2. Hyperbolic dynamical systems

The next great era of awakening of human intellect may well produce a method of understanding the qualitative content of equations. Today we cannot. Today we cannot see that the water flow equations contain such things as the barber pole structure of turbulence that one sees between rotating cylinders. Today we cannot see whether Schrödinger’s equation contains frogs, musical composers, or morality - or whether it does not. We cannot say whether something beyond it like God is needed, or not. And so we can all hold strong opinions either way.

—R. P. Feynman, *Feynman Lectures in Physics*

2.1 Generalities on dynamical systems

In this section we define some of the basic notions of dynamical systems. By a *dynamical system*, we shall mean a space $X$, generally with some additional structure, and a family of maps $f_t : X \to X$, $t \in A$, which preserve the additional structure on $X$. The index set $A$ is usually a group or a semigroup. An example of additional structure on $X$ is topology, in which case the map $f_t$ which preserves the topology is continuous. We shall mainly be interested in the case where the inverse $f_t^{-1} : X \to X$ exists, such that $f_t^{-1} \circ f_t = f_t \circ f_t^{-1} = 1_X$, where $1_X$ is the identity map on $X$, and $f_t^{-1}$ also preserves the additional structure. Such a map $f_t$ is an an automorphism of $X$, the notion of course depends on what additional structure we assume on $X$. For example, if the additional structure is a topology on $X$, then an automorphism is simply a homeomorphism, a continuous map with a continuous inverse. The set of automorphisms preserving some structure on $X$ clearly forms a group. We shall generally be interested in the case where the index set $A$ is a group, and the map $t \to f_t$ is a group homomorphism. The primary examples are:

(i) $A = \mathbb{Z}$, the integers, this is called a discrete time dynamical system, or for short a discrete dynamical system, Since $\mathbb{Z}$ is the free group on one generator, it is clear that $f_t = (f_1)^t$. In this thesis, we are primarily interested in discrete dynamical systems.

(ii) $A = \mathbb{R}$, the reals, this is called a continuous time dynamical system, or a flow. The interest in flows is obvious. In the real world, time can be
described by one real parameter. In this thesis, we shall make some mention of flows, although they are not essential to our main developments.

We shall not be interested in more general situations which are often studied, e.g., $A$ is a more general group, or $A$ is a semigroup. The study of dynamical systems involves question about asymptotic behavior of the family of maps $f_t$. In our case, this corresponds to $t \to \pm \infty$, where $t$ is either in $\mathbb{Z}$ or $\mathbb{R}$.

Our principal object of study will be differentiable dynamical systems. A differentiable dynamical system is a dynamical system $f_t : X \to X$, where $X$ is a differentiable manifold, and $f_t$, for each $t$ in $\mathbb{Z}$ or $\mathbb{R}$, is a diffeomorphism, a differentiable map with differentiable inverse. We shall also be interested in Borel measures on $X$ invariant under $f_t$. References for the basic theory of differentiable manifolds are [34], Kobayashi and Nomizu [35]. Good introductory references on dynamical systems are Eckmann and Ruelle [2], Arnold and Avez [1], and Arnold [11].

2.2 Uniformly hyperbolic systems

In this section, we introduce the notion of a hyperbolic dynamical system, and examine some simple examples. We shall in this section restrict to the simpler case of uniformly hyperbolic systems, where the results are simpler and easier to state. This case was historically analyzed first, and contains many important examples. These types of systems have strong structural stability properties. We shall mention this for completeness, although it is not central for our purposes. One of the main technical tools are stable and unstable manifolds, which we shall describe and illustrate in this somewhat simpler setting. The theory of stable manifolds will be essential for our later developments. We shall initially consider discrete time systems, and at the end of the section comment on the necessary modifications for flows. Some general references for this section are Smale [13], Anosov [12], Shub [16], Lanford [6], and Newhouse [36].
We introduce the notion of a hyperbolic set. To be more precise, the hyperbolic sets discussed in this section should be called a *uniform completely* hyperbolic set, or uniform hyperbolic set for short. Let $X$ be a (compact) differentiable manifold, and $f$ a diffeomorphism. Let $\Lambda$ be an invariant set in $X$, i.e., if $x \in \Lambda$, then $f(x) \in \Lambda$. We choose some Riemann metric on $M$, and denote by $\| \cdot \|$ the corresponding norm in the tangent space.

**Definition 2.1.** The set $\Lambda$ is called a *(uniform completely) hyperbolic set* for $f$ if there exists a continuous splitting of $TM|_{\Lambda}$, the restriction of the tangent bundle $TM$ to $\Lambda$, which is invariant under the action of the derivative map $Df$;

$$TM|_{\Lambda} = E^s \oplus E^u, \quad Df(E^s) = E^s, \quad Df(E^u) = E^u, \quad (2.1)$$

and for which there are constants $c$ and $\lambda$, $c > 0$ and $0 < \lambda < 1$, such that for $t \in \mathbb{Z}$,

$$\|Df^t|_{E_s}\| < c\lambda^t, \quad t \geq 0$$
$$\|Df^{-t}|_{E_u}\| < c\lambda^t, \quad t \geq 0. \quad (2.2)$$

The uniformity refers to the fact that the constants $c$ and $\lambda$ do not depend on $x \in \Lambda$. It is always possible to find a so-called *adapted* (or Lyapunov) Riemann metric on $X$, for which $c = 1$, see [16], p 21.

Anosov was the first to explicitly study systems of this type, see Anosov’s monograph [12]. An *Anosov diffeomorphism* is a differentiable dynamical system $(M, f)$ such that all of $M$ is a hyperbolic set.

**Example 2.2.** A simple example is the *cat map*, which was studied by Thom, and by Arnold. Let $M$ be the two-dimensional torus $T^2$. This can be realized as the quotient of two-dimensional Euclidean space $\mathbb{R}^2$ by the lattice of integers $\mathbb{Z}^2 \subset \mathbb{R}^2$. In other words, a point in $T^2$ is an equivalence class of points $(x, y) \in \mathbb{R}^2$, under the equivalence relation $(x, y) \sim (x + n, y + m)$, for $n, m \in \mathbb{Z}$. The general linear group $GL(2, \mathbb{R})$ acts on $\mathbb{R}^2$, it is simply the group of all invertible linear maps on $\mathbb{R}^2$. We recall that $SL(2, \mathbb{Z})$ denotes the subgroup of
$GL(2, \mathbf{R})$ which preserves the integer lattice $\mathbb{Z}^2$. $SL(2, \mathbf{Z})$ consists of the matrices with integer entries, and unit determinant. An element $A \in SL(2, \mathbf{Z})$ preserves equivalence classes in $\mathbf{R}^2$, so it induces a map $A : T^2 \to T^2$. The prototypical “cat map” is the induced map $A : T^2 \to T^2$, where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbf{Z}).$$

The derivative map, in the natural basis of the tangent space $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is equal to $A$ at all points of $T^2$. The eigenvectors of $A$ are orthogonal, since $A$ is self-adjoint, and the eigenvalues are $\lambda_\pm = \frac{1}{2}(3 \pm \sqrt{5})$, which satisfy $\lambda_+ > 1$, and $\lambda_- < 1$. The tangent space can be split at each point in $T^2$ as $E_s \oplus E_u$, where $E_s$ is the eigenspace of $\lambda_-$, and $E_u$ is the eigenspace of $\lambda_+$. This gives a splitting of the tangent bundle of $T^2$, which satisfies the conditions above for a hyperbolic set. Thus all of $T^2$ is a hyperbolic set.

