $[\Sigma] = 0 \in H_2(X, \mathbb{Z}_2)$; hence the corresponding $Y$ is always $S^2 \times S^2$. For $\Sigma = T_n$, Viro claims that it can be realized by $F$ with both $[\Sigma] = 0$ and $[\Sigma] \neq 0$. Therefore the quotients are $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ respectively. This completes the proof of the assertion and so the proposition. \(\Box\)

The discussions so far have relied on the classification of the real part $F^{-1}(0) \cap \mathbb{R}P^3$, which is available only for $\deg F \leq 4$. For $\deg F > 4$, no such explicit description of the quotients is possible. But we can still examine the surgeries relating different quotients. We start with a variant of Morse's lemma. Suppose $f : U \to \mathbb{C}$, $f(0) = 0$, is a holomorphic function defined in a neighborhood $U$ of the origin in $\mathbb{C}^n$. $f$ is called $c$-equivariant if $f(\overline{z}) = \overline{f(z)}$.

**Lemma 2.** If a $c$-equivariant function $f$ has only one critical point $0 \in U$ which is also nondegenerate, then there is a $c$-equivariant holomorphic coordinate transformation, in a possibly smaller neighborhood than $U$, such that $f = \varepsilon_1 y_1^2 + \varepsilon_2 y_2^2 + \cdots + \varepsilon_n y_n^2$ under the new coordinates, where $\varepsilon_i = \pm 1$.

**Proof.** The only difference from the nonequivariant case is in the induction step. Suppose by induction that there is a $c$-equivariant coordinate transformation such that

$$f = \varepsilon_1 u_1^2 + \varepsilon_2 u_2^2 + \cdots + \varepsilon_{r-1} u_{r-1}^2 + \sum_{i,j \geq r} u_i u_j H_{ij}(u_1, \ldots, u_n),$$

where $H_{ij}$ are symmetric polynomials in $u_1, \ldots, u_n$. Then $f$ is $c$-equivariant if and only if $H_{ij}$ are $c$-equivariant. This is true for $H_{ij} = \delta_{ij}$, the Kronecker delta function, and the claim follows by induction.
in which \( (H_{ij}(u)) \) is a symmetric matrix, \( c \)-equivariant in \( u \). We may as well assume \( H_{rr}(0) \neq 0 \), which is then a real number by the \( c \)-equivariance. Let \( \varepsilon_r = |H_{rr}(0)|/H_{rr}(0) (= \pm 1) \). Since \( \varepsilon_r H_{rr}(u) > 0 \) for \( u \in \mathbb{R} \), it is possible to choose a \( c \)-equivariant holomorphic branch \( g(u) \) of \( \sqrt{\varepsilon_r H_{rr}(u)} \). We introduce the new \( c \)-equivariant coordinates \( v_1, \ldots, v_n \) by

\[
  v_i = u_i \quad \text{for} \quad i \neq r, \quad \text{and} \\
  v_r(u_1, \ldots, u_n) = g(u) \left[ u_r + \sum_{i > r} u_i H_{ir}(u)/H_{rr}(u) \right],
\]

with which it is easy to verify that \( f \) can be expressed as

\[
  f = \sum_{i \leq r} (\varepsilon_r v_i^2) + \sum_{i, j > r} v_i v_j H'_{ij}(v_1, \ldots, v_n)
\]

where \( H'_{ij}(v) \) is \( c \)-equivariant in \( v \). \( \square \)

Any family of real polynomials \( F_t \) with the same degree can be perturbed so that only double (singular) points occur in the singular surfaces. Therefore it is sufficient to consider a family of real algebraic surfaces in which the only singular surface has one double point. More specifically, consider a family of degree \( d \) surfaces in \( \mathbb{CP}^3 \), parameterized by \( t \) within a neighborhood of \( 0 \in \mathbb{C} \). Suppose \( F(t; x) \) is \( c \)-equivariant with respect to \((t; x)\), so we have a family of real algebraic surfaces parameterized by real \( t \). Denote the zero locus of \( F(t; x) \) by \( X_t \), and when \( t \) is real, denote the quotient \( X_t/\text{conj} \) by \( Y_t \).

