5. Measure groupoids

The best notation is no notation; whenever it is possible to avoid the use of a complicated alphabetic apparatus, avoid it. A good attitude to the preparation of written mathematical exposition is to pretend that it is spoken. Pretend that you are explaining the subject to a friend on a long walk in the woods, with no paper available; fall back on symbolism only when it is really necessary.

—P. R. Halmos, How to Write Mathematics (1970)

5.1 Groupoids

We shall now present the formalism of Connes’ noncommutative integration theory. The main technical notion is that of a measure groupoid. This has its roots in Mackey’s virtual group program [29], [30]. The integration theory for nonstandard spaces described previously is more conveniently described in terms of a measure groupoid, and various generalizations are possible. This theory was developed by Connes in [27], where he uses it to prove a generalization of the Atiyah-Singer index theorem. Another reference, with slightly more detail on some points, is Kastler’s article [28].

We now define the notion of a groupoid, and describe some of its basic properties. References for groupoids are [31], [32], [27], [28], and [65] or [66] for the basics of category theory. We shall assume that all categories are small, meaning that the objects comprise a set, and the morphisms comprise a set.

**Definition 5.1.** A groupoid $G$ is a (small) category with inverses, i.e., every morphism has an inverse.

The objects of the category are a set denoted by $G(0)$. For $x, y \in G(0)$, the set of morphisms $\gamma : x \to y$ is denoted by $G^x_y$, which may be the empty set. As a set, $G$ is the disjoint union of all the morphisms. The composition $\gamma' \circ \gamma$ of the morphisms $\gamma'$ and $\gamma$ is defined if $\gamma : x \to y$ and $\gamma' : y \to z$, and then $\gamma' \circ \gamma : x \to z$. We shall abbreviate by omitting the composition symbol, and write $\gamma' \circ \gamma$ as $\gamma' \gamma$. The composition is associative. There is for each $y \in G(0)$ a unit $1_y \in G$, such that $1_y : y \to y$, and with the properties $1_y \gamma = \gamma$ and $\gamma' 1_y = \gamma'$, whenever these
compositions are defined. $1_y$ is unique for each $y \in G^{(0)}$. There is an injection

$$G^{(0)} \rightarrow G,$$

$$y \mapsto 1_y \quad (5.1)$$

and consequently a bijection between the set of objects $G^{(0)}$ and the set of units in $G$. By way of this identification, the set of objects $G^{(0)}$ becomes in a natural way a subset of the set of morphisms $G$. The preceding properties all follow just from the definition of a (small) category. We have in addition the property that the category has inverses. Each morphism $\gamma : x \rightarrow y$ has an inverse $\gamma^{-1} : y \rightarrow x$, such that $\gamma \gamma^{-1} = 1_y$, and $\gamma^{-1} \gamma = 1_x$. The inverse is unique.

We define for $\gamma : x \rightarrow y$ the source and range maps $s$ and $r$ by

$$s : G \rightarrow G^{(0)}$$

$$\gamma \mapsto x, \quad (5.2)$$

$$r : G \rightarrow G^{(0)}$$

$$\gamma \mapsto y. \quad (5.3)$$

We shall frequently use the notation $G^y$ to denote $G^y = r^{-1}(y) \subset G$. We define the set of multiplyable pairs $G^{(2)} \subset G \times G$ by the requirement $(\gamma, \gamma') \in G^{(2)}$ if $s(\gamma) = r(\gamma')$.

**Example 5.2.** Suppose $G^{(0)}$ has only one element, $G^{(0)} = x$. It is easy to check from the definitions that $G$ is then a group. First note that $G^x = G$, and $G^{(2)} = G \times G$. Now each element $\gamma \in G$ has an inverse, and the composition

$$G \times G \rightarrow G$$

$$(\gamma, \gamma') \mapsto \gamma \gamma'$$

is associative.  

\[ \square \]
Definition 5.3. A groupoid $G$ defines an equivalence relation $\sim$ on $G^{(0)}$ as follows: for $x, y \in G^{(0)}$, $x \sim y$ if there exists a $\gamma \in G$ such that $s(\gamma) = x$, and $r(\gamma) = y$.

It is easy to check that this relation is indeed an equivalence relation:

(i) $\sim$ is reflexive: $x \sim x$ for any $x \in G^{(0)}$, because there exists $1_x : x \to x$,

(ii) $\sim$ is symmetric: $x \sim y \Rightarrow y \sim x$ because each $\gamma : x \to y$ has an inverse,

(iii) $\sim$ is transitive: if $x \sim y$ and $y \sim z$, then $x \sim z$, because $\gamma' : y \to z$ can be composed with $\gamma : x \to y$, yielding $\gamma' \gamma : x \to z$.

We obtain a map

$$P : G \to G^{(0)} \times G^{(0)}$$

$$\gamma \mapsto (s(\gamma), r(\gamma)).$$

The range of $P$ is precisely the the graph (definition 3.9) of the equivalence relation $\sim$.

The groupoid of example 5.2 had one object, and many morphisms. There is a class of groupoids, the principal groupoids, which are at the opposite extreme, namely they have many objects, but at most one morphism between any pair of objects.