We now introduce stable manifolds. The main idea is that given a point $x \in M$, the set of points $y$ such that as $n$ tends to $\infty$, $f^n y$ approaches $x$ pursuant to certain conditions forms a submanifold of $M$. A class of results known as stable manifold theorems give sufficient conditions for this to happen. For $x$ in a uniform hyperbolic set, the corresponding stable manifold theorem provides much information. We first present some basic definitions. Let $f$ be a diffeomorphism of the manifold $M$, $x \in M$, and $\epsilon > 0$. Define the local stable and unstable sets of $x$

$$W^s_\epsilon(x) = \{ y \in M | \lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0, \quad d(f^n(x), f^n(y)) \leq \epsilon, \forall n \geq 0 \},$$

$$W^u_\epsilon(x) = \{ y \in M | \lim_{n \to -\infty} d(f^n(x), f^n(y)) = 0, \quad d(f^n(x), f^n(y)) \leq \epsilon, \forall n \leq 0 \}$$

(2.3)

The (global) stable and unstable sets of $x \in M$. are defined as

$$W^s(x) = \{ y \in M | \lim_{n \to +\infty} d(f^n(y), f^n(x)) = 0 \}$$

$$W^u(x) = \{ y \in M | \lim_{n \to -\infty} d(f^n(y), f^n(x)) = 0 \}$$

(2.4)

It is easy to verify that the stable sets $W^s$ form equivalence classes.
Proposition 2.3. The relation defined on $M$ by the global stable sets is an equivalence relation.

Proof:

(i) Reflexive: $x \in W^s(x)$ is obvious.

(ii) Symmetric: $x \in W^s(y) \Rightarrow y \in W^s(x)$ is also obvious.

(iii) Transitive: $x \in W^s(y)$, and $y \in W^s(z) \Rightarrow x \in W^s(z)$. To show this, use the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$.

For a subset $B \subset M$, define analogously to equation (2.4)

$$
W^s(B) = \{ y \in M | \lim_{n \to +\infty} d(f^n(y), f^n(B)) = 0 \}
$$

$$
W^u(B) = \{ y \in M | \lim_{n \to -\infty} d(f^n(y), f^n(B)) = 0 \}.
$$

(2.5)

The next theorem is a central result in the study of uniform hyperbolic systems.

Theorem 2.4. Stable manifold theorem for hyperbolic sets, see e.g., [16], theorem 6.2. Let $\Lambda$ be a closed (uniform) hyperbolic set for $f$, and assume $\Lambda$ is furnished with an adapted metric, as defined immediately after definition 2.1 . Then there is a positive $\epsilon$ such that for every point $x$ in $\Lambda$, $W^s_\epsilon(x)$ is an embedded disk of dimension equal to that of $E^s_x$: moreover, $T_x W^s_\epsilon(x) = E^s_x$; and similarly for the unstable case. The stable and unstable discs also satisfy the following:

(i) $d(f^n(x), f^n(y)) \leq \lambda^n d(x, y), \forall y \in W^s_{\epsilon}(x), \forall n \geq 0,$

$$
d(f^{-n}(x), f^{-n}(y)) \leq \lambda^n d(x, y), \forall y \in W^u_{\epsilon}(x), \forall n \leq 0,
$$

where $\lambda < 1$ is such that $\|Df|_{E^s}\| < \lambda$ and $\|(Df|_{E^u})^{-1}\| < \lambda$.

(ii) The embedding of $W^u_{\epsilon}(x)$ (respectively $W^s_{\epsilon}(x)$) varies continuously with $x.$

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(iii) \[ W^s_\epsilon(x) = \{ y | d(f^n(x), f^n(y)) \leq \epsilon, \quad \forall n \geq 0 \} \]
\[ W^u_\epsilon(x) = \{ y | d(f^n(x), f^n(y)) \leq \epsilon, \quad \forall n \leq 0 \} \]

(iv) The manifold \( W^u_\epsilon(x) \) (respectively \( W^s_\epsilon(x) \)) is as smooth as \( f \).

Proof The proof proceeds by reduction to the stable manifold theorem for hyperbolic maps on a Banach space, see [16] or [15].

The \( W^s_\epsilon(x) \) are known as local stable manifolds. In addition, one can show [16] that under the hypotheses of theorem 2.4, if \( x \in \Lambda \), then

\[
W^s(x) = \bigcup_{n \geq 0} f^{-n}(W^s_\epsilon(f^n(x))), \\
W^u(x) = \bigcup_{n \geq 0} f^n(W^u_\epsilon(f^{-n}(x))),
\]

and \( W^s(x) \) and \( W^u(x) \) are immersed submanifolds of \( M \). They are called respectively the global stable manifold and global unstable manifold of \( x \in M \).

Example 2.5. It is easy to construct the stable and unstable manifolds for the cat map of example 2.2. The stable manifolds consist of straight lines (geodesics in the flat metric) on \( T^2 \) whose tangent vectors are in the contracting eigenspace \( E_s \). Each stable manifold is diffeomorphic to \( \mathbb{R} \), as the slope of the lines are irrational, so they never close to be circles. Similarly, the unstable manifolds are the straight lines whose tangent vectors are in the expanding eigenspace \( E_u \). The global stable (unstable) manifolds in fact give a foliation of \( T^2 \).

It is a general result for Anosov systems \((M,f)\), that both the global stable and unstable manifolds form a foliation of \( M \). This foliation is not necessarily smooth, but in many instances it has the important property of absolute continuity, see [12]. We now discuss absolute continuity in the context of uniformly hyperbolic systems. Absolute continuity is also important in the nonuniformly hyperbolic theory (following section), which requires more sophisticated considerations. Consider smoothly embedded discs transverse to the foliation. If the
disks are close enough, one can define a homeomorphism from a subset of one disk to a subset of the other, by identifying points which lie on the same local piece of the foliation. This is called the successor mapping. Consider the Lebesgue measure on the disks. The foliation is called \textit{absolutely continuous} if the successor mapping is absolutely continuous, \textit{i.e.}, if it maps sets of Lebesgue measure zero to sets of Lebesgue measure zero.

**Theorem 2.6.** (Anosov, [12] Thm. 10) If an Anosov system of class $C^2$, then the stable and unstable foliations are absolutely continuous.

This theorem is valid for discrete time systems as well as flows, the definition of an Anosov flow and the necessary modifications are given below. Consider now a discrete time Anosov system which preserves a measure $\mu$ absolutely continuous to Lebesgue measure on $M$, \textit{i.e.}, with the same sets of zero measure. Anosov calls such an invariant measure an integral invariant, we shall call it a smooth invariant measure.

