Quantum Theory of the Electron Liquid

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Corrections and Notes

Chapter 1

p. 11, footnote 17: The unit of electronic mobility is cm^2/V.s, not cm^2 V/s.

p. 18, 6$^{\mathrm{th}}$ line: $v_q \propto q^{2-d}$ should be replaced by $v_q \propto q^{1-d}$.

p. 33, Eq. 1.93: The minus sign in the square brackets should be replaced by a + sign. (Giovanni Pizzi)

p. 48, 26$^{\mathrm{th}}$ line: The reference (Ceperley, 2003) should be corrected to (Ceperley, 2004).

p. 51, Table 1.5: The value of the coefficient c_1 in 3D was *not* obtained from a fit to the numerical QMC data, but was set equal to the coefficient of the corresponding classical energy reported on the second line of Table 1.3. The same coefficient, in 2D, was obtained from a fit to the numerical QMC data and therefore differs slightly from the coefficient of the corresponding classical energy, reported on the last line of Table 1.3 (Klaus Capelle, May 2009).

p. 59, Eq. 1.139: The fourth expression from the left should read $n\frac{\partial \mu}{\partial n}$.

p. 66, Exercise 1.8: In the formula for the contact interaction r_s should be replaced by r_s^d. Accordingly, the factor r_s^2 in the denominator of the second term in the square bracket of the expression for $\epsilon(r_s, p)$ in three dimensions should be removed.

Chapter 2

Chapter 3

p. 122, Eq. 3.59: This equation is valid under the assumption that it is possible to choose a complete set of real eigenfunctions for the hamiltonian. Contrary to what stated in the text, time-reversal invariance is in general not sufficient to guarantee this. For instance, the usual spin-orbit interaction is time-reversal invariant, yet leads to complex eigenfunctions. On the other hand it is

sometimes possible to find a real set of eigenfunctions even for hamiltonians that are not time-reversal invariant. A simple example of the latter case is that of the Zeeman coupling between spins and a *unidirectional* magnetic field. For spin $\frac{1}{2}$ particles, in view of the structure of the time reversal operator, the general form of the reciprocity relation that follows from time-reversal invariance is

$$\chi_{AB}(\omega, \vec{B}) = \chi_{^TB^TA}(\omega, -\vec{B}) \ ,$$

where we have defined ${}^T\hat{A} \equiv \hat{\sigma}_y({}^t\hat{A})\hat{\sigma}_y$ (σ_y is the *sum* of the Pauli matrices for each particle of the system) and $\vec{B}$ is the magnetic field. This result follows from the fact that if ψ_n is an eigenfunction of the hamiltonian with magnetic field $\vec{B}$ then $i\hat{\sigma}_y\psi_n^*$ is an eigenfunction of the hamiltonian with magnetic field $-\vec{B}$, with the same eigenvalue. The above equation has general validity. If the eigenfunctions of the hamiltonian can be chosen to be real, then Eq. (3.59) also holds.

p. 126, Eq. 3.75: $\coth\left(\frac{\beta\hbar\omega}{2}\right)$ must be replaced by $\tanh\left(\frac{\beta\hbar\omega}{2}\right)$. (Marco Polini)

p. 127, Eq. 3.77: Notice that for this equation to hold (with consequences 3.78 etc.) the function $\chi_{AB}(z)$ must be analytic not only in the upper half of the complex plane, but also on the real axis.

p. 128, Fig. 3.1: The axes labels for this figure should be $\Re e(z)$ and $\Im m(z)$. (Danny Dhokarh)

p. 135, Eq. 3.115: The sign of the second term on the right hand side should be changed from minus to plus. (Iacopo Torre)

p. 137, Eq. 3.130: $(\hat{n}_{\vec{q}})_{nm}$ should be replaced by $(\hat{n}_{-\vec{q}})_{nm}$ (Danny Dhokarh).

p. 140, Eq. 3.145: The factor m in the denominator should be removed. (Giovanni Pizzi)

p. 153, Eq. 3.210: The sign on the right hand side of these two equations should be reversed, i.e., the equations should read $\chi_{S_+S_-} = 2[\chi_{S_xS_x} - i\chi_{S_xS_y}]$ and $\chi_{S_-S_+} = 2[\chi_{S_xS_x} + i\chi_{S_xS_y}]$ (Danny Dhokarh).