Definition 5.4. We say a groupoid $G$ is **principal** if the map $P$ of equation (5.4) is injective.

There is a bijective correspondence between equivalence relations on $G^{(0)}$ and principal groupoids with set of units $G^{(0)}$. Any groupoid $G$ defines an equivalence relation on $G^{(0)}$ (definition 5.3). Conversely, given a space $G^{(0)}$ and an equivalence relation, we can construct a principal groupoid $G$, which consists of the elements $\gamma : x \to y$ for all equivalent $x, y \in G^{(0)}$.

Definition 5.5. Given an equivalence relation $\sim$ on $G^{(0)}$, we the unique principal groupoid $G$ corresponding to $\sim$ under the above bijection is called the principal groupoid **associated** to $\sim$. 
We now construct a few concrete examples of principal groupoids.

**Example 5.6.** Let \( G^{(0)} = I = [0, 1] \subset \mathbb{R} \), with the trivial equivalence relation \( x \sim y \) for all \( x, y \in G^{(0)} \). The graph of this equivalence relation is simply \( G = I \times I \). For any \((x, y) \in I \times I\), there exists a unique \( \gamma \in G \) such that \( \gamma : x \rightarrow y \). Denote this by \( \gamma = (x, y) \). In fact, the map \( P \) of eq. (5.4) is surjective. The range and source maps take the form

\[
    r(x, y) = x \quad s(x, y) = y.
\]

The elements \((z, y)\) and \((y', x)\) can be composed if \( y = y' \), then

\[
    (z, y) \circ (y, x) = (z, x).
\]

The units are of the form \((y, y)\). The inverse of \((x, y)\) is clearly \((y, x)\). Equation (5.1) gives the identification between \( I \) and the units in \( G \), which is simply the inclusion of the diagonal.

We now consider more general equivalence relations on \( I \). Let \( R \subset I \times I \) be the graph of an equivalence relation on \( I \). The set \( R \) has a natural groupoid structure as the principal groupoid associated to the equivalence relation (definition 5.5). This principal groupoid will thus be denoted also by \( R \). There is a natural inclusion

\[
    R \subset I \times I, \quad (5.5)
\]

preserving the groupoid structure. We thus consider \( R \) as a *subgroupoid* of \( I \times I \).

**Example 5.7.** Let \( \beta \) be a positive rational number, and \( \sim \) be the equivalence relation on \( I \) defined by \( x \sim y \) if there exist an integer \( n \in \mathbb{Z} \) such that

\[
    x = y + n\beta \quad (\text{mod } 1).
\]

Suppose \( \beta = p/q \), where \( p \) and \( q \) are positive integers with no common integer factors. Each equivalence class is finite, in fact each has \( q \) elements.
Example 5.8. Let $\alpha$ be a positive irrational number, and $\sim$ be the equivalence relation on $I$ defined by $x \sim y$ if there exist an integer $n \in \mathbb{Z}$ such that

$$x = y + n\alpha \pmod{1}.$$  

Each equivalence class is countably infinite. Denote the associated principal groupoid by $R_\alpha$.  

We give one more example.

Example 5.9. Consider the partition of the torus $T^2$ into lines of irrational slope $\alpha$, example 3.14. The associated principal groupoid $R_\alpha$ is a subgroupoid of $T^2 \times T^2$.

$$R_\alpha = \{(x, y, \hat{x}, \hat{y}) \in T^2 \times T^2|\hat{y} - y = \alpha(\hat{x} - x) \pmod{1}\}$$

The set of units is the diagonal $T^2 \subset T^2 \times T^2$. The groupoid $R_\alpha$ can also be realized as $T^2 \times \mathbb{R}$, with the set of units equal to $T^2 \times \{0\}$,

$$s : T^2 \times \mathbb{R} \to T^2$$

$$(x, y, z) \mapsto (x + z \pmod{1}, y + \alpha z \pmod{1}, 0)$$

$$r : T^2 \times \mathbb{R} \to T^2$$

$$(x, y, z) \mapsto (x - z \pmod{1}, y - \alpha z \pmod{1}, 0)$$

One easily checks that for any two points $p, q \in T^2$ lying on the same line of slope $\alpha$, there exists a unique point $t \in T^2 \times \mathbb{R}$ such that $s(t) = p$ and $r(t) = q$.  

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5.2 Measure groupoids

We now introduce a Borel structure on a groupoid $G$.

**Definition 5.10.** A measure groupoid is a groupoid $G$ with a Borel structure $B$ such that the following maps are all measurable:

\[
\begin{align*}
\text{inv} : G & \to G \\
\gamma & \mapsto \gamma^{-1},
\end{align*}
\] (5.6)

\[
\begin{align*}
\circ : G^{(2)} & \to G \\
(\gamma, \gamma') & \mapsto \gamma' \gamma,
\end{align*}
\] (5.7)

\[
\begin{align*}
r, s : G & \to G^{(0)},
\end{align*}
\] (5.8)

where $G \times G$ is given the product Borel structure, and $G^{(0)} \subset G$ and $G^{(2)} \subset G \times G$ the induced Borel structures.