**Theorem 2.7.** (Anosov, [12], Thm. 9) For a discrete time Anosov system with a smooth invariant measure, each leaf of the stable or unstable foliation (\textit{i.e.}, each global stable or unstable manifold) is everywhere dense. If we do not assume the existence a smooth invariant measure, one can prove that there exists a finite number of leaves of the expanding or contracting foliation, whose union is everywhere dense.

Analogous results hold for Anosov flows. An Anosov diffeomorphism is called \textit{transitive} if every leaf of the stable foliation, and every leaf of the unstable foliation, is everywhere dense. A point $x \in X$ is a \textit{periodic point} if there is an $n > 0$ such that $f^n(x) = x$.

**Theorem 2.8.** A transitive Anosov diffeomorphism has a finite invariant measure which is absolutely continuous with respect to the Riemann volume if and only if for an arbitrary periodic point $x$, $f^n(x) = x$, the determinant of the tangent map $\text{Det} (Df^n)_x = 1$. 

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Proof: See Livshits and Sinai,[37,38], and also de la Llave, Marco, and Moriyon [39] for questions of smoothness.

A foliation is called \textit{metrically transitive} if for any measurable set \( A \subset M \) which is a union of leaves, either \( \mu(A) = 0 \) or \( \mu(M \setminus A) = 0 \).

**Theorem 2.9.** For a discrete time transitive Anosov system, the stable and unstable foliations are metrically transitive.

\textit{Proof}: This is Thm. 11 in Anosov [12], which requires in addition the transitivity assumption to hold. [40].

The structural stability of Anosov systems is a very important result. Since it is not central for our purposes, we mention it only briefly. For simplicity, we define this notion only for discrete time systems. We first define topological conjugacy. Let \( f \) be a diffeomorphism of \( M \), and \( g \) a diffeomorphism of \( N \), assume both \( f \) and \( g \) are \( C^k \), \( k \geq 1 \). As discrete time dynamical systems, \((M,f)\) and \((N,g)\) are \textit{topologically conjugate} if there is a homeomorphism \( h : M \to N \) which satisfies \( hf = gh \). Denote by \( \text{Diff}^k(M) \) the group of \( C^k \) diffeomorphisms of \( M \), given the \( C^k \) topology ([16], p4). A \( C^k \) dynamical system \((M,f)\) is \textit{structurally stable} if there exists a neighborhood \( U \) of \( g \) in \( \text{Diff}^k(M) \) such that all \( g' \in U \) are topologically conjugate to \( g \). The following theorem holds for flows as well as discrete time systems, in either case for any \( C^k \), \( k \geq 1 \). See Anosov [12], p 11 for the definition of structural stability for flows.

**Theorem 2.10.** (Anosov, [12], Thm. 1) Anosov systems are structurally stable.

A more general type of hyperbolic system are the \textit{Axiom A} systems introduced by Smale, see \textit{e.g.} [13]. To define Axiom A, we must first introduce some auxiliary concepts. Let \( f : X \to X \) be a continuous map on a topological space. The set of periodic points is denoted by \( \text{Per}(f) \), and its closure is \( \overline{\text{Per}}(f) \). If a point \( x \in X \) has a neighborhood \( U \) such that \( f^n(U) \cap U \) is empty for all \( n > 0 \), then \( x \) is called a \textit{wandering} point. A point is \textit{nonwandering} if it is not wandering. The \textit{nonwandering set} is the set of all nonwandering points, and is denoted
by $\Omega(f)$ or $\Omega$ for short. One easily checks that the set of wandering points is open, so $\Omega$ is closed. It is straightforward to verify that $\Omega(f)$ is invariant under $f$. A periodic point is clearly nonwandering, so $\overline{\text{Per}(f)} \subset \Omega(f)$.

Let $f : X \to X$ be a diffeomorphism of class $C^r$, with $r > 1$. $f$ is said to satisfy axiom A if the following two conditions are satisfied

(i) $\Omega(f)$ is hyperbolic,

(ii) $\Omega(f) = \overline{\text{Per}(f)}$.

One can show that an Anosov diffeomorphism satisfies Axiom A, see e.g., [16]. A map $f$ on a topological space $X$ is topologically transitive if there is a point $x \in X$ such that $\bigcup_n f^n(x)$ is dense in $X$. Smale proved a spectral decomposition theorem for Axiom A systems, so called because “the decomposition of the manifold into invariant sets of the diffeomorphism is quite analogous to the decomposition of a finite dimensional vector space into eigenspaces of a linear map” [13].

**Theorem 2.11.** (Spectral decomposition theorem, Smale, [13]) Suppose $f : M \to M$ satisfies Axiom A. Then there is a unique way of writing $\Omega$ as the finite union of disjoint, closed, invariant subsets (or “pieces”) on each of which $f$ is topologically transitive:

$$\Omega = \Omega_1 \cup \ldots \cup \Omega_k.$$ 

The sets $\Omega_i$ in the above theorem are called basic sets.

**Corollary 2.12.** (Smale, [13]) If $f : M \to M$ is as above one can write $M$ canonically as a finite disjoint union of invariant subsets $M = \bigcup_{i=1}^k W^s(\Omega_i)$ (recall eqn. (2.5)).

There exist results on structural stability of Axiom A systems. The fundamental result is that Axiom A and no cycles (see [16]) is equivalent to structural stability. We refer to the literature, e.g., Smale [13], and Shub [16]. We shall
be interested in some results on Axiom A attractors, which we discuss in a later section.

2.3 Nonuniformly Hyperbolic Systems

In this section, we study stable manifolds for weaker forms of hyperbolicity. Specifically, we consider nonuniform partial hyperbolicity. Nonuniform refers to the fact that the bounds on the exponential growth cannot be specified uniformly on \( M \), and partial refers to the fact that there may be subspaces of the tangent space which neither contract nor expand exponentially. The Anosov and Axiom A systems are obviously a somewhat special class. Various ideas can be carried over to arbitrary differentiable dynamical systems with an invariant measure, however the results are weaker. There exist “partial nonuniform hyperbolic sets”, defined in terms of Lyapunov exponents. One can prove a stable manifold theorem not everywhere on such a “hyperbolic set”, but only almost everywhere with respect to an invariant measure. Much of this work is due to Pesin [14,18,20]. Other references are Ruelle [19], Ledrappier and Young [25,26], and the review article by Pesin and Sinai [41]. A brief elementary introduction to some of the basic ideas is given in [42].

A central result is the multiplicative ergodic theorem of Oseledec. We present the version most convenient for our application, if not necessarily the most general. Let \( M \) be a compact smooth \( m \)-dimensional manifold, and \( f \) a diffeomorphism of \( M \).

