4. An introduction to Connes’ non-commutative integration theory

Logic - the last refuge of a scoundrel.
—New Yorker cartoon, May 25, 1987

We are interested in making sense of an integral over a singular space, for example the quotient of the torus $T^2$ by the lines of irrational slope $\alpha \notin \mathbb{Q}$. More generally, we would like to study the quotient space of the global unstable manifolds for a hyperbolic dynamical system. In this chapter we present an “easy” version of Connes’ noncommutative integration theory. We start by studying integer valued functions on a standard measure space, a case in which Connes’ theory and the ordinary integration theory coincide. We then study a simple example of a “singular” measure space, for which the ordinary integration theory is inapplicable.

We consider a map $f : Y \to \Omega$, where $(Y, \mathcal{A})$ is a standard measure space, and $\Omega$ is a set. We can define a family $\mathcal{B}$ of subsets of $\Omega$ by $B \in \mathcal{B}$ if $f^{-1}(B) \in \mathcal{A}$, and a measure $f_*\mu$ on $(\Omega, \mathcal{B})$, so that $f_*\mu(B) = \mu(f^{-1}B)$. As a measure space, $(\Omega, \mathcal{B}, f_*\mu)$ will not, in general, have good properties, for example, it is not necessarily standard, or even countably separated. We have already seen an example of this for a (nonmeasurable) partition of a Lebesgue space, example 3.14. Let $\xi$ be a partition of $Y$, and $p : Y \to Y/\xi$. We have seen that if $\xi$ is a measurable partition, then $Y/\xi$ is Lebesgue, as in example 3.13. This follows essentially from the definition of a measurable partition and proposition 3.8 But if $\xi$ is a partition which is not measurable, as in example 3.14, then $Y/\xi$ is not countably separated, by definition. We denote by $\Omega_\alpha$ the quotient space of the partition of example 3.14, i.e., a point $x \in \Omega_\alpha$ is a line of slope $\alpha$ on $T^2$. We recall that $\Omega_\alpha$ with the quotient measure structure is not countably separated. In fact, the quotient Borel structure $\mathcal{B}$ consists of only two elements, the empty set, and $\Omega_\alpha$. Since $\Omega_\alpha$ contains many points, this indicates that the quotient Borel structure of $\Omega_\alpha$ is quite pathological. The only measurable functions on $(\Omega_\alpha, \mathcal{B})$ are the constant functions. Thus the usual apparatus of analysis becomes useless, since
all the usual spaces of functions, e.g. \( L^p(\Omega) \) are one-dimensional. We need some new tools if we wish to study such spaces as measure spaces.

The main idea of Connes’ noncommutative integration theory [27,61,28] is to redefine the notion of a function. Let us first consider positive integer valued functions on a standard space \((X, \mathcal{B})\). Here \( \mathbb{Z}^+ \) denotes the nonnegative integers \( \{0,1,2,\ldots\} \), and \( \overline{\mathbb{Z}}^+ \) denotes the extended nonnegative integers, \( \mathbb{Z}^+ \cup \infty = \{0,1,2,\ldots,\infty\} \). We need to consider both \( \mathbb{Z}^+ \) and \( \overline{\mathbb{Z}}^+ \), because the cardinality of the finite set \( \{0,1,\ldots,n-1,n\} \) is equal to \( n+1 \), and the cardinality of \( \mathbb{Z}^+ = \{0,1,2,\ldots\} \) is equal to \( \infty \in \overline{\mathbb{Z}}^+ \), but \( \infty \notin \mathbb{Z}^+ \). A \( \overline{\mathbb{Z}}^+ \) valued function \( f \) on \( X \) is a map \( f : X \to \overline{\mathbb{Z}}^+ \).

**Definition 4.1.** Let \( f \) be as above. The **subgraph** of \( f \) is the measure space

\[
\Gamma_f = \{(x,n) \in X \times \mathbb{Z}^+ | 0 \leq n \leq f(x)\},
\]

with the measure structure induced from the inclusion \( \Gamma_f \subset X \times \mathbb{Z}^+ \), where \( X \times \mathbb{Z}^+ \) has the product measure structure.