There exist slight variations in terminology, the terms measurable groupoid or measured groupoid are sometimes used. The idea of a measure groupoid originated in Mackey’s work on virtual groups, see e.g. [31], [32].

**Example 5.11.** The groupoid $R_\alpha$ of example 5.9, with the Borel structure given by the product of the usual Borel structures on $T$ and $R$ is a measure groupoid. 

We now discuss principal groupoids with Borel structures arising from equivalence relations of a Borel space. Consider a space $M$ with a Borel structure $\mathcal{A}$, and an equivalence relation $\sim$ on $M$. We form the principal groupoid $G$ associated to the equivalence relation as in definition 5.5.

**Definition 5.12.** Let $G$ be the principal groupoid associated to an equivalence relation on a Borel space $(M, \mathcal{A}) = (G^{(0)}, \mathcal{A})$. The Borel structure $B$ on $G$ generated by sets of the form $s^{-1}(A_1) \cap r^{-1}(A_2)$, where $A_1, A_2 \in \mathcal{A}$ will be called the *Borel structure associated to $\mathcal{A}$*. 

Note that this coincides with the Borel structure induced from the inclusion $G \subset G^{(0)} \times G^{(0)}$, where $G^{(0)} \times G^{(0)}$ has the product Borel structure. We shall sometimes abbreviate, and say that $(G, \mathcal{B})$ is the principal (measure) groupoid associated to the equivalence relation $\sim$ on $(M, \mathcal{A})$.

The following proposition gives necessary and sufficient conditions for a principal groupoid associated to an equivalence relation on a Borel space to be a measure groupoid. This will be important in the sequel.

**Proposition 5.13.** Let $(G^{(0)}, \mathcal{A})$ be a Borel space, and $\sim$ an equivalence relation on $G^{(0)}$. The associated principal groupoid $(G, \mathcal{B})$ is a measure groupoid if and only if the graph of $\sim$ is a Borel subset of $G^{(0)} \times G^{(0)}$.

**Proof:** ($\Rightarrow$): Suppose $(G, \mathcal{B})$ as defined above is a measure groupoid. By hypothesis, $G \subset G^{(0)} \times G^{(0)}$ has the Borel structure induced by the inclusion. Now $G = r^{-1}G^{(0)}$, and by hypothesis $r$ is a measurable map, so $G$ is a Borel set.

($\Leftarrow$): It is clear that the graph has a natural groupoid structure, we need to check that the maps $inv$, $\circ$, $r$, and $s$ of definition 5.10 are measurable. We first check that equation $s$ is measurable. Since $G^{(0)} \in \mathcal{A}$, for any $A \in \mathcal{A}$, the set

$$s^{-1}(A) \cap r^{-1}(G^{(0)}) = s^{-1}(A) \subset G$$

is in $\mathcal{B}$, thus $s$ is measurable. Similarly,

$$s^{-1}(M) \cap r^{-1}(A) = r^{-1}(A) \subset G,$$

so $r$ is measurable.

We now verify that the map $inv$ of equation (5.6) is measurable. The sets of the form $B = s^{-1}(A_1) \cap r^{-1}(A_2)$, $A_1, A_2 \in \mathcal{A}$ generate $\mathcal{B}$. This is mapped to

$$inv : B \mapsto B' = s^{-1}(A_2) \cap r^{-1}(A_1),$$

and $B'$ belongs to the generating family. Thus each Borel set is mapped to a Borel set, and since the map $inv$ is an involution, the inverse image of each Borel set is Borel.
Finally we check that the multiplication, equation (5.7), is measurable. Define the maps

\[ m : G \times G \to G^{(0)} \times G^{(0)} \]
\[ (\gamma, \gamma') \mapsto (s(\gamma), r(\gamma')) \]

\[ \bar{m} : G \times G \to G^{(0)} \times G^{(0)} \]
\[ (\gamma, \gamma') \mapsto (r(\gamma), s(\gamma')) \]

The maps \( m \) and \( \bar{m} \) are clearly measurable if we take the product Borel structures on \( G \times G \) and \( G^{(0)} \times G^{(0)} \). We have the natural inclusion \( G \subset G^{(0)} \times G^{(0)} \). The set of multipliable pairs was defined as

\[ G^{(2)} = m^{-1}G. \]

We thus see that \( G^{(2)} \) is Borel in \( G \times G \).

Consider the subset \( B = s^{-1}(A_1) \cap r^{-1}(A_2) \), with \( A_1, A_2 \in A \). The set \( B \) belongs to the generating family of subsets of \( G \). We see that \( (\circ)^{-1}(B) \), which is a subset \( G \times G \), equals

\[ (\circ)^{-1}(B) = G^{(2)} \cap \bar{m}^{-1}(A_2 \times A_1), \]

where \( A_2 \times A_1 \) is a Borel subset of \( G^{(0)} \times G^{(0)} \). Thus \( (\circ)^{-1}(B) \) is the intersection of Borel subsets of \( G \times G \), and is Borel. Since the Borel structure is generated by sets of the form \( B \), we see that the map \( \circ \) is measurable.

Corollary 5.14. The principal groupoid \( G \) associated to an equivalence relation on a standard space \( G^{(0)} \) is a measure groupoid if and only if \( G \) is a standard space.