Chapter 4

p. 160, Eq. 4.14: The transverse spin-spin response function given in this equation should be multiplied by 2, i.e., the factor $\frac{2}{L^d}$ on the right hand side of the equation should be replaced by $\frac{4}{L^d}$.

p. 161, Eq. 4.19: The sum symbol $\sum_\sigma$ should be removed. (Giovanni Pizzi)

p. 162, Eq. 4.25: The sign of the expression for $\frac{\Im m\chi_{0\sigma}(q,\omega)}{N_\sigma(0)}$ in 1D should be reversed.

p. 173, Eq. 4.45:

- The expression for $\frac{\Im m\chi_{0\sigma}(q,\omega)}{N_\sigma(0)}$ in 3D should be divided by 2. (Marco Polini)

- The sign of the expression for $\frac{\Im m\chi_{0\sigma}(q,\omega)}{N_\sigma(0)}$ in 2D should be reversed. (Marco Polini)

p. 186, Eq. E4.1: The transverse spin-spin response function given in this equation should be multiplied by 2, i.e., $2N_\uparrow(0)$ on the right hand side of the equation should be replaced by $4N_\uparrow(0)$ and $2N_\downarrow(0)$ by $4N_\downarrow(0)$.

Chapter 5

p. 224, first line: The reader should refer to Fig. 1.21, not Fig. 1.22.

p. 271, Exercise 5.4: In the equation for q_c, k_3 should be replaced by κ_3 – the 3-dimensional Thomas-Fermi wave vector defined in Eq. (5.36).

Chapter 6

p. 278 – 279, Section 6.2.2: Although the final result of this section [Eqs. (6.27) and (6.28)] is correct, the derivation contains two errors that compensate each other. The first error is in Eq. (6.13): the correct form of the connection between the Heisenberg picture operator $\hat{A}(t)$ and the Schrödinger-picture operator $\hat{A} \equiv \hat{A}(-\infty)$ is

$$\hat{A}(t) = \hat{U}^{-1}(t,-\infty)\hat{A}\hat{U}(t,-\infty) \ , \qquad (6.13)$$

where the time-evolution operator $\hat{U}(t,-\infty)$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,-\infty) = \hat{H}(t)\hat{U}(t,-\infty) \ . \qquad (6.14)$$

These two equations should replace Eqs. (6.13) and (6.14), respectively. The second error occurred in the first line of Eq. (6.23), in which the operator $U_I(0,-\infty)$ is placed on the extreme right following a manipulation that is not allowed without the protection of the time-ordering operator. The correct derivation, following from Eqs. (6.13) and (6.14) and up to Eq. (6.23) included is as follows:

"The operator $\hat{U}(t,-\infty)$ can be more conveniently expressed as

$$\hat{U}(t,-\infty) = e^{-\frac{i\hat{H}_0 t}{\hbar}}\hat{U}_I(t,-\infty) \ , \qquad (6.15)$$

where $\hat{U}_I(t,-\infty)$ represents the part of the time evolution that is due to the electron-electron interaction (as opposed to the "trivial" time-evolution, which is due to $\hat{H}_0$). It is easy to verify that $\hat{U}_I(t,-\infty)$ satisfies the equation of motion

$$i\hbar\frac{\partial}{\partial t}\hat{U}_I(t,-\infty) = \hat{H}_{1I}(t)e^{-\eta|t|}\hat{U}_I(t,-\infty) \qquad (6.16)$$

with the initial condition $\lim_{t\to-\infty}\hat{U}_I(t,-\infty) = \hat{1}$. This equation has the formal solution

$$\hat{U}_I(t,-\infty) = T\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{t}\hat{H}_{1I}(t')e^{-\eta|t'|}dt'\right) \ , \qquad (6.?)$$