**Theorem 2.13.** There exists a measurable set \( \Gamma \subset M \) which satisfies:

(i) \( \Gamma \) is an invariant set, and for every invariant Borel measure \( \mu \) satisfying \( \mu(M) = 1, \mu(\Gamma) = 1. \)

(ii) For every \( x \in \Gamma \), there is a splitting of the tangent space

\[
T_x M = E^1(x) \oplus E^2(x) \oplus ... \oplus E^{s(x)}(x),
\]

and real numbers \( \lambda_1(x) < \lambda_2(x) < ... \lambda_{s(x)}(x) \) such that for any Riemann
norm $\| \cdot \|$ on $M$

(a) $\lambda_i(x)$, and $s(x)$ are invariant measurable functions, and $E^i(x)$ is a measurable field of vector spaces (with the measure structure induced from the tangent bundle);

(b) For $v \in E^i(x)$,

$$\lim_{n \to \pm \infty} \frac{1}{n} \ln \| (Df^n)_x(v) \| = \lambda_i(x); \quad (2.7)$$

(c)

$$\lim_{n \to \pm \infty} \frac{1}{n} \ln |\text{Det} (Df^n)_x| = \sum_{i=1}^{s(x)} \lambda_i(x) \dim E^i(x) \quad (2.8).$$

Proof: See Oseledec [43], or Ruelle [19].

There is also a covariance property of the subspaces. Let $x \in M$, and $k$ and integer satisfying $1 \leq k \leq s(x) = s(f x)$. Define

$$V^k(x) = E^1(x) \oplus \ldots \oplus E^k(x) \subset T_x M.$$ Then we have ([19])

$$(Df^n)_x V^k(x) \subset V^k(f x).$$

Definition 2.14. With the notation of theorem 2.13, a point $x$ is called regular if it is in $\Gamma$, and the real numbers $\lambda_i(x)$ are called Lyapunov exponents.

It is easy to see that if there exist any tangent vectors with nonzero Lyapunov exponents, then there cannot exist an invariant Riemann metric. Negative Lyapunov exponents correspond to exponentially “shrinking” directions in the tangent space. One might expect this to yield a stable manifold theorem. Such a theorem comes out of the work of Pesin [14,18]. Before stating the theorems, we present a less technical discussion.
We are dealing with a weaker form of hyperbolicity than in the uniformly hyperbolic case, and the stable manifold theorem is correspondingly weaker, but still contains much information. Let \((M, f, \mu)\) be a differentiable dynamical system on a compact manifold, with invariant measure \(\mu\) (not necessarily Lebesgue). Recall the definitions (2.3) and (2.4) of the local and global stable sets for uniformly hyperbolic systems. It turns out, that in the uniformly hyperbolic case, all points in the stable manifold of \(x\) approach \(x\) exponentially. In the general case, we have to stipulate this explicitly. For \(x \in M\), \(\epsilon > 0\) and \(\lambda < 0\), we define the local stable set

\[
W^s(x, \epsilon, \lambda) = \{y \in M | d(f^n x, f^n y) \leq \epsilon \exp(\lambda n), \forall n \geq 0\}
\]

\[
= \{y \in M | \frac{1}{n} \ln d(f^n x, f^n y) \leq \ln \lambda + \frac{1}{n} \ln \epsilon, \forall n \geq 0\}. \tag{2.9}
\]

The stable manifold theorems state that if \(\lambda_{i-1}(x) < \lambda < \lambda_i(x)\), then for \(\mu\)-almost all \(x \in M\), the local stable set \(W^s(x, \epsilon, \lambda)\) is a piece of a submanifold of \(M\) for small enough \(\epsilon\). The result is weaker than theorem 2.4 for uniformly hyperbolic systems. The local stable manifolds do not exist at every point \(x\), only on a set of full measure. Also, the \(\epsilon\) depends on \(x\), and in particular, cannot be uniformly bounded below for all \(x\). In particular, the size of a local stable manifold may shrink along the orbit of a point. It turns out that one can show that such shrinking is subexponential, i.e., slower than \(\exp(\epsilon n)\) for any \(\epsilon > 0\). See [42] for an elementary discussion. The technical part of Pesin’s work consists to a large degree of showing that certain quantities, though not bounded, shrink or grow subexponentially. Also, the local stable manifolds do not necessarily depend continuously on \(x\), compare with part \((iii)\) of theorem 2.4, but only measurably. See Pesin and Sinai [41] or Pugh and Shub [42] for further introductory discussion.

We now present Pesin’s local stable manifold theorem [14,18], with the unnecessary assumption of invariant Lebesgue measure removed, see [19,44]. We shall assume all diffeomorphisms are \(C^2\), although the stable manifold theorems below are valid more generally for \(C^{1,\alpha}\) diffeomorphisms, i.e. diffeomorphisms satisfying a Hölder condition on the derivative map [44,18]. We forego the added generality.
because some of the results presented in the following section on invariant measures are valid only for $C^2$, not $C^{1,\alpha}$. We use the notation of theorem 2.13, and definition 2.14. Let $[a, b] \subset \mathbb{R}$ be a compact interval, and $\Gamma_{a,b} \subset \Gamma$ the invariant set consisting of points $x$ such that none of the Lyapunov exponents $\lambda_i(x)$ at $x$ are in the interval $[a, b]$. Define for $x \in \Gamma_{a,b}$

$$E^s_x = \bigoplus_{\lambda_i(x)<a} E^i(x), \quad E^u_x = \bigoplus_{\lambda_i(x)>b} E^i(x) \quad (2.10).$$

We use the notation $B^s(x, \delta)$ for the ball of radius $\delta$ in $E^s$, similarly for $B^u(x, \delta)$, and $\exp_x : T_x M \to M$ for the Riemannian exponential map ([35]).

**Theorem 2.15.** Given real constants $\epsilon > 0$ and $\rho \in (\exp a, \exp b)$, there exist:

(i) Borel functions $\delta_{\epsilon, \rho} : \Gamma_{a,b} \to (0, \infty)$, and $\gamma_{\epsilon, \rho} : \Gamma_{a,b} \to [1, \infty)$

(ii) For every $x \in \Gamma_{a,b}$, a Lipschitz map

$$\psi_x : B^s(x, \delta_{\epsilon, \rho}(x)) \to E^u_x$$

such that:

(a) $\psi_x$ depends measurably on $x$, and $W_{s,\rho}^r(x) = \exp_x(\text{graph } \psi_x)$ is a small Lipschitz disc which depends measurably on $x$.

(b) For $y, z \in W_{s,\rho}^{r^\circ}(x)$, and for all $n \geq 0$,

$$d(f^n(y), f^n(z)) \leq \gamma_{\epsilon, \rho}(x) \rho^n d(y, z).$$

(c) If $z$ satisfies

$$d(f^n(y), f^n(z)) \leq \delta_{\epsilon, \rho}(x) \rho^n$$

for all $n \geq 0$, and

$$d(z, x) \leq \frac{\delta_{\epsilon, \rho}(x)}{\gamma_{\epsilon, \rho}(f(x))}$$

then $z \in W_{s,\rho}^{r^\circ}(x)$. 