Proof: \( (\Rightarrow) \): \( G^{(0)} \) is a standard space by hypothesis, so \( G^{(0)} \times G^{(0)} \) is standard by proposition 3.5. By proposition 5.13, \( G \subset G^{(0)} \times G^{(0)} \) is Borel, and thus standard by proposition 3.3.

\( (\Leftarrow) \): By proposition 3.3, \( G \) is a Borel subset of \( G^{(0)} \times G^{(0)} \). Thus \( G \) is a measure groupoid by proposition 3.5.
5.3 Kernels

We now develop the notion of a kernel [27] [28]. A kernel is in a sense a generalization of the canonical system of conditional measures for a measured partition. We shall only require the more specialized notion of a $G$-kernel, where $G$ is a measure groupoid, but for clarity, we first define the general notion. Consider two Borel spaces $(Y, \mathcal{B})$ and $(Y', \mathcal{B}')$. Let $\mathcal{F}^+(Y)$ and $\mathcal{F}^+(Y')$ respectively denote the positive extended real valued measurable functions on $Y$ and $Y'$, i.e., the measurable maps into $\mathbb{R}^+ = [0, +\infty]$. (Connes [27] uses the notation $\mathcal{F}^+$ to denote the subset of $\mathcal{F}^+$ consisting of nowhere vanishing functions, i.e., maps into $(0, +\infty]$. We shall instead follow Kastler’s notation [28], and explicitly specify when a function $f \in \mathcal{F}^+$ is nowhere vanishing.)

**Definition 5.15.** A *kernel* $\lambda$ from $Y$ to $Y'$, abbreviated by $\lambda : Y \to Y'$, is a map

$$\lambda : \mathcal{F}^+(Y) \to \mathcal{F}^+(Y'),$$

which has the following properties:

(i) $\lambda$ is affine, i.e.,

$$\lambda(\alpha f + \beta g) = \alpha \lambda(f) + \beta \lambda(g),$$

for $\alpha, \beta \in \mathbb{R}^+$, and $f, g \in \mathcal{F}^+(Y)$,

(ii) $\lambda$ is normal, i.e.

$$f_n \nearrow f \Rightarrow \lambda(f_n) \nearrow \lambda(f),$$

for a countable family $f_n \in \mathcal{F}^+(Y)$, $n \in \mathbb{Z}$.

The notation $f_n \nearrow f$ denotes the function which at the point $x \in Y$ takes the values $f(x) = \sup_{n \in \mathbb{Z}} f_n(x)$. Condition (ii) makes sense because $f$ is in $\mathcal{F}^+(Y)$ (Halmos [67] Thm. 20A).
Example 5.16. Consider a Lebesgue space \((M, \mathcal{B}, \mu)\) and a measurable partition \(\xi\). There is a natural kernel

\[
\lambda_\xi : M \to M/\xi.
\]

For notational simplicity, we write \(\lambda_\xi\) as simply \(\lambda\). Let \(f \in \bar{\mathcal{F}}^+(M)\), and \(C \subset M\) be an element of the partition \(\xi\), then \(\lambda(f) \in \bar{\mathcal{F}}^+(M/\xi)\) is the function which at the point \(C \in M/\xi\) takes the value (recall definition 3.11)

\[
(\lambda(f))(C) = \int_{C \subset M} f|_C \ d\mu_C.
\]

One readily checks that \(\lambda(f)\) is indeed a measurable function on \(M/\xi\).  

A kernel \(\lambda : Y \to Y'\) is alternately described as a measurable map

\[
Y' \to \bar{\mathcal{M}}^+(Y)
\]

\[
y' \mapsto \lambda^{y'},
\]

(5.9)

where \(\bar{\mathcal{M}}^+(Y)\) is the set of positive measures on \(Y\), and \(\lambda^{y'}\) is the measure on \(Y\) defined by

\[
\lambda^{y'}(S) = (\lambda(1_S))(y'), \quad S \in \mathcal{B}.
\]

Here \(1_S\) is the characteristic function of the set \(S \subset M\). The map \(Y' \to \bar{\mathcal{M}}^+\) must be measurable, in the sense that for \(S \in \mathcal{B}\), the function on \(Y'\) which at the point \(y' \in Y'\) takes the value \(\lambda^{y'}(S)\) should belong to \(\bar{\mathcal{F}}^+(Y')\). This is equivalent to the requirements of definition 5.15.

Example 5.17. Consider the kernel \(\lambda : M \to M/\xi\) associated to the measurable partition \(\xi\) of \(M\) in example 5.16. For \(C \in M/\xi\), the associated measure \(\lambda^C \in \bar{\mathcal{M}}^+(M)\) is simply equal to \(\mu_C\), the conditional measure on \(C \subset M\), definition 3.11.
Let $B \in \mathcal{B}$ be the largest Borel subset of $Y$ such that $\lambda^{y'}(B) = 0$ for all $y' \in Y'$. The support of a the kernel $\lambda : Y \to Y'$ is defined to be $Y - B$, the complement of $B$ in $Y$. If $Y'$ consists of a single point, this just reduces to the usual notion of the support of a measure.