This is the only measure structure we will use for \( \Gamma_f \), and it shall be assumed without explicit mention.

We shall often use proposition 3.3, which states that a subset of a standard space is standard if and only if it is measurable, as well as the basic result that the product of two standard spaces is standard (proposition 3.5).

**Proposition 4.2.** Let the space \( X \) with measure structure \( \mathcal{B} \) be a standard measure space, and \( f \) a function \( f : X \to \overline{\mathbb{Z}}^+ \). Consider the subgraph \( \Gamma_f \) with measure structure as above. Then \( \Gamma_f \) is a standard space if and only \( f \) is a measurable function of \( X \).

**Proof** (\( \Leftarrow \)): Suppose \( f \) is a measurable function. Then for \( n \in \overline{\mathbb{Z}}^+ \), the subset of \( X \) given by \( f^{-1}(n) \) is measurable. Trivially, the subset of \( X \times \mathbb{Z}^+ \) given by \( f^{-1}(n) \times \{n\} \) is measurable. The subgraph \( \Gamma_f \) is the countable disjoint union of
measurable subsets

\[ \Gamma_f = \bigcup_{n \in \mathbb{Z}^+} \left( \bigcup_{k \leq n} f^{-1}(k) \right) \times \{n\}, \]

and is thus a measurable subset of \( X \times \mathbb{Z}^+ \). By proposition 3.3, \( \Gamma_f \) is standard.

\((\Rightarrow)\): Suppose that \( \Gamma_f \) is a standard space. By proposition 3.3, \( \Gamma_f \) is a measurable subset of \( X \times \mathbb{Z}^+ \). The set \( B_m \subset X \) is defined by

\[ \Gamma_f \cap (X \times \{m\}) = B_m \times \{m\} \subset X \times \mathbb{Z}^+. \]

Since \( \Gamma_f \cap (X \times \{m\}) \) is the intersection of two measurable sets, it is measurable, from which it follows that \( B_m \) is measurable. Now \( B_m \backslash B_{m-1} \) is a measurable set in \( X \), and moreover, \( B_m \backslash B_{m-1} = f^{-1}(m) \). It then follows that \( f \) is a measurable map.

Proposition 4.3. Suppose \( X \) is standard, and the subgraph \( \Gamma_f \) of the function \( f : X \to \bar{\mathbb{Z}}^+ \) is a standard space (equivalently \( f : X \to \bar{\mathbb{Z}}^+ \) is measurable). Then the map \( p : \Gamma_f \to X \) is measurable.

Proof: Let \( B \) be a measurable subset of \( X \). The inverse image takes the form

\[ p^{-1}(B) = \Gamma_f \cap (B \times \mathbb{Z}^+) \subset X \times \mathbb{Z}^+, \]

which is the intersection of two measurable sets, and thus a measurable subset of \( X \times \mathbb{Z}^+ \). Since for every measurable subset \( B \subset X \), we have \( p^{-1}(B) \) is a measurable subset of \( \Gamma_f \), it follows that \( p \) is a measurable map.
Suppose, more generally, that \( X \) and \( Y \) are standard, and we have a function \( f : X \to \mathbb{Z}^+ \) such that \( \Gamma_f \) is measurably isomorphic to \( Y \). We comment that \( f \) is necessarily a measurable function by proposition 4.2. Let \( \psi : Y \to \Gamma_f \) be such a measurable isomorphism. Then it is trivial that the map \( p \circ \psi = \pi : Y \to X \) is measurable, and for \( x \in X \), the cardinality of the fiber \( \pi^{-1}(x) \) is \( f(x) \). In fact, we have

**Proposition 4.4.** Let \( X \) and \( Y \) be standard measure spaces, and \( \pi \) a measurable map \( \pi : Y \to X \), with \( \pi^{-1}(x) \) countable for all \( x \in X \). Denote by \( f(x) \in \mathbb{Z}^+ \) the cardinality of \( \pi^{-1}(x) \). Then there exists a measurable isomorphism \( \psi : Y \to \Gamma_f \), such that \( p \circ \psi = \pi \), where \( \Gamma_f \) is the subgraph of \( f \).