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(d) For all $x \in \Gamma_{a,b}$, define $j(x) = \delta_{\epsilon,\rho}(f(x))/\gamma_{\epsilon,\rho}(f(x))$. Then

$$f(W_{\text{loc}}^{s,\rho}(x)) \cap B(f(x), j(x)) \subset W_{\text{loc}}^{s,\rho}(f(x)).$$

(e) For $x \in \Gamma_{a,b}$, and for all $n \in \mathbb{Z}$,

$$\gamma_{\epsilon,\rho}(f^n(x)) \leq \gamma_{\epsilon,\rho}(x) \exp(\epsilon|n|),$$

$$\delta_{\epsilon,\rho}(f^n(x)) \leq \delta_{\epsilon,\rho}(x) \exp(\epsilon|n|). \quad (2.11)$$

(f) For all $v \in E_x^s$ such that $\|v\| \leq \delta_{\epsilon,\rho}(x)$,

$$\frac{1}{\gamma_{\epsilon,\rho}(x)} d(x, \exp_x(v + \psi_x(v)) \leq \|v\| \leq \gamma_{\epsilon,\rho}(x) d(x, \exp_x(v + \psi_x(v))).$$

**Proof:** See [44], or [18], or [19].

One can further show that $W_{\text{loc}}^{s,\rho}(x)$ satisfies certain smoothness conditions. Note that by (2.11), the growth of $\gamma_{\epsilon,\rho}$ and $\delta_{\epsilon,\rho}$ is subexponential. In the uniformly hyperbolic case, their analogues were uniformly bounded. Also note that by part (a) of theorem 2.15 the local stable manifold depends measurably on $x$, whereas in the uniformly hyperbolic case, part (iii) of theorem 2.4, the local stable manifolds depend continuously on $x$. The measurable dependence will be all that is necessary for later sections.

We now discuss the global stable manifolds. We define the global stable set as follows, compare with the uniformly hyperbolic case (2.4), and with (2.9) above.

**Definition 2.16.** The *global stable set* $W^s(x)$ of a point $x \in M$ is given by

$$W^s(x) = \{y \in M| \limsup_{n \to +\infty} \frac{1}{n} \ln(d(f^n x, f^n y)) < 0\}. \quad (2.12)$$

The *global unstable set* $W^u(x)$ is defined as $W^s(x)$ for $f^{-1}$. It is straightforward to verify from (2.12) that that the sets $W^s$ are equivalence classes.
Proposition 2.17. The relation $\sim$ defined by $x \sim y$ if $x \in W^s$ is an equivalence relation;

Proof:

(i) Reflexive: $x \in W^s(x)$ is clear.

(ii) Symmetric: $x \in W^s(y) \Rightarrow y \in W^s(x)$ is clear.

(iii) Transitive: $x \in W^s(y)$, and $y \in W^s(z) \Rightarrow x \in W^s(z)$. To show this, let

$$C_1 = \limsup_{n \to \infty} \frac{1}{n} \ln(d(f^nx, f^ny)),$$

$$C_2 = \limsup_{n \to \infty} \frac{1}{n} \ln(d(f^ny, f^nz)).$$

By hypothesis, $C_1 < 0$ and $C_2 < 0$. For any $\epsilon > 0$, we can find $N_1, N_2 \in \mathbb{Z}$ such that

$$\frac{1}{n} \ln(d(f^nx, f^ny)) < C_1 + \epsilon, \quad \forall \ n > N_1,$$

$$\frac{1}{n} \ln(d(f^ny, f^nz)) < C_2 + \epsilon, \quad \forall \ n > N_2.$$

Take $N = \max(N_1, N_2)$, $C = \max(C_1, C_2)$. Using the triangle inequality for the metric, $d(x, z) \leq d(x, y) + d(y, z)$, and the fact that the ln function is monotonically increasing, we see that for $n > N$

$$\frac{1}{n} \ln(d(f^nx, f^nz)) \leq \frac{1}{n} \ln(d(f^nx, f^ny) + d(f^ny, f^nz))$$

$$\leq \frac{1}{n} \ln(\exp(n (C_1 + \epsilon) + \exp(n (C_2 + \epsilon)))$$

$$\leq \frac{1}{n} \ln(2 \exp(n (C + \epsilon))) = \frac{1}{n} \ln 2 + C + \epsilon,$$

so that if we take $0 < \epsilon < -C$,

$$\limsup_{n \to \infty} \frac{1}{n} \ln(d(f^nx, f^nz)) \leq C + \epsilon < 0,$$

thus showing that $x \in W^s(z)$. ■
The analogous result applies of course for $W^u$.

The Lyapunov exponents of points $x \in \Gamma_{a,b}$ are outside the interval $[a, b]$. The local stable manifold $W_{s, \rho, \text{loc}}^{s, \rho}(x)$ of theorem 2.15 superficially appears to depend on the constants $\epsilon$ and $\rho$. In fact, as the following theorem shows, particular choices of these constants are not important. The theorem shows that the global stable manifold is independent of these choices. Choose $\rho$ and $\epsilon$ such that $\log \rho + 2\epsilon < 0$. The global stable manifolds, defined for all $x \in \Gamma$, correspond to the subspaces

$$E^s_x = \bigoplus_{\lambda_i < 0} E^i(x)$$

(2.13).

Note that this definition of $E^s_x$ differs from equation (2.10), wherein we defined $E^s_x$ and $E^u_x$ for $x \in \Gamma_{a,b}$. Similarly, we shall define

$$E^u_x = \bigoplus_{\lambda_i > 0} E^i(x)$$

(2.14).

For a regular point $x$, define $\lambda(x)$ as the largest Lyapunov exponent $\lambda_i(x)$ which is strictly negative. We then have the following stable manifold theorem.

**Theorem 2.18.** For $x \in \Gamma$, the set $W^s(x)$ is the image of a $C^1$ injective immersion of a euclidean space, satisfying:

(i) $\dim W^s(x) = \dim E^s_x$.

(ii) $T_xW^s(x) = E^s_x$.

(iii) $W^s(x) = \bigcup_{n \geq 0} f^{-n}(W_{s, \rho, \text{loc}}^{s, \rho}(f^n(x)))$, for $\ln \rho \in (\lambda, 0)$.

(iv) $W^s(x) = \{y \in M | \limsup_{n \to \infty} \frac{1}{n} \ln(d(f^n x, f^n y)) < \lambda(x)\}$.

**Proof:** See [44], p197, or [19].

One can, furthermore, obtain results on the differentiability of the global stable (and unstable) manifolds, [44], p199. The ideas which have been presented in this section are based on the fundamental work of Pesin [14,18,20].
2.4 Physical measures

We now study invariant measures on hyperbolic differential systems. There is a class of measures which appears naturally in both the mathematical analysis of hyperbolic systems, and in physical applications. These measures are often associated to an attractor. We shall follow [2] in calling these \textit{SRB measures}, after Sinai, Ruelle, and Bowen, who made contributions to their study, and stressed their importance. In this section, we first discuss the evolution of these ideas. We stress the mathematical developments, and mention the physical applications, attractors, etc., only briefly. This discussion serves to motivate the introduction of Connes’ noncommutative integration theory to the study of hyperbolic dynamical systems, and SRB measures in particular. For the rest of this section, unless otherwise stated, we consider discrete time $C^2$ dynamical systems $(M, f)$ on compact manifolds, \textit{i.e.} $f$ is a diffeomorphism with two derivatives. Some of the results we quote hold more generally, in particular, have analogues for flows.