The notion of a kernel $\lambda : Y \to Y'$ is more general than that of a measurable partition and an associated system. For example, if $\lambda_\xi : Y \to Y'$ is the kernel associated to a measurable partition $\xi$ of $Y$, and $Y' = M/\xi$, then for two distinct points $z, u \in Y'$, the supports of the measures $\lambda^z, \lambda^u \in \mathcal{M}^+(Y)$ are disjoint. An arbitrary kernel need not have this property.

**Definition 5.18.** Suppose $\pi : Y \to Y'$ is a measurable surjection. We say that the kernel $\lambda : \bar{\mathcal{F}}^+(Y) \to \bar{\mathcal{F}}^+(Y')$ is fibered by $\pi$ if for each $y' \in Y'$, the support of $\lambda^{y'}$ is carried by the fiber $\pi^{-1}y' \subset Y$.

Equivalently, $\lambda$ is fibered by $\pi$ if for $f \in \bar{\mathcal{F}}^+(Y)$ and $g \in \bar{\mathcal{F}}^+(Y')$,

$$\lambda(f \cdot \pi^* g) = g \cdot \lambda(f).$$

The dot ($\cdot$) denotes multiplication of functions, and we recall that by definition, $\pi^* g = g \circ \pi$.

A kernel $\lambda$ is called $\sigma$-finite if there exists a sequence $B_n$ of measurable subsets of $Y$, with $\lambda(1_{B_n})$ finite for each $n$, and $\bigcup_n B_n = Y$. A kernel $\lambda$ is called proper if there exists a sequence $B_n$ of measurable subsets of $Y$, with $\lambda(1_{B_n})$ for each $n$ a bounded function on $Y'$, and $\bigcup_n B_n = Y$. It is easily verified that an equivalent condition for $\lambda$ to be $\sigma$-finite is that there exist a nowhere vanishing $f \in \bar{\mathcal{F}}^+(Y)$ such that $\lambda(f) \in \bar{\mathcal{F}}^+(Y')$ is finite. Similarly, $\lambda$ is proper if and only if there exists a nowhere vanishing $f \in \bar{\mathcal{F}}^+(Y)$ such that $\lambda(f) \in \bar{\mathcal{F}}^+(Y')$ is bounded. A kernel $\lambda : Y \to Y'$ fibered by a measurable surjection $\pi : Y \to Y'$ is $\sigma$-finite if and only if it is proper [27], [28]. We shall in the sequel assume all kernels are proper unless explicitly mentioned otherwise. We remark that the notion of a canonical system of conditional measures for a measurable partition $\xi$ on a
normalised measure space \((M, \mathcal{B}, \mu)\), example 5.16 and example 5.17, corresponds to a finite kernel \(\lambda_\xi : M \to M/\xi\) fibered by the natural projection. By finite, we mean that \(\lambda(1_M) \in \mathcal{F}^+ M/\xi\) is bounded. A finite kernel is clearly \(\sigma\)-finite and proper.

The notion of kernel also generalizes the notion of a canonical system of conditional measures to the case where the measure cannot be normalised, i.e., is infinite.

**Example 5.19.** Consider the kernel \(\lambda : \mathbb{R}^2 \to \mathbb{R}\) defined by

\[
\lambda : \mathcal{F}^+(\mathbb{R}^2) \to \mathcal{F}^+(\mathbb{R})
\]

\[
f(x, y) \mapsto \int_{y=-\infty}^{+\infty} f(x, y) dy,
\]

where \(\mathbb{R}^2\) and \(\mathbb{R}\) have the usual (Haar) measures. They are standard spaces, but not Lebesgue, since the measure is infinite. This kernel has the appropriate properties of a canonical system of conditional measures with respect to the partition defined by the projection

\[
p : \mathbb{R}^2 \to \mathbb{R}
\]

\[(x, y) \mapsto x.
\]

But the Haar measure on \(\mathbb{R}\) is not the pushforward under \(p\) of the usual measure on \(\mathbb{R}^2\). The pushforward measure on \(\mathbb{R}\) is either infinite or zero on all measurable sets of \(\mathbb{R}\).

We note however that merely the use of a kernel instead of a canonical system of conditional measures does not solve the difficulties associated with the torus partitioned into lines of irrational slope. The torus has a normalised measure, and the desired kernel is fibered by the projection on the quotient, thus the notion of a kernel is not in this case any more general. The problems associated to the pathological measure structure of the quotient remain.
We now introduce the notion of $G$-kernel, where $G$ is a measure groupoid.

**Definition 5.20.** A $G$-kernel is a kernel $\lambda : G \to G^{(0)}$ fibered by $r : G \to G^{(0)}$.

For each $x \in G^{(0)}$, the measure $\lambda^x$ is carried by $G^x = r^{-1}(x)$.