We omit the proof, see however [64]. Thus we see that measurable \( \mathbb{Z}^+ \) valued functions on a standard space \( X \) can be equivalently described by the pairs \( \mathcal{Y} = (Y, \pi) \), where \( Y \) is a standard space, and \( \pi : Y \to X \) has countable fibers.

**Definition 4.5.** We shall call the pair \( \mathcal{Y} = (Y, \pi) \), \( Y \) a standard space, and \( \pi : Y \to X \) a measurable map with countable fibers, a generalized (\( \mathbb{Z}^+ \) valued) function on the standard space \( X \).

There is an obvious notion of equivalence.

**Definition 4.6.** Two generalized functions \( \mathcal{Y}_1 = (Y_1, \pi_1) \) and \( \mathcal{Y}_2 = (Y_2, \pi_2) \) on the standard space \( X \) are called equivalent if there exists a measurable isomorphism \( \psi : Y_2 \to Y_1 \), satisfying \( \pi_2 = \pi_1 \circ \psi \).

Equivalence ensures that for each \( x \in X \), the fiber \( \pi_2^{-1}(x) \) is mapped by \( \psi \) isomorphically onto the fiber \( \pi_1^{-1}(x) \).

**Proposition 4.7.** Let \( \mathcal{Y}_1 = (Y_1, \pi_1) \) and \( \mathcal{Y}_2 = (Y_2, \pi_2) \) be two generalized functions on the standard space \( X \). Then the set \( R(\mathcal{Y}_1, \mathcal{Y}_2) \subset Y_1 \times Y_2 \) defined by

\[
R(\mathcal{Y}_1, \mathcal{Y}_2) = \{(y_1, y_2) \in Y_1 \times Y_2 | \pi_1(y_1) = \pi_2(y_2)\}
\]

is measurable.
Proof: There exist by proposition 4.4 measurable functions \( f_i : X \to \mathbb{Z}^+ \), \( i = 1, 2 \), such that the generalized function \( \mathcal{G}_i = (\Gamma_{f_i}, p_i) \) is equivalent to \( \mathcal{Y}_i \). It thus suffices to prove the proposition for the special case \( \mathcal{Y}_1 = \mathcal{G}_1 \) and \( \mathcal{Y}_2 = \mathcal{G}_2 \). We need to show that \( R(\mathcal{G}_1, \mathcal{G}_2) \subset \Gamma_1 \times \Gamma_2 \) is measurable. From the proof of proposition 4.2

\[
\Gamma_{f_i} = \bigcup_{n \in \mathbb{Z}^+} B_{i,n} \times \{n\},
\]

where \( B_{i,n} = \bigcup_{k \leq n} f_i^{-1}(k) \). We write \( \Gamma_{f_1} \times \Gamma_{f_2} \) as a countable union of measurable sets,

\[
\Gamma_{f_1} \times \Gamma_{f_2} = \bigcup_{m \in \mathbb{Z}^+} \bigcup_{n \in \mathbb{Z}^+} C_{m,n},
\]

where

\[
C_{m,n} = B_{1,m} \times \{m\} \times B_{2,n} \times \{n\} \subset (X \times \mathbb{Z}^+) \times (X \times \mathbb{Z}^+).
\]