where the time-ordering acts on the power-series expansion of the exponential. The operator

$$\hat{H}_{1I}(t) \equiv e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{H}_1 e^{-\frac{i\hat{H}_0 t}{\hbar}} \ , \qquad (6.17)$$

is the interaction hamiltonian in the so-called *interaction picture* of the time evolution. We will henceforth attach a subscript I to operators that evolve in time according to this picture; e.g., we set

$$\hat{A}_I(t) \equiv e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{A}e^{-\frac{i\hat{H}_0 t}{\hbar}} \ . \qquad (6.18)$$

Substituting Eqs. (6.15) and (6.13) in Eq. (6.12) we obtain

$$\chi^T_{AB}(t) = -\frac{i}{\hbar}\langle\Phi_0|T[\hat{U}_I^{-1}(t,-\infty)\hat{A}_I(t)\hat{U}_I(t,-\infty)U_I^{-1}(0,-\infty)\hat{B}U_I(0,-\infty)]|\Phi_0\rangle \ . \qquad (6.19)$$

The right-hand side of this expression can be greatly simplified with the help of two rather obvious properties of the time-evolution operator:

$$(i) \qquad \hat{U}_I(t_2,t_1) = \hat{U}_I^{-1}(t_1,t_2) = \hat{U}_I^{\dagger}(t_1,t_2) \ , \qquad (6.20)$$

(i.e., the inverse of the operator that propagates the state from t_1 to t_2 is the operator that propagates the state from t_2 to t_1), and

$$(ii) \qquad \hat{U}_I(t_3,t_2)\hat{U}_I(t_2,t_1) = \hat{U}_I(t_3,t_1) \ . \qquad (6.21)$$

Let us focus on the case $t > 0$ first. This allows us to disregard the time-ordering operator in Eq. (6.12) (the operators are already in chronological order), and hence in Eq. (6.19). After writing $\hat{U}_I^{-1}(t,-\infty) = \hat{U}_I(-\infty,t) = \hat{U}_I(-\infty,\infty)\hat{U}_I(\infty,t)$, and noting that the operators that appear in the expansion of

$$U_I(0,-\infty) = T\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{0}\hat{H}_{1I}(t')e^{-\eta|t'|}dt'\right) \qquad (6.22)$$

are all at times earlier than 0, we see that Eq. (6.19) can be rewritten in the form

$$\chi^T_{AB}(t) = -\frac{i}{\hbar}\langle\Phi_0|\hat{U}_I(-\infty,\infty)\hat{U}_I(\infty,t)\hat{A}_I(t)\hat{U}_I(t,0)\hat{B}_I\hat{U}_I(0,-\infty)|\Phi_0\rangle$$

$$= -\frac{i}{\hbar}\langle\Phi_0|\hat{U}_I(-\infty,\infty)T[\hat{U}_I(\infty,t)\hat{A}_I(t)\hat{U}_I(t,0)\hat{B}_I\hat{U}_I(0,-\infty)]|\Phi_0\rangle \ . \quad (6.23)$$

Notice that in the second line we introduced a time-ordering operator acting on the block $[\hat{U}_I(\infty,t)\hat{A}_I(t)\hat{U}_I(t,0)\hat{B}_I\hat{U}_I(0,-\infty)]$. This could be done only because those operators are already time-ordered."

From this point on the original text resumes.

p. 280, line after Eq. 6.26: $\langle\Phi_0|U_I(\infty,-\infty) = e^{-i\alpha}\langle\Phi_0|$ should be replaced by $\langle\Phi_0|U_I(-\infty,\infty) = \exp^{-i\alpha}\langle\Phi_0|$.

p. 281, Eqs. 6.33 and 6.34: The factor $e^{-\eta|t|}$ should be removed. (Giovanni Pizzi).

p. 283, sixth line after Eq. 6.40: "disconnected" should be replaced by "connected".