We briefly review some of the physical motivation for studying hyperbolic dynamical systems. Many dissipative physical systems are well described by a differentiable dynamical system. The asymptotic time evolution often concentrates on subsets, generally called attractors. The definition of an attractor is not completely universal. Loosely speaking, an attractor is a set to which most points accumulate. The lore of attractors in physical systems has become widespread, notably the “strange” attractors, these are attractors which have sensitive dependence on initial conditions, \textit{i.e.}, hyperbolicity, leading to chaotic behavior. Many attractors of interest locally look like a product of a smooth submanifold with a a cantor set or “fractal” structure. In particular, in many cases, one can show that an attractor contains the global unstable manifolds of its points. The attractor can then be smooth in the direction of separation, the unstable manifolds, and may be rough in complementary directions.

More detailed information than the geometry of an attractor can be provided by an invariant measure supported on the attractor. We shall in the sequel discuss
several criteria for choosing a “physical” measure, a measure which best conveys
the relevant information about the behavior of the dynamical system. Measures
concentrated on attractors of the type described above will generally be singular
with respect to Lebesgue measure on $M$. We collect some mathematical results
regarding invariant measures for hyperbolic dynamical systems, stressing ideas
which are related to “physical” measures. We refer to Eckmann and Ruelle [2] for
a discussion of the physical aspects of the theory, as well as references to relevant
experimental and computer results.

The detailed description of some of the measure theoretic ideas used in this
section is deferred to the next chapter. We now briefly mention the basics of
the necessary ideas. We first describe the conditional measures for a measurable
partition. This subject is covered in detail in chapter 3. Consider a measure space
$M$ with a probability measure $\mu$, i.e., $\mu(M) = 1$. If $M$ is partitioned into pieces
in a “measurable” way, then the measure $\mu$ is decomposed (or “disintegrated”) in
a canonical way into probability measures supported $\mu_C$ on each piece $C$ of the
partition. These are called the conditional measures. The partition of a manifold
$M$ into global stable or unstable manifolds for a diffeomorphism $f$ is generally
not a measurable decomposition. There do exist measurable partitions for which
the elements are subsets of stable manifolds, or unstable manifolds, and these
have proven extremely useful, particularly the unstable case.

The notion of metric entropy enters in some of the results presented in this
section. We present here a very informal discussion of entropy, and refer to
Rohlin [45] or Sinai [23] for the details. Roughly, the entropy $h_\mu$ gives a mea-
sure of the rate of information creation in a dynamical system $(M, f)$ with an
invariant probability measure $\mu$. The entropy is defined in terms of measurable
partitions. Positive entropy signifies that information is being created. As we
shall see, for differentiable dynamical systems, positiveness of entropy is related
to hyperbolicity, more specifically, to the Lyapunov exponents.

There is an upper bound on the metric entropy in terms of the Lyapunov
exponents. Let $J^u(x)$ be the rate of expansion of the unstable manifold at the point $x$, relative to some Riemann metric;

$$J^u(x) = \|\text{Det}(Df)|_{E^u(x)}\|,$$  \hspace{1cm} (2.15)

where for typographic reasons we use $\| \cdot \|$ for absolute value. Define the function $\lambda_+$ which gives the asymptotic rate of expansion of the unstable manifolds;

$$\lambda_+(x) = \sum_{\{i: \lambda_i(x) > 0\}} \lambda_i(x) \dim E^i(x).$$  \hspace{1cm} (2.16)

Using theorem 2.13, one can show that for an invariant probability measure $\mu$,

$$\int \ln J^u \, d\mu = \int \lambda_+ \, d\mu.$$  \hspace{1cm} (2.17)

The following proposition is due to Ruelle [46], earlier versions are attributed to Margulis (unpublished).

Proposition 2.19. For a $C^1$ differentiable dynamical system $(M, f)$ and every invariant probability measure $\mu$, we have

$$h_\mu \leq \int \lambda_+ \, d\mu.$$

As we shall see, the measures which saturate this inequality have many interesting properties.

We now discuss attracting sets, and one possible definition of attractors. Though attractors had been defined previously in similar contexts, see [47] for a brief history of the related ideas, the work of Smale on axiom A systems brought them to prominence. Smale [13] defined attractors for Axiom A systems. Further study of axiom A and more general examples has led to more refined notions of attractors and attracting sets. We follow the exposition of Ruelle [48,49,2]. This is based in part on the study of axiom A diffeomorphisms [13], and on the work of Conley [50]. One distinguishes between attracting sets and attractors.
Definition 2.20. An *attracting set* for a dynamical system \((M, f)\) as a closed set \(\Lambda\) which satisfies:

(i) Invariance, \(f\Lambda = \Lambda\);

(ii) There exist a neighborhood \(U\) of \(\Lambda\) (a *fundamental neighborhood*) such that for every neighborhood \(V\) of \(\Lambda\), \(f^n U \subset V\) for sufficiently large \(n\);

The open set \(W = \bigcup_{n \geq 0} f^{-n} U\) is called the *basin of attraction* of \(\Lambda\).

We need the notion of chain recurrence. Choose a Riemann metric on \(M\), fixing a distance function \(d(\cdot, \cdot)\). For \(a, b \in M\), and for any \(\epsilon > 0\), if there exists a *chain* \(a = x_0, x_1, \ldots x_n = b\) such that \(d(x_k, f(x_{k-1})) < \epsilon\) for \(k = 1, \ldots n\), we say \(a\) *goes to* \(b\), written as \(a \rightarrow b\). If for every \(\epsilon > 0\) there is a chain \(a \rightarrow a\) of length \(\geq 1\), then \(a\) is called *chain recurrent*. For \(a \in M\) chain recurrent, its *basic class* \([a] = \{ b \in M | a \rightarrow b \rightarrow a \}\). The basic classes are equivalence classes.

**Definition 2.21.** A basic class \([a]\) is called an *attractor* if for any \(x \in M\), \(a \rightarrow x\) implies \(x \rightarrow a\).

These definitions were motivated by the study of Axiom A systems. See Ruelle [48,49] for discussion. If a dynamical system satisfies Axiom A and the no-cycles condition, (see *e.g.* Smale [13] or Shub [16]), then the basic sets \(\Omega_i\) of theorem 2.11 are the basic classes defined above, and the notion of attractors above coincides with the Axiom A attractors, see [13]. The following applies in general, not just for axiom A. If \(\Lambda\) is a compact attracting set, it is an attractor if and only if it is a basic class. Recall the definition (2.4) of the stable and unstable sets, which are submanifolds in cases of interest. If \([a]\) is an attractor, then \(W^u(a) \subset [a]\). An attracting set contains at least one attractor. An attractor can be equivalently described as an intersection of all attracting sets containing it.

**Proposition 2.22.** ([48], corollary 4.3) If an ergodic probability measure has compact support, this support is contained in a basic class.