Suppose that for any fixed \( t \neq 0 \) the surface \( F(t; x) \) is nonsingular everywhere, and that \( F(0; x) \) has only one double point \( a = [1, 0, 0, 0] \in \mathbb{CP}^3 \). If moreover \( \partial F(t; x)/\partial t \) is nonzero at \((0; a)\), so that the equation \( F(t; x) = 0 \) can be solved locally around \((0, a)\) by \( t = f(x) \). Then using the affine coordinates of \( \mathbb{CP}^3 \), Lemma 2 implies that there is a \( c \)-equivariant holomorphic coordinate transformation such that \( f = -y_1^2 + y_2^2 + y_3^2 \) or \( y_1^2 + y_2^2 + y_3^2 \). For convenience, we say that \( a \) is a standard or nonstandard double point of \( F(0; x) \) corresponding to these two forms of \( f \). Thus when the real \( t \neq 0 \) switches signs, for standard double point, the genus of the real part \( X_t \cap \mathbb{RP}^3 \) is changed by one (or two if the real part is nonorientable), but for nonstandard double point a new sphere is created or lost in the real part. (This is because the real zero locus of \( z_0^2 = -x_1^2 + x_2^2 + x_3^2 \) in \( \mathbb{RP}^3 \) is \( S^1 \times S^1 \) and that of \( x_0^2 = x_1^2 + x_2^2 + x_3^2 \) is \( S^2 \).) We can now state one of the two main results in the paper.

**Theorem 3.** Suppose that \( F(t; x) \) is a \( c \)-equivariant family of algebraic surfaces in \( \mathbb{CP}^3 \) and only \( F(0; x) \) has a singular point, which is a standard double point. For small real numbers \( r, s \) with \( rs < 0 \), if the genus of the real part of \( X_r \) is smaller than that of \( X_s \), then \( Y_r \) is diffeomorphic to \( Y_s \# \mathbb{CP}^2 \). In particular \( Y_r \) is always nonspin.
Proof. Without loss of generality, assume that the double point is \( a = [1, 0, 0, 0] \in \mathbb{C}P^3 \). Then as above through a \( c \)-equivariant coordinate change

\[
F(t; x) = 0 \iff tx_0^2 = -x_1^2 + x_2^2 + x_3^2
\]

in a neighborhood \( N \) of \( (0; a) \). By rescaling one can assume that \([-1, 1] \times U \subset N\), where \( U = \{ y \in \mathbb{C}^3 \mid \| y \| < 2 \} \) (using affine coordinates), and it is enough to show \( Y_{-1} \cong Y_1 \# \mathbb{C}P^2 \) to prove the theorem.

Since there is no singular point other than \( a \), \( X_1 \setminus U \cong X_{-1} \setminus U \) even equivariantly. Hence \( Y_1 \setminus K_+ \cong Y_{-1} \setminus K_+ \), where \( K_\pm \) is the quotient of

\[
M_\pm = \{ y \in U \mid -y_1^2 + y_2^2 + y_3^2 = \pm 1 \}
\]

under the conjugation. It remains to examine \( K_\pm \).

First one can show that \( M_\pm \) are both diffeomorphic to the total space \( TS^2 \) of the tangent bundle of \( S^2 \). In fact, \( M_+ \) is simply the set of solutions to the equations in \( \mathbb{R}^6 \)

\[
\begin{align*}
-u_1^2 + u_2^2 + u_3^2 + v_1^2 - v_2^2 - v_3^2 &= 1, \\
-u_1v_1 + u_2v_2 + u_3v_3 &= 0, \\
\sum u_j^2 + \sum v_j^2 &< 4,
\end{align*}
\]

where \( y_j = u_j + iv_j \). Such a system of equations can be transformed to the system

\[
\| w \|^2 = \| z \|^2 - 1, \quad \langle w, z \rangle = 0, \quad 1 \leq \| z \|^2 < 5/2
\]

where \( w = (-u_1, u_2, v_3) \) and \( z = (v_1, u_2, u_3) \), whose solution set is readily seen to be the tangent bundle of the 2-sphere \( S^2 = \{ (z, 0) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \| z \| = 1 \} \) in \( \mathbb{R}^6 \).

Similarly, \( M_- \) is given as the solution set of

\[
\begin{align*}
-u_1^2 - u_2^2 - u_3^2 + v_1^2 + v_2^2 + v_3^2 &= 1, \\
-u_1v_1 + u_2v_2 + u_3v_3 &= 0, \\
\sum u_j^2 + \sum v_j^2 &< 4,
\end{align*}
\]

which can be changed into the system of equations above through a simple transformation.