We shall identify \( X \times \{m\} \times X \times \{n\} \subset X \times X \) for notational simplicity. Let \( \Delta \) denote the diagonal in \( X \times X \). Now

\[
R(\mathcal{G}_1, \mathcal{G}_2) \cap C_{m,n} = \Delta \cap (B_{1,m} \times X) \cap (X \times B_{2,n})
\]

is the intersection of three measurable subsets in \( X \times X \), and is thus measurable. \( R(\mathcal{G}_1, \mathcal{G}_2) \) is the countable disjoint union of the measurable sets

\[
\bigcup_{(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+} R(\mathcal{G}_1, \mathcal{G}_2) \cap C_{m,n}
\]

and is thus measurable. □

Corollary 4.8. Let \( \mathcal{Y} = (Y, \pi) \) be a generalized function on a standard space \( X \). Then the set

\[
R_1(\mathcal{Y}) = R(\mathcal{Y}, \mathcal{Y}) \subset Y \times Y
= \{(y_1, y_2) \in Y \times Y | \pi(y_1) = \pi(y_2)\}
\]

is measurable.
Proof: Take $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Y}$ in proposition 4.7.

We now consider integrating $\mathbb{Z}^+$ valued functions on a standard space $X$. We recall from definition 4.5 that generalized function on $X$ is a pair $\mathcal{Y} = (Y, \pi)$, where $Y$ is a standard space, and $\pi : Y \rightarrow X$ is a measurable map with countable fibers.

**Definition 4.9.** We shall say that two generalized function $Y_1$ and $Y_2$ are *compatible* if the set $R(\mathcal{Y}_1, \mathcal{Y}_2)$ is measurable.

By proposition 4.7, if $X$ is standard, all generalized functions are compatible. When $X$ is standard, there is nothing gained by introducing the apparatus of generalized functions. They become useful when $X$ is not standard, where not all generalized functions are compatible.

We now restate the idea of a measure in the setting of generalized $\mathbb{Z}^+$ valued functions.

**Definition 4.10.** Given two generalized functions $\mathcal{Y}_i = (Y_i, \pi_i), i = 1, 2$, on a standard space $X$, their sum

$$\mathcal{Y}' = \mathcal{Y}_1 + \mathcal{Y}_2 = (Y', \pi')$$

is defined as the disjoint union

$$Y' = Y_1 \cup Y_2,$$

and the map $\pi' : Y' \rightarrow X$ is specified by demanding that for any $x \in X$,

$$\pi'^{-1}(x) = \pi_1^{-1}(x) \cup \pi_2^{-1}(x) \subset Y'.$$

Note that $\mathcal{Y}_1 + \mathcal{Y}_2$ is equivalent as a generalized function (definition 4.6) to $\mathcal{Y}_2 + \mathcal{Y}_1$.  

51
**Definition 4.11.** A generalized measure on a standard space $X$ is a map $\Lambda$ from the generalized $\mathbb{Z}^+$ valued functions on $X$ to $\mathbb{R}^+$, which satisfies the properties:

(i) countable additivity: For a countable family of generalized functions $\mathcal{Y}_n$, $n \in \mathbb{Z}^+$,

$$\Lambda\left(\sum_{n \in \mathbb{Z}^+} \mathcal{Y}_n\right) = \sum_{n \in \mathbb{Z}^+} \Lambda(\mathcal{Y}_n)$$

(ii) invariance under equivalence: For $\mathcal{Y}_1$ and $\mathcal{Y}_2$ equivalent generalized functions, $\Lambda(\mathcal{Y}_1) = \Lambda(\mathcal{Y}_2)$.

Since on a standard space $X$, every generalized function is equivalent to the subgraph of some measurable function, the notion of generalized measure coincides with an ordinary measure. We give this definition because it is formally similar to the one which will be required for nonstandard spaces.

The motivation for developing this apparatus is to make sense of measurable functions on a space $\Omega$, with a measure structure $\mathcal{B}'$, which is not standard. We must first consider what properties a generalized function on $\Omega$ should have. There are two obvious candidates for generalized function on $\Omega$, and we shall see that only one of these will deserve to be called a generalized function. First consider an ordinary $\mathbb{Z}^+$ valued function on $\Omega$. We can construct as before the subgraph $\Gamma_f \subset \Omega \times \mathbb{Z}^+$, and the usual projection $p : \Gamma_f \rightarrow \Omega$. Denote this pair by $\mathcal{Y}' = (Y', \pi') = (\Gamma_f, p)$. The space $\Gamma_f$, with the measure structure induced from the product measure structure on $\Omega \times \mathbb{Z}^+$, is not in general standard. A trivial example is the constant function $f_0(\omega) = 0$ for all $\omega \in \Omega$. The subgraph of $f_0$ consists of the space $\Omega$ with the usual measure structure, which is by hypothesis not standard.