This definition of an attractor is further motivated by results on stochastic
perturbations of dynamical systems. One natural candidate for a physical measures is a zero noise limit of a stationary measures for a stochastic process. A physical experimental system includes a (hopefully) small amount of stochastic noise, and is not completely deterministic. Similarly, computer calculations have some roundoff error, may in some instances act as a small noise term, although this is a delicate point in general. The deterministic dynamical systems are in some sense idealizations of the real process of interest, in the limit where the noise term goes to zero. Eckmann and Ruelle [2] attribute this idea to Kolmogorov (unpublished).

We consider a discrete time dynamical system $(M, f)$. A point $x$ is mapped to $f(x)$, and then there is a small random perturbation, which moves $x_1 = f(x)$ to some nearby point $\bar{x}_1$ with a certain probability distribution. One can realize such a stochastic perturbation by associating to a dynamical system $(M, f)$ a diffusion, which an affine map $F$ on the space of probability measures on $M$ [48]. Let $\delta_x$ be the Dirac delta measure at $x \in M$. For $\epsilon > \alpha > 0$, an $(\epsilon, \alpha)$ diffusion $F$ associated to $(M, f)$ has the property that the support of the measure $F\delta_x$ lies in an open $\epsilon$ neighborhood of $f(x) \in M$, and contains an open $\alpha$ neighborhood. In effect, the delta measure $\delta_x$ is smeared out over a small region around $f(x)$. Denote by $B(x, \alpha)$ the open $\alpha$ neighborhood of $x \in M$. Ruelle [48] shows that for a probability measure $\nu$, for a sufficiently small $\epsilon$, if the basic class $[x]$ of a point $x \in M$ is not an attractor, then

$$\lim_{n \to \infty} (F^n\nu)(B(x, \delta)) = 0. \quad (2.18)$$

Suppose one can find a stationary measure $\mu$, $F\mu = \mu$. Then (2.18) shows that if a point $x$ is not on an attractor, then $x$ is not in the support of $\mu$. This gives further credence to the above definition of an attractor. Suppose we consider the limit of vanishing stochastic perturbation, i.e., $(\epsilon, \alpha)$ diffusions associated to $(M, f)$ as $\epsilon \to 0$, and corresponding stationary measures $\mu_\epsilon$. In some cases, as $\epsilon \to 0$, the (vague) limit $\mu_\epsilon \to \mu_0$ exists. This gives us one possible notion of a
physical measure. For Axiom A attractors, the physical measure $\mu_0$ is an SRB measure, to be defined in the next paragraph. This is a result of Kifer [51], based on earlier work by Sinai [38] on Anosov systems.

SRB measures are invariant measures with the property of smooth conditional measures along unstable manifolds. SRB stands for Sinai, Ruelle, and Bowen, who originated most of the associated ideas. The complete definition of a measurable partition is given in the next chapter.

**Definition 2.23.** A measurable partition $\xi$ is said to be subordinated to the partition by unstable manifolds $W^u$ if for $m$-almost everywhere $x$, the element $\xi(x)$ containing $x$ satisfies:

(i) $\xi(x) \subset W^u(x)$,

(ii) $\xi(x)$ contains an open neighborhood of $x$ (in the submanifold topology on $W^u(x)$).

The partition into entire unstable manifolds is not generally measurable, so such partitions consist of strict subsets. We say a measure $\mu$ has smooth conditional measures along the unstable manifolds if for every measurable partition $\xi$ subordinate to $W^u$, for almost all $x \in M$, the conditional measures $\mu_{\xi(x)}$ are absolutely continuous with respect to the Lebesgue measure on $W^u$ induced by the Riemann metric ($\xi(x)$ is the piece of the partition containing the point $x$).

**Definition 2.24.** An SRB measure is an invariant measure with smooth conditional measures along the unstable manifolds.

The SRB measures have remarkable mathematical properties, some of which we shall describe in this section. Attractors are often smooth sets along the unstable directions, and fractal along the complementary directions, so one might expect SRB measures associated to such attractors to be common. In many instances, one can show that attractors consists of entire unstable manifolds. One can also make some heuristic arguments that “physical” measures should be smooth along unstable manifolds [2,42].
We discuss another natural criterion for a physical measure. Let $g$ be a continuous function on $M$. For some initial point $x \in M$, the we call the quantity

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^k x),$$

if it exists, the time average of $g$. One might like a physical measure $\mu_x$ to satisfy

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^k x) = \int g \mu_x$$

for all continuous functions $g$. This is no good if for each $x \in M$ the measure $\mu_x$ is different. We would like $\mu_x$ to coincide for a large subset of $M$. Let $\mu$ be an invariant probability measure. We call $x$ a generic point for $\mu$ if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^k x) = \int g d\mu.$$

Recall that the ergodic theorem states that if $\mu$ is ergodic for $f$, then (2.21) holds for $\mu$-almost all points $x \in M$. But the measure $\mu$ associated to an attractor is often singular with respect to Lebesgue, so $\mu$-almost all points can be a set of zero Lebesgue measure.

One can show in some specific instances that there exists a subset $A \subset M$ with positive Lebesgue measure, such that all $x \in A$ are generic for $\mu$.

**Theorem 2.25.** (Pugh and Shub, quoted in [2] p 640, see also [42]) Let $f$ be a twice differentiable diffeomorphism of a manifold $M$, and $\mu$ an SRB measure with all Lyapunov exponents nonzero. Then there is a set $A \subset M$ with positive Lebesgue measure such that for all $x \in A$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^k x) = \int g d\mu.$$

In other words, every $x \in A$ is generic for $\mu$. 

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Physical measures of this type, when they exist, are not necessarily SRB measures. See [52] for a counterexample. One can also base a definition of attractors on this property, see [42]. This is physically sensible, since Lebesgue measure plays a distinguished role for differentiable dynamical system, the notion of “typical” initial conditions is reasonably described in terms of Lebesgue measure. See Milnor [47] for a discussion of the privileged role of Lebesgue measure in differentiable dynamical systems, and a notion of attractor based on this.

For Anosov and Axiom A systems, many useful results have been obtained using symbolic dynamics. We briefly review the basic ideas. Symbolic dynamics focuses on the study of spaces of the type $S_l = \prod_{i \in \mathbb{Z}} B_i$, where $B_i$ is a set of finite cardinality $l$. An element $y \in S$ consists of a sequence $\{b_i \in A, i \in \mathbb{Z}\}$. The shift map $\sigma : S \to S$ maps the sequence $\{b_i\}$ into $\{b_{i+1}\}$. More precisely, a certain type of $\sigma$-invariant subspace of $S_l$ is important. The Markov partitions relate the shift maps on these subspaces of $S_l$ to the ergodic theory of uniformly hyperbolic dynamical systems. Markov partitions are measurable partitions whose pieces consist of “rectangles” [53], the edges of which are pieces of stable and unstable manifolds. In addition, the union of the “stable” edges must be mapped by $f$ into itself, and similarly for the “unstable” edges under $f^{-1}$. A Markov partition with $l$ elements provides a continuous surjection $g : S \to M$, where $S$ is a subset of $S_l$ constructed canonically from the Markov partition, which satisfies $f \circ g = g \circ \sigma$. In this way, properties of the dynamical system $(M, f)$ can be determined from the corresponding properties of the dynamical system $(S, \sigma)$, which is easier to analyze.