Next consider the involutions \( \sigma_\pm \) on \( TS^2 \cong M_\pm \) inherited from the conjugation on \( M_\pm \). Clearly \( \sigma_- \) acts on \( S^2 \) by rotating \( \pi \)-angle around the north and south poles, and on \( TS^2 \) the fixed point set \( \text{Fix} \sigma_- \) consists of the two fibres over the poles. Thus \( K_- = TS^2/\sigma_- \) is a \( \mathbb{R}^2 \)-bundle on \( S^2 \) with Euler class \( e = e(TS^2)/2 = 1 \) (from the Chern–Weil formula of \( c_1 = e \)). Furthermore by identifying \( K_- \) with its open disk bundle, the boundary or more precisely the end \( \partial K_- \) is the circle bundle on \( S^2 \) with \( e = 1 \), namely the Hopf line bundle. Thus \( K_- \cong S^3 \) and \( K_- \cong \mathbb{C}P^2 \), i.e., \( \mathbb{C}P^2 \) with a disk removed. It follows that \( Y_{-1} = (Y_{-1} \setminus K_-) \cup S^3 \mathbb{C}P^2 \) or \( (Y_{-1} \setminus K_-) \cup S^3 \mathbb{C}P^2 \), where the choice of orientations on \( \mathbb{C}P^2 \) will be determined below.

One can easily trace back to see that the induced involution \( \sigma_+ \) on \( TS^2 \cong M_+ \) fixes a circle \( (u_1 = 0) \), over whose fibers \( \sigma_+ \) sends \( \xi \) to \( -\xi \). Thus \( K_+ \cong B^4 \), a 4-ball, and \( Y_1 = (Y_1 \setminus K_+) \cup S^3 B^4 \). Since \( Y_1 \setminus K_+ \cong Y_{-1} \setminus K_- \), the above two glueings give
either $Y_{-1} = Y_1 \# \mathbb{CP}^2$ or $Y_{-1} = Y_1 \# \overline{\mathbb{CP}}^2$. To rule out the first possibility notice that $b_2^+(Y_1) = b_2^+(Y_{-1})$, where $b_2^+(Y_{\pm 1})$ is the dimension of a maximal positive subspace of $H^2(Y_{\pm 1})$ under the intersection pairing. (As given in the next section, it is not difficult to prove that $b_2^+(Y_1)$ and $b_2^+(Y_{-1})$ both equal the geometric genus of the complex surface $X_1$.) □

The case of a nonstandard double point has a new feature, as the boundaries of $K_\pm$ are not $S^3$ and the surgery is no longer given by connected sums. We need to define such a surgery operation. Let $T_4$ be the orientable disk bundle over $S^2$ with Euler class $e = 4$ and $N_1$ be the nonorientable disk bundle of the tangent bundle of $\mathbb{RP}^2$ (with Euler number 1). Then as bases of 4-fold coverings of the Hopf line bundle $S^3 \to S^2$, the circle bundles $\partial T_4, \partial N_1 \cong L(4, 1) = S^3/\mathbb{Z}_4$, where $L(4, 1)$ denotes a lens space. (A related lucid discussion is contained in Lawson [9].)

**Definition 4.** Suppose that a 4-manifold $Z$ can be written as $Y \cup_{L(4,1)} T_4$, then we call the operation $Z \to Y \cup_{L(4,1)} N_1$ to be an $N$-surgery.

In other words, an $N$-surgery first splits $T_4$ then glues back $N_1$. Since $\pi_1(N_1) = \mathbb{Z}_2$, the $N$-surgery can change the fundamental group of a manifold. (The letter N means nonstandard in this sense.)

**Theorem 5.** Suppose all conditions in Theorem 3 are satisfied, except that $F(0; x)$ has a nonstandard double point. Then the quotient $Y_t$ is changed by $N$-surgery when $t$ passes through 0.

**Proof.** We outline the proof, which is similar to the proof of Theorem 3. Without loss of generality, one can again assume that $F = 0$ is given by $tx_0^2 = x_1^2 + x_2^2 + x_3^2$ near the nonstandard double point $[1, 0, 0, 0]$. It is then enough to identify the quotients $L_\pm$ of $H_\pm = \{ y \mid y_1^2 + y_2^2 + y_3^2 = \pm 1 \text{ and } \|y\| < 2 \}$ under the conjugation.