Consider now the pair $\mathcal{Y} = (Y, \pi)$, where $Y$ is a standard space, $\pi : Y \rightarrow \Omega$ has countable fibers, and such that the set

$$R_1(\mathcal{Y}) = \{(y, y') \in Y \times Y | \pi(y) = \pi(y')\}$$
is measurable. We can define the function \( f : \Omega \to \mathbb{Z}^+ \) so that for \( \omega \in \Omega \), the cardinality of \( \pi^{-1}(\omega) \) is equal to \( f(\omega) \). Suppose that this \( f \) is the same as the \( f \) used in the construction of \( \mathcal{Y}' \) in the preceding paragraph. We recall that if \( \Omega \) is standard, there exists a measurable isomorphism \( \psi : Y' \to Y \), which preserves fibers, i.e., \( \pi \circ \psi = \pi' \). This is not generally true for nonstandard \( \Omega \). We have already seen that \( Y' \) is not generally standard, while \( Y \) is standard by definition, so there does not generally exist a measurable isomorphism. If we forget the measure structure on \( Y' \) and \( Y \), and consider them merely as sets, we can construct a bijection of sets \( \psi : Y' \to Y \) which preserves fibers, using the fact that the fibers are isomorphic as sets.

The question arises as to how we should define generalized functions. By hypothesis, \( \mathcal{Y}' \) and \( \mathcal{Y} \) have fibers of the same cardinality, so if both are admissible as generalized functions, they should be considered equivalent in some sense. Then if we admit both \( \mathcal{Y}' \) and \( \mathcal{Y} \) as generalized functions, the notion of equivalence would have to be broadened, since there exist no measurable isomorphism \( \psi : Y' \to Y \) preserving fibers. If, on the other hand, we do not wish to broaden the notion of equivalence, then not both \( \mathcal{Y}' \) and \( \mathcal{Y} \) can be admissible generalized functions. We will be interested, for example, in the case where \( \Omega \) is the quotient of \( T^2 \) by the lines of irrational slope. We have seen that the ordinary theory of measurable functions is pathological, and should be replaced with something else (this is true both for \( \mathbb{Z}^+ \) and \( \mathbb{R}^+ \) valued functions). The pairs \((\Gamma_f, p)\), of the type of \( \mathcal{Y}' \), are in exact correspondence to the ordinary measurable functions \( f \) on \( \Omega \), and do not provide a generalization, so we can give them up without grief. This leaves us to consider the generalized functions of the type \( \mathcal{Y} \).

The following definitions are formally similar to those already given for a standard space. We do not, however, now assume any measure structure at all on \( \Omega \).

**Definition 4.12.** A \((\mathbb{Z}^+ \text{ valued})\) generalized function on a space \( \Omega \) is a pair \( \mathcal{Y} = (Y, \pi) \), where \( Y \) is a standard space, and \( \pi : Y \to \Omega \) has countable fibers,
such that \( R_1(Y) \subset Y \times Y \) (defined as before) is measurable.

If \( \Omega \) is standard, then \( G = (\Gamma_f, p) \), where \( \Gamma_f \) is the subgraph of any ordinary measurable function \( f : \Omega \to \mathbb{Z}^+ \), and \( p \) is the natural projection, is a generalized function in the sense of definition 4.12. Thus definition 4.12 generalizes definition 4.5 from standard measure spaces to arbitrary sets with no measure structure assumed.

Equivalence and compatibility are defined analogously. We write this out for completeness.

**Definition 4.13.** Two generalized functions \( Y_1 = (Y_1, \pi_1) \) and \( Y_2 = (Y_2, \pi_2) \) on \( \Omega \) are called *equivalent* if there exists a measurable isomorphism \( \psi : Y_2 \to Y_1 \), satisfying \( \pi_2 = \pi_1 \circ \psi \).