There is a class of measures on $S$ invariant under $\sigma$ for which the ergodic theory is well understood, see e.g.[17,38]. Equilibrium states for a function $\phi$, are measures which satisfy the variational principle

$$P(f, \phi) = \sup_{\mu}(h_{\mu}(f) + \int \phi d\mu),$$

(2.22)

where the sup is taken over invariant probability measures. $P$ is the topological pressure, see e.g.[17]. There is a class of functions on the shift space for which
equilibrium states are unique, the so-called Gibbs measures. These properties can be carried over to the study of measures on $M$. This was first done by Sinai [53,38] for Anosov diffeomorphisms.

**Theorem 2.26.** (Sinai, [53] thm 1) One can construct for a transitive Anosov diffeomorphism an invariant measure $\mu^u$ which is uniquely determined by the conditions:

1. As a metrical automorphism of the measure space $M$ with either one of these measures, the diffeomorphism is a $K$ automorphism.
2. $\mu^u$ is an SRB measure.

We note that $\mu^u$ is not necessarily absolutely continuous with respect to Lebesgue measure on $M$.

We now discuss some results for the Axiom A case. For simplicity, we restrict attention to discrete time systems of class $C^2$, although many of the results extend to flows. We recall definition 2.21 of attractors, and note that for Axiom A systems, an attractor is a basic set $\Omega_i$ of theorem 2.11, and an attractor in the sense of Smale [13]. Let $\Omega_i$ be an attractor for a $C^2$ axiom A dynamical system $(M,f)$. There exists a measure $\mu$ supported on $\Omega_i$, uniquely specified by the variational principle (2.22) for the function $J^u$. In fact, the topological pressure $P(f,J^u)$ vanishes, so [22,17]

\[ h_\mu + \int \ln(J^u) \, d\mu = 0, \]

\[ h_\mu + \int \lambda_+ \, d\mu = 0. \]  

(2.23)

We have used (2.17) to obtain the second equation. One can further show that $\mu$ is the unique SRB measure supported on $\Omega_i$. The measure $\mu$ also satisfies another of the properties discussed above. Under the same hypotheses as above, we have;

**Theorem 2.27.** (see [17], Thm. 4.11) For Lebesgue almost all $x \in W^s(\Omega_i)$, $x$ is a generic point for $\mu$. 

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In other words, for all continuous functions \( g: M \rightarrow \mathbb{R} \), and \( x \in W^s(\Omega_i) \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(f^k x) = \int g \, d\mu.
\]

For Axiom A systems, \( W^s(\Omega_i) \) is the basin of attraction of \( \Omega_i \). The above result is similar to theorem 2.25. One can in the axiom A case say something about the union of the basins of all the attractors. Recall that corollary 2.12 states that \( M = \bigcup_i W^s(\Omega_i) \). See \[22,17\] for details, \[54\] for axiom A flows.

We consider now the general case, of a \( C^2 \) diffeomorphism of \( M \). This was initially developed by Pesin \[14,18,20\]. The following theorem was first proved by Pesin \[20\] with the additional assumption that the invariant measure is smooth. Let \( f \) be a \( C^2 \) diffeomorphism of a compact manifold \( M \), and \( m \) a invariant probability measure.

**Theorem 2.28.** (Ledrappier and Young, \[25\] thm 1.5) \( m \) is an SRB measures if and only if

\[
h_m = \int \lambda_+ \, dm. \tag{2.24}
\]

The equation (2.24) is sometimes known as *Pesin’s entropy formula*. Note that this saturates the inequality of proposition 2.19. See also \[33\], and \[55\]. Ledrappier and Young \[26\] give a generalization of this formula, calculating the entropy for an arbitrary invariant probability measure as an integral over the manifold of a function constructed from the positive Lyapunov exponents, and a function which measures the “dimension” of the measure relative to the unstable manifolds.

This theorem is proved using a measurable partition with some special properties. There exist measurable partitions \( \xi \) subordinate to the partition by unstable manifolds which also have properties resembling those of the Markov partitions used in the uniformly hyperbolic case , \[25\], Lemma 3.1.1. Partitions of this kind were introduced by Pesin in \[56\], and used to prove the entropy formula (2.24)
We shall at this point not be more specific about the condition satisfied by the partition $\xi$. These partitions will be used in chapter 6. The following theorem provides information about the density of the conditional measures on an unstable manifold, relative to the measure $\mu_x$ induced on the unstable manifold of $x \in M$ by a Riemann metric.

**Proposition 2.29.** ([25], Cor. 6.1.4, Cor. 6.2) Let $m$ be a measure satisfying Pesin’s formula, let $\xi$ be as above, and let $\rho$ be the density of the conditional measure $m_{\xi(x)}$ with respect to the Riemann volume on the unstable manifold. Then at $m$-a.e. $x$, $\rho$ is a strictly positive function on $\xi(x)$ satisfying

$$\frac{\rho(y)}{\rho(x)} = \prod_{i=1}^{\infty} \frac{J^u_i(f^{-i}x)}{J^u_i(f^{-i}y)}.$$  

(2.25)

In particular, $\log \rho$ is Lipschitz along $W^u$ leaves, and it can even be shown that it is $C^\infty$ [40].

Despite the existence of these results, the general case is considerably more complicated than the uniformly hyperbolic case. For the uniformly hyperbolic case, the geometric structure of the stable and unstable manifolds is simpler, and reasonably well understood. The “physical” measures are also better understood. We summarize some of the results. For $C^2$ axiom A diffeomorphisms, there is a measure $\mu$ characterized by the following equivalent properties;

(i) $\mu$ is an SRB measure.

(ii) The Pesin entropy formula holds.

(iii) There is an open set $U \subset M$ such that Lebesgue almost every point $x \in U$ is generic for $\mu$.

(iv) $\mu$ can be approximated by measures that are stationary for certain stochastic perturbations.

In the sequel, we develop the Connes’ noncommutative integration theory, which we shall apply to the study of hyperbolic dynamical systems. This theory allows us to consider the “partition” into entire $W^u$ manifolds, even though
this is not measurable in the ordinary sense. Our program is to reformulate in terms of the noncommutative integration theory those aspects of the theory of differentiable dynamical systems where the unstable manifolds play an important role. We shall consider separately the uniformly hyperbolic systems, for which the partition by unstable manifolds has the structure of a foliation, and the more difficult general case. The SRB measures fit rather naturally into this framework. We shall attempt to understand various aspects of the SRB measures. For example, conditions for existence or nonexistence of SRB measures in a given system. Measurable partitions subordinate to $W^u$, though technically awkward, are necessary for the entropy calculations. One may hope that the noncommutative integration theory will provide a more conceptual understanding of the metric entropy for differentiable dynamical systems.

2.5 Notes

This chapter is entirely expository, and contains no new results. We have attempted to collect here the important results related to stable manifold theory. We hope thereby to survey the field for the nonspecialist reader, and to provide some motivation for our later constructions.