As in the proof of Theorem 3, through $TS^2 \cong H_+$, the conjugation acts on $TS^2$ by rotating all fibres a $\pi$ angle; therefore the quotient $L_+$ is a bundle on $S^2$ with Euler number twice of that of $TS^2$, namely 4. Thus $L_+ \cong T_4$. Similarly via $TS^2 \cong H_-$, conjugation acts freely on $TS^2$ and as the antipodal map on the base $S^2$, the quotient is thus the tangent bundle of $\mathbb{RP}^2$, namely $N_1$ introduced above. So $Y_1 = (Y_1 \setminus L_+) \cup T_4$ and $Y_{-1} = (Y_{-1} \setminus L_-) \cup N_1$, that is, $Y_1$ is changed into $Y_{-1}$ under the $N$-surgery. □

There are a couple of examples explaining the $N$-surgery. A basic one is the family of quadric surfaces $P(t; x) = tx_0^2 + x_1^2 + x_2^2 + x_3^2$: it is not difficult to check that for $t < 0$ the quotient $Y_t$ is $\overline{\mathbb{CP}}^2$ and for $t > 0$, $Y_t$ is $S^2 \times S^2/\sigma_0$ where $\sigma_0(u, v) = (-u, -v)$. Using the language of $N$-surgery, this says that we cut $T_4$ from $\overline{\mathbb{CP}}^2 = N_1 \cup_{L(4,1)} T_4$ (see Lawson [9]) and paste back $N_1$ so get $\overline{N_1} \cup_{L(4,1)} N_1$ which is indeed $S^2 \times S^2/\sigma_0$.

A more subtle example of $N$-surgery is a real K3 surface. By Proposition 1 above, there is a real degree four polynomial $F$ such that the corresponding quotient $P^{-1}(0)/\text{conj}$ is $\mathbb{CP}^2 \# 10\overline{\mathbb{CP}}^2$ with real part $S^2$. Such a polynomial can be deformed, via one double
point, into a real polynomial with empty real part, therefore giving the Enriques surface as quotient. Thus the N-surgery in this case changes $\mathbb{CP}^2 \# 10\overline{\mathbb{CP}}^2$ into the Enriques surface. Since the Enriques surface can be identified with the Dolgachev surface $D_{2,2}$, the N-surgery here is realized as first splitting off a $\mathbb{CP}^2$ from $\mathbb{CP}^2 \# 10\overline{\mathbb{CP}}^2$ and then composing with the $(2,2)$-logarithmic transform on the elliptic surface $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ to get $D_{2,2}$.

These examples show that N-surgery can indeed change manifolds substantially, e.g., fundamental groups. It appears interesting to investigate such surgery in some details. Also according to Hodgeson and Rubinstein [6], the diffeomorphism group of $L(4,1)$ up to isotopy is $\mathbb{Z}_2$ (which send $1 \in \mathbb{Z}_4 = \pi_1(L(4,1))$ to $1$ or $-1$), thus there are two ways to glue $T_4$ and $N_1$. It is not clear to the author how they affect the surgery.

3. Other related constructions

Take $X$ to be the covering of $\mathbb{CP}^2$ branched over the zero locus of a real polynomial $f(x_0, x_1, x_2)$ of even degree. Then lifting the conjugation on $\mathbb{CP}^2$ endows $X$ with two antiholomorphic involutions. For degree up to six, where the classification of real algebraic curves $f^{-1}(0)$ is available, one concludes that the quotients of $X$ under these involutions are standard, namely $S^4$, $S^2 \times S^2$ or connected sums of $\mathbb{CP}^2$ with its reverse. More generally for higher degree, Akbulut [1] has proved that the quotients are again standard if the real part $f^{-1}(0) \cap \mathbb{RP}^2$ consists of $\frac{1}{2} \deg f$ number of concentric circles in $\mathbb{RP}^2$. With the help of the analogues of Theorems 3 and 5, one would extend this result to other cases, see [1].

There is an immediate extension for the construction above. Instead of $\mathbb{CP}^2$, one can start with any complex surface $Z$ with antiholomorphic involution. If $C \subset Z$ is any complex curve invariant under the involution, generating a homology class divisible by 2 in $H_2(Z, \mathbb{Z})$, then the covering of $Z$ branched over $C$ inherits two antiholomorphic involutions. The point is that with a simple $Z$ such as $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$, one can arrive at extremely complicated branched coverings together with antiholomorphic involutions, by deforming the complex curve $C$. It is a subtle issue what Kahler metrics the branched covering inherits from $Z$. (For the case of $\mathbb{CP}^2$ over real algebraic curves, there is a canonical choice for the branched covering which has a natural Kahler structure.)