**Definition 4.14.** We shall say that two generalized functions \( Y_1 \) and \( Y_2 \) are *compatible* if the set \( R(Y_1, Y_2) \) is measurable.

One difference that arises when \( \Omega \) is not standard, is that not all generalized functions are necessarily compatible.

**Example 4.15.** Let \( \Omega_\alpha \) be the quotient of the torus \( T^2 \) by the lines of irrational slope \( \alpha \). Let \( M \) be the submanifold of \( T^2 \) defined by \( M = \{(x, y) \in T^2 | x = 0 \} \). This is a closed transversal to the lines of slope \( \alpha \). We define two generalized functions \( Y_i = (Y_i, \pi_i), i = 1, 2 \), where \( Y_i \) is a measurable subset of \( M \), and \( \pi_i : Y_i \to \Omega_\alpha \) maps the point \( y \in Y_i \) to the line with slope \( \alpha \) passing through \( y \).

**Proposition 4.16.** \( Y_1 \) and \( Y_2 \) are generalized functions on \( \Omega_\alpha \), and are, moreover, compatible.

**Proof:** For any integer \( n \in \mathbb{Z} \), the set

\[
K_n = \{(x, y) \in T \times T | y = x + n\alpha \pmod{1}\}
\]
is clearly measurable. Thus the countable union
\[
K(T) = \{(x, y) \in (T)^2 | y = x + n\alpha \pmod{1}, n \in \mathbb{Z}\} \\
= \bigcup_{n \in \mathbb{Z}} K_n
\]
is measurable. Now
\[
R(\mathcal{Y}_i, \mathcal{Y}_j) = K(T) \cap (Y_i \times T) \cap (T \times Y_j) \subset Y_i \times Y_j,
\]
i and j taking the values 1 or 2, is the intersection of measurable sets, and is thus measurable. If i = j, then the measurability follows, if i ≠ j, the compatibility follows.

We return to example 4.15. Denote by μ the usual measure on T, which is invariant under \( x \to x + c \pmod{1} \), this is the Haar measure of the group T. Suppose that \( \mu(Y_i) > 0, \ i = 1, 2 \). Then for any \( \omega \in \Omega_\alpha \), the cardinality of the fiber \( \pi_i^{-1}(\omega) \) is countably infinite. Then if we consider the ordinary \( \mathbb{Z}^+ \) valued function \( f_i \) on \( \Omega_\alpha \) defined by the cardinality of \( \pi_i^{-1}(\omega) \), we find that \( f_1 \) is equal to \( f_2 \). But we shall now see that as generalized functions, \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) are not equivalent if \( \mu(Y_1) \neq \mu(Y_2) \). The notion of generalized function on a nonstandard space such as \( \Omega_\alpha \) contains more information than an ordinary function.

**Proposition 4.17.** Let \( \mathcal{Y}_i \) be as above, and \( \psi : Y_1 \to Y_2 \) a bijection preserving fibers, i.e., \( \pi_1 = \psi \circ \pi_2 \). If \( \psi \) is a measurable isomorphism, then \( \mu(Y_1) = \mu(Y_2) \).

**Proof:** For each \( n \in \mathbb{Z} \), define the set
\[
P_n = \{y \in Y_1 | \psi(y) = y + n\alpha \pmod{1}\}.
\]
Since \( \psi \) preserves fibers, every \( y \in Y_1 \) belongs to some \( P_n \), and the \( P_n \) are disjoint, thus
\[
Y_1 = \bigcup_{n \in \mathbb{Z}} P_n.
\]
Define the function \( g : Y_1 \to \mathbb{Z} \) such that \( g(y) = n \) for \( y \in P_n \). The function
$g$ is measurable, since $\psi(y) = y + g(y)\alpha \pmod{1}$, and $\psi$ is measurable by hypothesis. It follows that the sets $P_n$ are measurable. The action of $\psi$ on $P_n$ is a translation, and $\mu$ is translation invariant, so $\mu(P_n) = \mu(\psi(P_n))$. Since $Y_1$ and $Y_2$ are the disjoint countable unions of $P_n$ and $\psi(P_n)$ respectively, we see that $\mu(Y_1) = \mu(Y_2)$.