p. 305, line above Eq. 6.89: "Eq. (6.89) can be written more compactly as" should be replaced by "The above equation can be written more compactly as"

p. 309, Eq. 6.99: The second term on the right hand of this equation should be multiplied by a factor $-i$.

p. 309, Eqs. 6.100: The second term on the right hand sides of both equations should be multiplied by a factor $-2i$.

p. 310, Eq. 6.101: The second term on the right hand of this equation should be multiplied by a factor $-i$.

p. 321, Eq. 6.127: A factor π is missing in the definition of the Matsubara frequencies. The correct formulas should read

$$\begin{aligned} \omega_k &= \frac{2k\pi}{\hbar\beta} \qquad (even\ f) \\ \omega_k &= \frac{(2k+1)\pi}{\hbar\beta} \qquad (odd\ f) \end{aligned} \quad (1)$$

($k = 0, \pm1, \pm2.....$). The same typo is present in Eq. (6.142) on p. 323 and in the second and third line of p. 324 (see corrections below).

p. 321, Eq. 6.128: The upper limit of the integral should be changed from β to $\hbar\beta$.

p. 323, Eq. 6.142: The expression for the Matsubara frequency should be corrected to

$$\omega_k = \frac{(2k+1)\pi}{\hbar\beta} \ .$$

p. 324, second line: The expression for the Matsubara frequency should be corrected to $\omega_k = \frac{(2k+1)\pi}{\hbar\beta}$.

p. 324, third line: The expression for the Matsubara frequency should be corrected to $\omega_k = \frac{2k\pi}{\hbar\beta}$.

p. 324, Rule (vii): $\left(\frac{1}{\hbar}\right)^{n+1}(-1)^L$ should be replaced by $\left(-\frac{1}{\hbar}\right)^{n+1}(-1)^L$.

Chapter 7

p. 331, Fig. 7.2: A line is missing.

p. 332, footnote 5: It is not true that the Hohenberg-Kohn theorem implies the invertibility of the linear density-density response function. In fact, the theorem leaves open the possibility of a non-constant potential $V(r)$ producing no change in density to first order, while changing the density in higher orders. However, such a scenario would imply the non-existence of the functional derivative a $F[n]$ with respect to the density at constant particle number – a possibility we have informally discarded at the end of Section 7.2.1 (see however footnote 2).

p. 342: The definition of the ionization potential $I(M)$ in the paragraph following Eq. (7.58) has the wrong sign. The correct definition is $I(M) = -\mu_+(M-1)$.

p. 345, Eq. 7.67: The correct expression for the "Kohn-Sham gap" is

$$E_{g,KS} = \varepsilon_{M+1} - \varepsilon_M$$

following from the fact that for the Kohn-Sham system, as for any noninteracting system, $E(M+1) - E(M) = \varepsilon_{M+1}$ and $E(M) - E(M-1) = \varepsilon_M$. Thanks to Klaus Capelle for pointing this out.

Chapter 8

p. 432, Eq. 8.76: The factor n on the right hand side of this equation should be replaced by m^2, where m is the mass. (Alessandro Principi)

p. 450, Eq. 8.133: This equation contained several errors. The correct

equation should read

$$n_{\vec{k},\sigma} \simeq \frac{1}{2}\begin{cases} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_{10} x^5 & (x < \sqrt{2}) \\ \frac{4g(0)r_s^2}{x^6} + \left(a_7 + a_8 x + a_9 x^2\right) \exp\left\{-\frac{x-\sqrt{2}}{a_5} - \frac{(x-\sqrt{2})^2}{a_6^2}\right\} & (x > \sqrt{2}) \end{cases}$$

with $x = \sqrt{2}\frac{k}{k_F}$. $g(0)$ is given by Eq. A4.11 and Table A4.1. Thanks to Dr. Neil Drummond for pointing this out.

p. 451, Table 8.3: The values of Z on the last line of this table should be corrected as follows:

r_s	1	2	5	10	20	40
Z	0.878	0.732	0.466	0.267	0.103	0.027

Thanks to Dr. Neil Drummond for pointing this out.

p. 478, end of footnote 51: The complete reference is Simion G. and Giuliani G. F. *Many-body local fields theory of quasiparticle properties in a three-dimensional electron liquid*, Physical Review B **77**, 035131 (2008).

p. 498, Exercise 8.18: In the equation for $\frac{1}{\tau^{(e)}_{x,\vec{k}\sigma}}$, the symbol *theta'* should be replaced by θ'_f.

