Math 331 - Summary of Chapter VI -

1. Eigen Values and Eigen Vectors

$V$ is a vector space over $\mathbb{R}$, the real numbers (or $\mathbb{C}$, the complex numbers). Let $T: V \rightarrow V$ be a linear transformation.

A scalar $\lambda$ is called an eigenvalue of $T$ if there is a non-zero vector $v$ in $V$ such that $T(v) = \lambda v$. This non-zero vector $v$ is called an eigenvector of $T$ with the eigenvalue $\lambda$.

If $A$ is a square matrix of size $n$ over $\mathbb{R}$ (or $\mathbb{C}$, the complex numbers), then the eigenvalues and eigen vectors of $A$ are the eigenvalues and the eigen vectors of the linear transformation on $\mathbb{R}^n$ (or $\mathbb{C}^n$ defined by multiplication by $A$. Let $V = \mathbb{R}^n$ (or $\mathbb{C}^n$.

A scalar $\lambda$ is called an eigenvalue of $A$ if there is a non-zero vector $v$ in $V$ such that $Av = \lambda v$. This non-zero vector $v$ is called an eigenvector of $A$ with the eigenvalue $\lambda$.

The characteristic polynomial $P_A(\lambda)$ of $A$ is the polynomial

$$ \text{det}(A - \lambda I) = (-1)^n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 $$

For a matrix $A$, and a scalar $\lambda$,

$E_\lambda(A) = \{v|A(v) = \lambda v\}$ is the subspace of all eigen vectors with eigen value $\lambda$. This is called the eigen space of $\lambda$.

Thus, dimension of $E_\lambda(A)$ is not zero if and only if $\lambda$ is an eigenvalue of $A$.

**Theorem 1.** $x$ is an eigen value of $A$ if and only if $x$ is the root of the characteristic polynomial of $A$, that is if and only if $P_A(x) = 0$.

**Theorem 2.** Similar matrices have the same characteristic polynomial and hence the same eigen values.

**Theorem 3.** $A$ and $A^t$, the transpose of $A$, have the same characteristic polynomial and the same eigen values.

As a result of Theorem 2, the eigen values and eigen vectors of a linear transformation $T$ as defined above can be computed by finding those of the matrix of $T$ with respect to some basis $B$ of the vector space $V$. Theorem 2 says that the eigen values and eigen vectors do not depend on the choice of the basis.

To Compute the eigen values and the eigen vectors of a matrix $A$.

1. Compute $A-xI$.
2. Compute determinant $(A-xI) = P_A(x)$
3. Solve $P_A(x) = 0$ to find the roots $x = \lambda_1, \cdots, \lambda_t$
4. Proceed to find the eigen vectors. $E_{\lambda_i}(A) = \text{Null space of } (A - \lambda_i I)$.

Use Gaussian elimination to find a basis for the null space of $A - \lambda_i I$ for each $i$.

**Theorem 4.** If $v_1, v_2, \cdots, v_n$ are eigen vectors of $A$ with distinct eigen values $\lambda_1, \cdots, \lambda_n$, then $v_1, v_2, \cdots, v_n$ are linearly independent.

An eigen value $\lambda$ of a matrix $A$ is said to be of multiplicity $k$ if $(\lambda - x)^k$, divides the characteristic polynomial $P_A(x)$ and $(\lambda - x)^{k+1}$ does not divide $P_A(x)$. That is, $\lambda$ is an eigen value with multiplicity $k$ if $\lambda$ is a root of $P_A(x) = 0$ with multiplicity $k$. 

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Theorem 5: If \( \lambda \) is an eigen value of \( A \) with multiplicity \( k \) then \( \dim E_\lambda(A) \leq k \).

To compute the multiplicity of the eigen value \( \lambda \) and \( \dim E_\lambda(A) \),

Factor the characteristic polynomial \( P_A(x) \). Then the multiplicity of \( \lambda \) equals the power of \( (x - \lambda) \) in the factorization of \( P_A(x) \).

Compute a row echelon form of matrix \( A - \lambda I \). Dimension of the eigne space of \( \lambda \) is the number of columns without a leading 1.

2. Diagonalization

A matrix \( A \) is **diagonalizable** if there is an invertible matrix \( X \) such that \( X^{-1}AX \) is a diagonal matrix.

**Theorem 6.** An \( n \times n \) matrix \( A \) is diagonalizable over \( \mathbb{R} \) (or \( \mathbb{C} \)) if and only if there is a basis for \( \mathbb{R}^n \) (respectively \( \mathbb{C}^n \)) consisting of eigen vectors of \( A \).

To determine if a given matrix \( A = (a_{ij})_{n \times n} \) is diagonalizable:

1. Compute the eigen values and their multiplicities as in the previous section.
2. Let \( \lambda_1, \cdots, \lambda_t \) be the distinct eigen values of \( A \) with multiplicities \( k_1, k_2, \cdots k_n \) respectively.

Compute the basis for the eigen spaces \( E_{\lambda_i}(A) \) as before. If \( \dim (E_{\lambda_i}(A)) < k_i \) for any \( i \), then \( A \) is not diagonalizable. If \( \dim E_{\lambda_i}(A) = k_i \) for all \( i \), then \( A \) is diagonalizable. Collect all the vectors in the bases of \( E_{\lambda_i}(A) \) for each \( i \), to get a \( v_1, v_2, \cdots v_n \) consisting of eigen vectors. Let \( X = v_1, v_2, \cdots v_n \). Then \( X^{-1}AX \) is diagonal.

3. Orthogonal diagonalization

A matrix \( A \) is said to be orthogonally diagonalizable if there is an orthogonal matrix \( X \) such that \( XAX^{-1} \) is diagonal. \( A \) is orthogonally diagonalizable if there is an orthonormal basis consisting of eigen vectors.

**Theorem 7.** Suppose that \( A \) is a symmetric matrix. If \( v_1 \) and \( v_2 \) are eigen vectors with distinct eigen values \( t_1 \) and \( t_2 \) respectively, then \( v_1 \) and \( v_2 \) are orthogonal to each other.

**Theorem 8.** If \( A \) is orthogonally diagonalizable then \( A \) must be symmetric.

Suppose that \( A \) is symmetric. To orthogonally diagonalize \( A \): 1. Proceed as in section 2 to find the eigen values and bases for eignen spaces.
2. Use Gram-Schmidt process to find orthonormal basis for each eigen space. Putting these together gives us an orthonormal basis of eigen vectors.