**Example 5.21.** We first consider $G = I \times I$, the principal groupoid of example 5.6 above. We define a $G$-kernel $\lambda$, such that for $x \in G^{(0)}$, $\lambda^x = \mu_x$, where $\mu_x$ is the usual measure on $r^{-1}x = I$. We recall that $r^{-1}x$ is the set $\{x\} \times I \subset I \times I$. Thus for $f(x, y) \in \bar{\mathcal{F}}^+(G)$, we have at the point $z = (z, z) \in G^{(0)}$

$$\lambda(f)(z) = \int_{x=0}^1 f(x, z) dx.$$  

A $G$-kernel $\lambda$ allows us to define the convolution $f^*_\lambda g$ of two elements $f, g \in \bar{\mathcal{F}}^+(G)$, with respect to $\lambda$;

$$^*_\lambda : \bar{\mathcal{F}}^+(G) \times \bar{\mathcal{F}}^+(G) \to \bar{\mathcal{F}}^+(G)$$
$$\quad (f, g) \mapsto f^*_\lambda g$$
$$\quad (f^*_\lambda g)(\gamma) = \int d\lambda^{r(\gamma)}(\gamma') f(\gamma') g(\gamma'^{-1}\gamma).$$  

**Example 5.22.** Define $G$ and $\lambda$ as in example 5.21. We compute the convolution of two functions $f, f' \in \bar{\mathcal{F}}^+(G)$. Choose a point $(\hat{x}, \hat{y}) \in G$. Then

$$\quad (f^*_\lambda f')(\hat{x}, \hat{y}) = \int_{y=0}^1 dy \ f(\hat{x}, y) \cdot f'(\hat{x}, y)^{-1} \circ (\hat{x}, \hat{y})$$
$$\quad = \int_{y=0}^1 dy \ f(\hat{x}, y) \cdot f'(y, \hat{y}) $$
$$\quad = \int_{y=0}^1 dy \ f(\hat{x}, y) \cdot f'(y, \hat{y}).$$

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Consider
\[ f(x, y) = \exp(2\pi i(mx + ny)), \quad f'(x, y) = \exp(2\pi i(m'x + n'y)), \]
\[ (f \ast f')(\hat{x}, \hat{y}) = \exp(2\pi i(m\hat{x} + n'\hat{y})) \delta_{0,n+m'}. \]

It is useful to define another type of convolution, namely the convolution of a G-kernel \( \lambda \) and a function \( f \in \mathcal{F}^{+}(G) \). This takes the form
\[ (\lambda * f)(\gamma) = \int d\lambda^r(\gamma')(\gamma')f(\gamma'^{-1}\gamma). \]

We define the product of \( f \) and \( \lambda \) as the G-kernel \( (f\lambda) \), which has the property that for \( y \in G^{(0)} \), \( (f\lambda)^y = f \cdot \lambda^y \). The convolution of functions \( f \ast g, f, g \in \mathcal{F}^{+} \), is the same as the convolution \( (f\lambda) \ast g \) of the G-kernel \( f\lambda \) and the function \( g \). There is a technical point we must keep in mind: the function \( \lambda \ast f \) is measurable if and only if \( \lambda \) is proper [28], [27].

### 5.4 Transverse functions

We now introduce the important notion of a transverse function on \( G \). A transverse function \( \nu \) is a G-kernel which is in a certain sense invariant under the action of \( G \). A transverse function on a groupoid is the analogue of a Haar measure on a group, and in fact coincides with Haar measure for a groupoid which is a group. We shall follow the approach and terminology of Connes [27], [28]. Earlier work, including an existence proof, had been done by Hahn [32], building on the work of Mackey [29,30] and of Ramsay [31] on virtual groups. We shall use the term transverse function to refer to these objects, as this is much more suitable term in the context of integration over singular spaces.

**Definition 5.23.** A G-kernel \( \nu \) is called a transverse function if \( \nu^x = \gamma \nu^y \) for any \( \gamma : y \to x \), where \( \gamma \in G \), and \( x, y \in G^{(0)} \).
This equation should be interpreted in the sense that the set $G^y = r^{-1}(y)$ is mapped by $\gamma$ to the set $G^x = r^{-1}(x)$, the pushforward $\gamma_*$ mapping the measure $\nu^y$ to the measure $\gamma \nu^y$. The measure $\nu^y$ is supported on $G^y$, the measure $\gamma \nu^y$ is supported on $G^x$. We shall denote the set of all proper (or equivalently $\sigma$-finite) transverse functions by $\mathcal{E}^+$. We shall in the sequel assume that all transverse functions are proper unless explicitly stated otherwise.

**Example 5.24.** For the case of $G$ a group, example 5.2, there is only one object $x \in G^{(0)}$, $G = r^{-1}(x)$, and thus we only consider one measure $\nu = \nu^x$. The property $\gamma \nu = \nu$ is simply the (left) invariance under the group action, so a transverse function on a group is precisely a Haar measure.

**Example 5.25.** Consider the groupoid $G = I \times I$ of example 5.6. Consider two elements $x, y \in G^{(0)}$. Recall that $G^x = \{x\} \times I$, and similarly for $G^y$. The element $\gamma = (x, y) \in G$ maps $(y, z)$ to $(x, z)$,

$$\gamma : G^y \to G^x$$

$$(y, z) \mapsto (x, y) \circ (y, z) = (x, z),$$

see figure 1. It follows from the existence of $\gamma^{-1}$ that the map $\gamma : G^y \to G^x$ is a Borel isomorphism. The pushforward $\gamma_* \nu^y$ is a measure whose support is $G^x$. By definition 5.24, $\nu$ is a transverse function if $\nu^x = \gamma_* \nu^y = \gamma \nu^y$. The kernel $\nu$ is of the form

$$\nu : \mathcal{F}^+(G) \to \mathcal{F}^+(G^{(0)})$$

$$f(x, y) \mapsto \frac{1}{y=0} \int f(x, y) d\nu^y,$$

where the measure $\nu = \nu^x$ for any $x \in G^{(0)}$. There is then a natural bijection between the transverse functions on $I \times I$ and the measures on $I$.