Chapter 9

p. 508, Eq. 9.18 (ii): Replace $2\pi v_F$ by $2\pi\hbar v_F$. (Giovanni Pizzi)

p. 509, Eq. 9.21: Replace $-\hat{H}_{int}$ by $\hat{H}_{int}$. (Giovanni Pizzi)

p. 510, Eq. 9.25: The expression on the right hand side should include a factor $\frac{1}{\hbar v_F}$. (Giovanni Pizzi)

p. 516, Eq. 9.49 The first two lines of this equation should be modified as follows:

$$\begin{aligned} A^{-2} &= \langle 0| e^{-\sum_{q>0} \tanh\varphi_q \hat{b}_q \hat{b}_{-q}} e^{-\sum_{q>0} \tanh\varphi_q \hat{b}_q^\dagger \hat{b}_{-q}^\dagger} |0\rangle \\ &= \int \prod_{q\neq 0} \frac{d\Re e\varphi_q d\Im m\varphi_q}{\pi} e^{-\sum_{q>0}\left[|\varphi_q|^2 + |\varphi_{-q}|^2 + \left(\varphi_q\varphi_{-q} + \varphi_q^*\varphi_{-q}^*\right)\tanh\varphi_q\right]} \end{aligned}$$

The last line remains unchanged.

p. 517, Eq. 9.52 The integral should run from $-\infty$ to $+\infty$. Accordingly, the right hand side should be multiplied by 2:

$$\frac{E}{L} \simeq \frac{\pi}{6}\frac{(k_B T)^2}{\hbar c} \ .$$

p. 517, Eq. 9.53 The heat capacity should be multiplied by 2:

$$c_v(T) = L\frac{\pi}{3}\frac{k_B^2 T}{\hbar c} \ .$$

Chapter 10

p. 553, Footnote 7: The value of the flux quantum must be corrected as follows:

$$\frac{hc}{e} \simeq 4.13 \times 10^{-7} \text{ G.cm}^2 = 4.13 \times 10^{-15} \text{ T.m}^2 \, .$$

p. 555, line immediately under Eq. 10.10: The correct expression for the operator $\hat{H}_Z$ is the following: $\hat{H}_Z = -\frac{g\mu_B}{2} B\hat{\sigma}_z$.

p. 608, Derivation of Eq. 10.152 for the thermodynamic gap of a fractional quantum Hall liquid: The text beginning after Eq. (10.154) and ending with Eq. (10.157) (included) contains several errors and should be replaced as follows:
"Let us now reduce the magnetic field from $B + \frac{B}{N}$ to B at constant particle number $(N+1)$. The removal of M flux quanta amounts, according to the discussion of Section 10.9, to the creation of M quasi-electrons. The energy of the resulting state is therefore

$$E_0(N+1, B) = E_0\left(N+1, B + \frac{B}{N}\right) + M\varepsilon^e \ , \qquad (10.155)$$

where ε^e is the energy of the quasi-electron (the quasi-electrons are put infinitely far from one another so as not to incur an electrostatic energy cost). Combining Eqs. (10.154) and (10.155) we obtain

$$E_0(N+1, B) - E_0(N, B) = \frac{3}{2}\epsilon(\nu)\frac{e^2}{\epsilon_b \ell} + M\varepsilon^e \ . \qquad (10.156)$$

By a completely parallel argument we also arrive at

$$E_0(N-1, B) - E_0(N, B) = -\frac{3}{2}\epsilon(\nu)\frac{e^2}{\epsilon_b \ell} + M\varepsilon^h \ , \qquad (10.157)$$

where ε^h is the energy of a quasi-hole."