**Corollary 4.18.** If $\mu(Y_1) \neq \mu(Y_2)$, then $Y_1$ and $Y_2$ are not equivalent as generalized functions.

The preceding discussion gave a prototype of a measure on generalized functions for a specific example. We now formalize this idea. We wish to define a measure on the generalized function of a (not necessarily standard) space $\Omega$. We first need to define the class of functions which will be integrable. Recall that for a standard space, the class of all generalized functions, modulo equivalence, coincides with the ordinary measurable functions. For generalized functions on a standard space, it is easy to see that compatibility is an equivalence relation. Thus given one measurable generalized function, we can define the class of integrable functions as the saturation of the equivalence relation defined by compatibility. For generalized functions on a nonstandard space, the relation defined by compatibility is not in general an equivalence relation, as it not need be transitive. (To see this, we notice that for standard spaces $X$, $Y$ and $Z$, one can construct a subset of $X \times Y \times Z$ such that the images of the projections onto $X \times Y$ and onto $Y \times Z$ are measurable subsets, but the image of the projection onto $X \times Z$ is not.) We thus consider the class of generalized functions which are compatible with a given generalized function $Y$. Not all the functions in this class are necessarily compatible with each other. The definition of a generalized measure is then formally identical to the standard case. The sum of generalized functions on an arbitrary set is defined in the same way as for generalized functions on a standard measure space, definition 4.10.

**Definition 4.19.** A *generalized measure* on a space $\Omega$ is a map $\Lambda$ from a class of integrable generalized $\mathbb{Z}^+$ valued functions on $X$ to $\mathbb{R}^+$, which satisfies
the properties:

(i) countable additivity: For a countable family of generalized functions $\mathcal{Y}_n$, $n \in \mathbb{Z}^+$,

\[ \Lambda\left( \sum_{n \in \mathbb{Z}^+} \mathcal{Y}_n \right) = \sum_{n \in \mathbb{Z}^+} \Lambda(\mathcal{Y}_n) \]

(ii) invariance under equivalence: For $\mathcal{Y}_1$ and $\mathcal{Y}_2$ equivalent generalized functions, $\Lambda(\mathcal{Y}_1) = \Lambda(\mathcal{Y}_2)$.

We return to example 4.15, the space $\Omega_\alpha$ of lines of irrational slope $\alpha$ on $T^2$. We wish to have the generalized function $(M, \pi)$, where $M$ is, as above, the full transverse circle, in the domain of the integral. We thus consider the class of generalized functions which are compatible with $(M, \pi)$. The generalized functions $\mathcal{Y}_i$, as defined as above in example 4.15 and proposition 4.16 are then in the class. We check easily that the standard measure $\mu$ on $M$, as in proposition 4.17 defines a generalized measure on this class by

\[ \Lambda(\mathcal{Y}_i) = \mu(\omega_i). \]

We have so far only considered integer valued generalized functions. Real ($\mathbb{R}^+$) valued generalized functions are constructed analogously. Instead of assigning to each point $\omega \in \Omega$ a countable set, we assign a standard measure space with smooth measure, i.e., with no points of positive measure. The value of the generalized function at a point is taken to be the total measure of the fiber over that point, which can be any number in $\mathbb{R}^+$. The construction is quite analogous to the $\mathbb{Z}^+$ valued case. In the next section, we develop the theory of measure groupoids and transverse measures, which provides a suitable formalism for dealing with these objects. Consequently, we shall not develop at this point the analogue of the $\mathbb{Z}^+$ generalized functions presented above.
4.1 Notes

This chapter is based on ideas outlined by A. Connes [27,61]. We have endeavored to provide a much more detailed discussion than appears in these references, where many of these results are implicit, but not explicit. All the proofs presented in this chapter are original.