In general, the following proposition explains in one way why quotients of Kahler surfaces under antiholomorphic involutions are interesting objects:

**Proposition 6.** Suppose $X$ is a Kahler surface with an antiholomorphic and metric-preserving involution $\sigma$. If the real part $\Sigma = \text{Fix } \sigma$ is orientable with genus bigger than 1, then except the case $b_2^+(X) = 3$, the quotient smooth manifold $Y = X/\sigma$ never admits a compatible Kahler structure.

**Proof.** Let $T, N$ be respectively the tangent and normal bundles of $\Sigma$ in $X$. Then $\chi(\Sigma) = c_1(T)$ and $c_1(N) = \Sigma \circ \Sigma$ (selfintersection). One can easily verify that by restricting to $TX|_{\Sigma} = T \oplus N$, the almost complex structure on $X$ induces an orientation-reversing isomorphism $T \to N$. Thus $c_1(T) = -c_1(N)$ and hence $\chi(\Sigma) = -\Sigma \circ \Sigma$. 

Since the projection $p: X \to Y$ is a double cover branched over $\Sigma' = p(\Sigma)$, the following formulae are familiar:

$$\chi(X) = 2\chi(Y) - \chi(\Sigma), \quad \tau(X) = 2\tau(Y) - \Sigma \circ \Sigma,$$

where $\chi$ and $\tau$ stand for Euler characteristic and signature respectively. See [2] and [5]. Also a simple application of Hodge decomposition shows that $b_1(X) = 2b_1(Y)$. One can then verify directly that $b_2^+(X) = 1 + 2b_2^+(Y)$, from these formulae and $\chi(\Sigma) = -\Sigma \circ \Sigma$ showed above. In other words $b_2^+(Y)$ equals the geometric genus of $X$ (so independent of $\sigma$).

Since $b_2^+(X) \neq 3$, $b_2^+(Y) \neq 1$. So either $b_2^+(Y)$ is even, in which case $Y$ can not have Kähler structures by Hodge decomposition, or $b_2^+(Y)$ is odd and bigger than 1, in which case $Y$ still can not have Kähler structures by the following argument. As showed above, $\Sigma' \circ \Sigma' = 2\Sigma \circ \Sigma = -2\chi(\Sigma) > -\chi(\Sigma)$, namely $\Sigma \circ \Sigma > -\chi(\Sigma')$. This inequality for the surface $\Sigma'$ in $Y$ violates the conclusion of the main theorem of Kronheimer and Mrowka [7]. Therefore the key assumption for the nonvanishing of the Donaldson invariant is not satisfied for $Y$. (The other assumptions in their theorem are obviously satisfied: $\Sigma' \circ \Sigma' > 0$, $b_2^+(Y) > 1$ is odd.) Once the Donaldson invariant of $Y$ vanishes, a theorem of Donaldson [3] implies that $Y$ does not admit Kähler structures. \(\square\)

We have seen in Proposition 1 that the quotients of K3 surfaces under antiholomorphic involutions are actually rational complex surfaces. So the assumption $b_2^+(X) \neq 3$ can not be dropped in the theorem above.

The situation here is of course in sharp contrast with quotients under holomorphic involutions, where for typical cases such as algebraic surfaces, the quotients always have Kähler structures.

Proposition 6 covers abundant examples. For instance any even degree real algebraic surfaces in $\mathbb{CP}^3$ and coverings of $\mathbb{CP}^2$ over any curves of degree $2(1 + 2d)$ all satisfy the conditions in Proposition 6. (The restrictions on the degree guarantee that the fixed point set is orientable.) Therefore none of the quotients from these Kähler manifolds have Kähler structures, and it would be very interesting to see whether these quotients can actually be decomposed into connected sums.

Finally we propose another construction which may also lead to non-Kähler 4-manifolds. Take any Kähler surface with antiholomorphic involution. Suppose the real part generates a homology class divisible by two, then we can take the covering space branched over the real part. Such a branched covering appears to be interesting to study. For example the branched covering of $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ over the standard $\mathbb{RP}^2 \# \mathbb{RP}^2$ is the Hopf surface $S^1 \times S^3$, which does not admit any Kähler structures again!

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