Sorry about this!

p. 620, first line: "elastic potential energy – a function of the displacement field and its derivatives" should be replaced by "elastic potential energy – a function of the displacement field and its spatial derivatives".

p. 623, Eq. (10.221): The quantity $q\ell$ on the right hand side of this equation should be replaced by $q\ell^2$.

p. 663, Exercise 10.5 The following integral (Gradshteyn and Ryzhik, 1965, 7.377) comes handy $(j, m \geq 0)$

$$\int_{-\infty}^{\infty} e^{-x^2} H_{j+m}(x-a) H_j(x-b) dx = (-1)^m 2^{j+m} \pi^{1/2} j! a^m L_j^m(-2ab) \ .$$

Appendix 1

Appendix 2

p. 672, Eq. (A2.19) and **p. 673, Eq. (A2.21):** in both equations $\vec{k} + \frac{\vec{q}}{2}$ must be replaced by $\vec{k} - \frac{\vec{q}}{2}$

Appendix 3

Appendix 4

p. 682, between eqs. (A4.3) and (A4.4): The modulus of $\vec{r}_2 - \vec{r}_1$ has an opening |, but no closing |. (Klaus Capelle)

p. 683, caption of Fig. A4.1, third line: "G. Senatoze" should be replaced by "G. Senatore".

p. 687, Eq. (A4.23): This formula gives the spin-resolved static structure factor for the ground-state of the non-interacting electron gas, which is *paramagnetic* ($n_\uparrow = n_\downarrow = \frac{n}{2}$). In the general case of a non-interacting *spin polarized* electron gas ($n_\uparrow \neq n_\downarrow$) the structure factor is given by the formula

$$S^{(0)}_{\sigma\sigma'}(\vec{q}) \ = \ \delta_{\sigma\sigma'} \begin{cases} \frac{n_\sigma}{n} \left[\frac{3}{4} \frac{q}{k_{F\sigma}} - \frac{1}{16} \frac{q^3}{k^3_{F\sigma'}} \right] \ , & 3D, \\ \frac{2n_\sigma}{\pi n} \left[\sin^{-1} \frac{q}{2k_{F\sigma}} + \frac{q}{2k_{F\sigma}} \sqrt{1 - \left(\frac{q}{2k_{F\sigma}} \right)^2} \right] \ , & 2D, \\ \frac{n_\sigma}{n} \frac{q}{2k_{F\sigma}} \ , & 1D, \end{cases} \qquad \text{(A4.23)}$$

for $|q| \leq 2k_{F\sigma}$, and

$$S^{(0)}_{\sigma\sigma'}(\vec{q}) = \frac{n_\sigma}{n} \delta_{\sigma\sigma'} \ , \qquad \text{(A4.24)}$$

for $|q| > 2k_{F\sigma}$ in all dimensions.

Appendix 5

Appendix 6

Appendix 7

p. 693, Eq. (A7.2): $\frac{1}{L^{2d}}$ should be replaced by $\frac{1}{L^d}$.

Appendix 8

Appendix 9

Appendix 10

Appendix 11

p. 702, Eq. (A11.3): $\beta_\pm$ should be replaced by β_+.
p. 702, Eq. (A11.4): $\beta_\pm$ should be replaced by β_-.

Appendix 12

Appendix 13

Appendix 14

Appendix 15

Appendix 16

Appendix 17

Appendix 18

Appendix 19

Appendix 20

Appendix 21

Appendix 22

p. 734, Eq. (A22.11): π^2 should be replaced by π^3.

Appendix 23

Appendix 24

Appendix 25
References

p. 758, 1st line: The correct reference should read Rajagopal, A. K., Kimball J. C. (1977). Correlations in a two-dimensional electron system, *Physical Review B* **15**, 2819–2825.

p. 758, 3rd line: The correct reference should read, Rajagopal, A. K., Kimball J. C., and Banerjee, M. (1978). Short-ranged correlations and the ferromagnetic electron gas, *Physical Review B* **18**, 2339–2345.