Let $\nu$ be a transverse function on $G$, and $B \subset G$ the support of the kernel $\nu$. The *support on $G^{(0)}$* of a transverse function is the set $A \subset G^{(0)}$ of points $x \in G^{(0)}$ such that the intersection $r^{-1}(x) \cap B$ is not empty. A transverse function $\nu$ on $G$ is called *faithful* if the support of $\nu$ on $G^{(0)}$ equals $G^{(0)}$. 

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Recall the principal groupoid $R_\alpha$ of example 5.8, which is a subgroupoid of the groupoid $I \times I$ of example 5.6. A $G$-kernel on $I \times I$ with support on $R_\alpha \subset I \times I$ becomes naturally a $G$-kernel on $R_\alpha$. Moreover, if the $G$-kernel is a transverse function on $I \times I$, it is automatically a transverse function on $R_\alpha$. However, it is easy to verify that there exists no transverse function on $I \times I$ with support on $R_\alpha$. Suppose such a transverse function $\nu$ exists. For $x \in I$, $\nu^x$ should be supported on the set of points $p \in r^{-1}(x)$ which satisfy $s(p) = x + n\alpha \pmod{1}$ for some integer $n$. But now consider $y \in I$ such that there exists no integer $m$ so that $y \neq x + m\alpha \pmod{1}$. Take $\gamma = (x, y)$. The pushforward $\gamma_* \nu^y$ is supported on $G^x$, as is $\nu^x$, but the intersection of the supports is null. Thus there are no transverse functions on $R_\alpha$ which come from transverse function on $I \times I$. We can however construct a transverse function on $R_\alpha$.

**Example 5.26.** Let $R_\alpha$ be as above, the groupoid of example 5.8. For $x \in G^{(0)} = I$, the set $G^x = r^{-1}(x)$ consists of a countable number of points. Any measure $\nu^x$ on $G^x$ thus consists of a sum of Dirac $\delta$ measures, with a real coefficient for each point

$$\nu^x = \sum_{n \in \mathbb{Z}} A_n(x) \delta(y_n), \quad y_n = x + n\alpha \pmod{1}.$$  

The invariance under the action of $R_\alpha$ requires that for $\hat{x} = x + m\alpha \pmod{1}$,

$$A_n(\hat{x}) = A_{n+m}(x).$$  

Clearly if $A_n(x) = A$ is constant for all $x \in I$ and all $n \in \mathbb{Z}$, then the map

$$I \to \bar{\mathcal{M}}^+(I)$$  

$$x \mapsto \nu^x$$

is measurable, and defines a $G$-kernel. The invariance under $R_\alpha$ is obvious, hence $\nu$ is a transverse function. One can in fact show that this is (up to overall normalization) the unique transverse function on $R_\alpha$.  

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The transverse functions are characterized by various interesting properties. For \( f \in \mathcal{F}^+(G) \), define the function
\[
\tilde{f}(\gamma) = f(\gamma^{-1}).
\] (5.12)

Recall that the convolution of functions on a group with respect to a measure is associative and symmetric if and only if the measure is Haar measure. By symmetric, we mean \((f \ast g) \tilde{\gamma} = g \ast f \tilde{\gamma} \). The generalization to groupoids is valid:

**Proposition 5.27.** The following are equivalent:

(i) \( \nu^r(\gamma) = \gamma \nu^s(\gamma) \), i.e. \( \nu \in \mathcal{E}^+ \);

(ii) \((f \ast g) \tilde{\gamma} = g \ast f \tilde{\gamma} \), for \( f, g \in \mathcal{F}^+(G) \);

(iii) \( \nu \ast f = \nu(\tilde{f}) \circ s \), \( f \in \mathcal{F}^+(G) \).

If these equivalent conditions are satisfied, then the convolution \((\ast)\) is also associative.

**Proof** See [28] Proposition 3.

We introduce some preliminaries to the definition of a convolution of two \(G\)-kernels. A kernel \( R_\lambda : G \to G \) is associated canonically to each \(G\)-kernel \( \lambda : G \to G^{(0)} \), defined by
\[
R_\lambda : \mathcal{F}^+(G) \to \mathcal{F}^+(G)
\]
\[
(R_\lambda(f))(\gamma) = \int d\lambda^s(\gamma') f(\gamma \gamma').
\] (5.13)

**Example 5.28.** We use the groupoid of example 5.6, and an arbitrary \(G\)-kernel \( \lambda \). For \( f \in \mathcal{F}^+(G) \),
\[
(R_\lambda(f))(x, y) = \int_{z=0}^1 d\lambda^s(z) f(x, z).
\]

We now define the convolution of two \(G\)-kernels \( \lambda \) and \( \mu \). The \(G\)-kernel \( \lambda \ast \mu \)
is defined as the composition

$$\lambda \ast \mu = \lambda \circ R_\mu. \quad (5.14)$$

The convolution of two proper $G$-kernels is not necessarily proper [27, 28]. The convolution of a transverse function with a $G$-kernel has some useful properties.

**Proposition 5.29.** ([27] p39, prop. 6)

(i) Let $\nu$ be a transverse function and $\lambda$ a $G$-kernel, then $\nu \ast \lambda$ is a transverse function.

(ii) Let $\nu$ and $\nu'$ be proper transverse functions on $G$ with support $\nu' \subset \text{support } \nu$, then there exists a proper $G$-kernel $\lambda$ such that $\nu' = \nu \ast \lambda$.

This proposition is useful in determining when a transverse function is unique (up to normalization). If one can show that for each $G$-kernel $\lambda$, there is a number $c$ such that $\nu \ast \lambda = c\nu$, it follows that $\nu$ is up to normalization the unique transverse function on $G$.

### 5.5 Transverse Measures

In this section, we develop the notion of transverse measures on a measure groupoid $G$. A transverse measure allows transverse functions to be integrated. A result of Connes gives a bijection between transverse measures on $G$ and ordinary measures on $G^{(0)}$ satisfying certain conditions.

We first need to consider homomorphisms of the groupoid $G$ into $\mathbf{R}^*$, the multiplicative group of positive real numbers. A *groupoid homomorphism* is defined in the obvious manner. For two groupoids $G$ and $G'$, a map $h : G \to G'$ is a groupoid homomorphism if for any $\gamma_1, \gamma_2 \in G$ such that $\gamma_1 \circ \gamma_2 \in G$ we have $h(\gamma_1 \circ \gamma_2) = h(\gamma_1) \circ h(\gamma_2) \in G'$. A group is naturally a groupoid. A homomorphism $\delta : G \to \mathbf{R}^*$ thus satisfies the condition $\delta(\gamma_1 \circ \gamma_2) = \delta(\gamma_1)\delta(\gamma_2)$. We shall
define the *trivial homomorphism* to be the map

\[ G \rightarrow 1 \subset \mathbf{R}^*. \]

**Definition 5.30.** Let \( \delta \) be a homomorphism of the groupoid \( G \) into \( \mathbf{R}^* \). A *transverse \( \delta \)-measure* \( \Lambda \) on \( G \) is a linear map

\[ \Lambda : \mathcal{E}^+ \rightarrow \bar{\mathbf{R}}^+ \]

which is

(i) normal, *i.e.*, for a sequence of transverse functions \( \{\nu_n\} \) majorized in \( \mathcal{E}^+ \), we have \( \Lambda(\sup \nu_n) = \sup \Lambda(\nu_n) \);

(ii) a \( \delta \)-module, *i.e.*, for every pair \( \nu, \nu' \in \mathcal{E}^+ \), and each \( G \)-kernel \( \lambda \) satisfying \( \lambda^y(1) = 1 \) for each \( y \in G^{(0)} \), we have

\[ \nu * \delta \lambda = \nu' \Rightarrow \Lambda(\nu) = \Lambda(\nu'). \]

For a sequence \( \nu_n \) of transverse functions, if there is a transverse function \( \nu \) such that for any \( f \in \bar{\mathcal{F}}^+(G) \), we have \( \nu(f) = \sup_n \nu_n(f) \), we say that \( \nu \) majorizes the sequence \( \nu_n \).

When \( \delta \) is the trivial homomorphism, condition (ii) is in effect invariance under the (right) action of \( G \) acting on \( \mathcal{E}^+ \) [27].

For \( f \in \bar{\mathcal{F}}^+(G^{(0)}) \) and \( \nu \) a transverse function, define

\[ \Lambda_\nu(f) = \Lambda((s^* f) \cdot \nu) \in \bar{\mathbf{R}}^+. \]

The following theorem is the central result characterizing transverse measures.
Theorem 5.31. Let $\nu$ be a proper transverse function and $A$ its support. The map $\Lambda \to \Lambda_\nu$ is a bijection between the set of transverse measures with module $\delta$ on $G_A$ and the set of positive measures $\mu$ on $G^{(0)}$ satisfying the following equivalent conditions:

(i) $\delta(\mu \circ \nu)^\sim = \mu \circ \nu$, i.e., for any $g \in \mathcal{F}^+(G)$, 

$$(\mu \circ \nu) \tilde{g} = (\mu \circ \nu) \delta^{-1} g.$$ 

(ii) $\lambda, \lambda'$ proper $G$-kernels, $\nu \ast \lambda = \nu \ast \lambda' \in \mathcal{E}^+ \Rightarrow \mu(\delta^{-1} \lambda(1)) = \mu(\delta^{-1} \lambda'(1)).$

Proof: See Connes [27] Thm. 3. 

5.6 Notes

This chapter presents the groupoid approach to the noncommutative integration theory. It is for the most part based on Connes [27] and Kastler [28], illustrated with some examples. However, proposition 5.13 and corollary 5.14, which are crucial for the applications of the next chapter, are original.