1. Find the solution to the initial-value problem:

\[
\begin{align*}
\frac{dx}{dt} &= 2x + y, \\
\frac{dy}{dt} &= 2x + 3y,
\end{align*}
\]

\[x(0) = 0, \ y(0) = 3.\]

**ANSWER:**

Let \( A \) be the coefficient matrix. We have

\[
\lambda I - A = \begin{pmatrix} \lambda - 2 & -1 \\ -2 & \lambda - 3 \end{pmatrix}.
\]

So \(|\lambda I - A| = (\lambda - 2)(\lambda - 3) - 2 = \lambda^2 - 5\lambda + 4\). We have \(\lambda = 1, 4\).

For \(\lambda = 1\), an eigenvector is

\[
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

For \(\lambda = 4\), an eigenvector is

\[
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

So all the solutions to the linear system are given by

\[
\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

Setting \(t = 0\) and using the initial conditions, we obtain

\[
\begin{pmatrix} 0 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix},
\]

i.e., \(-c_1 + c_2 = 0\) and \(c_1 + 2c_2 = 3\). Hence \(c_1 = 1\) and \(c_2 = 1\). Therefore the solution is

\[
\begin{pmatrix} x \\ y \end{pmatrix} = e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix},
\]

i.e., \(x = -e^t + e^{4t}\) and \(y = e^t + 2e^{4t}\).
1. Let \( y(t) \) be the solution to the initial-value-problem:
\[
y' = y - t, \quad y(0) = 0.01.
\]
Using the backward Euler’s method with step size \( h = 0.1 \), estimate \( y(0.2) \).

**ANSWER:**
We have \( h = 0.1, \ t_0 = 0, t_1 = 0.1, t_2 = 0.2, y_0 = y(0) = 0.01, \) and \( f(t, y) = y - t \). Recall the formula for the backward Euler’s method:
\[
y_{k+1} = y_k + hf(t_{k+1}, y_{k+1}).
\]
In our case, this simplifies to \( y_{k+1} = y_k + 0.1(y_{k+1} - t_{k+1}) \). With \( k = 0 \), we get
\[
y_1 = 0.01 + 0.1(y_1 - 0.1).
\]
Solving \( y_1 \), we have \( y_1 = 0 \). Next, with \( k = 1 \), we obtain
\[
y_2 = y_1 + 0.1(y_2 - t_2),
\]
i.e., \( y_2 = 0.1(y_2 - 0.2) \). Hence \( y_2 = -1/45 \). So \( y(0.2) \sim y_2 = -1/45 = -0.022 \).

2. Find all the eigenvalues of the following matrix:
\[
\begin{bmatrix}
1 & 3 \\
-2 & -3
\end{bmatrix}
\]

**ANSWER:**
We have
\[
\lambda I - A = \begin{bmatrix}
\lambda - 1 & -3 \\
2 & \lambda + 3
\end{bmatrix}.
\]
So \( |\lambda I - A| = (\lambda - 1)(\lambda + 3) + 6 = \lambda^2 + 2\lambda + 3 \). We have \( \lambda = -1 \pm \sqrt{2}i \).
1. Find the Laplace transform of $f(t)$ where

$$f(t) = \int_0^t e^{-(t-\tau)} \sin(\tau) \, d\tau.$$ 

**ANSWER:**

Note that by the definition of convolution integral,

$$f(t) = g(t) * h(t)$$

where $g(t) = e^{-t}$ and $h(t) = \sin(t)$. Therefore,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t) * h(t)\} = \mathcal{L}\{g(t)\} \cdot \mathcal{L}\{h(t)\} = \frac{1}{s+1} \cdot \frac{1}{s^2 + 1} = \frac{1}{(s+1)(s^2+1)}.$$ 

2. Find the Laplace transform of the solution to the initial-value-problem:

$$y'' + 3y' + 2y = u_4(t), \quad y(0) = 1, \quad y'(0) = 0.$$ 

**ANSWER:**

We apply the Laplace transform to both sides of the equation:

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{u_4(t)\}.$$ 

Put $Y = \mathcal{L}\{y\}$. Then again, $\mathcal{L}\{y'\} = sY - 1$, and $\mathcal{L}\{y''\} = s^2Y - s$. Since $\mathcal{L}\{u_4(t)\} = e^{-4s}/s$, we have

$$(s^2Y - s) + 3(sY - 1) + 2Y = \frac{e^{-4s}}{s}.$$ 

It follows that

$$\mathcal{L}\{y\} = Y = \frac{e^{-4s}}{s(s^2 + 3s + 2)} + \frac{s + 3}{s^2 + 3s + 2}.$$
3. Solve
\[ y'' + 3y' + 2y = \delta(t - 3), \quad y(0) = 0, \quad y'(0) = 0. \]

**ANSWER:** (use the other side if necessary)

Apply the Laplace transform to both sides of the equation:
\[ \mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t - 3)\}. \]

Put \( Y = \mathcal{L}\{y\} \). Then, \( \mathcal{L}\{y'\} = sY \), and \( \mathcal{L}\{y''\} = s^2Y \). Recall that \( \mathcal{L}\{\delta(t - 3)\} = e^{-3s} \).

So we have
\[ s^2Y + 3sY + 2Y = e^{-3s}, \]
and \( Y = e^{-3s}H(s) \) where \( H(s) = 1/(s^2 + 3s + 2) \). Let \( h(t) = \mathcal{L}^{-1}\{H(s)\} \). Note that \( s^2 + 3s + 2 = (s + 1)(s + 2) \), and \( H(s) = 1/(s + 1) - 1/(s + 2) \). So \( h(t) = e^{-t} - e^{-2t} \).

Thus,
\[ y = \mathcal{L}^{-1}\{Y\} = u_3(t)h(t - 3) = u_3(t)(e^{-(t-3)} - e^{-2(t-3)}). \]
QUIZ 7—MATHEMATICS 4100

Your last name: ____________________________
Your first name: ____________________________

1. Find $L^{-1}\{H(s)\}$ where 
   
   \[
   H(s) = \frac{e^{-6s}}{s^2}.
   \]

   **ANSWER:**
   
   By the formula in section 6.3, if we put $F(s) = 1/s^2$, then we have
   
   
   \[
   L^{-1}\{H(s)\} = L^{-1}\{e^{-6s} \cdot F(s)\} = u_6(t)f(t - 6)
   \]
   
   where $f(t) = L^{-1}\{F(s)\}$. Recall that $L^{-1}\{t\} = 1/s^2 = F(s)$. So $f(t) = t$. Hence,
   
   
   \[
   L^{-1}\{H(s)\} = u_6(t)f(t - 6) = u_6(t)(t - 6).
   \]

2. Using Laplace transform, solve the initial-value-problem:

   \[
   y'' + 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 0.
   \]

   **ANSWER:**
   
   Apply the Laplace transform to both sides of the equation:
   
   \[
   L\{y''\} + 3L\{y'\} + 2L\{y\} = 0.
   \]
   
   Put $Y = L\{y\}$. Then, $L\{y'\} = sY - 1$, and $L\{y''\} = s^2Y - s$. So we have
   
   
   \[
   (s^2Y - s) + 3(sY - 1) + 2Y = 0,
   \]
   
   and $Y = (s + 3)/(s^2 + 3s + 2)$. Note that $s^2 + 3s + 2 = (s + 1)(s + 2)$. Thus, $Y$ can be written as
   
   \[
   Y = \frac{s + 3}{s^2 + 3s + 2} = \frac{A}{s + 1} + \frac{B}{s + 2}.
   \]
   
   From $(s + 3) = A(s + 2) + B(s + 1)$, we see that $A = 2$ and $B = -1$. Hence
   
   \[
   Y = \frac{2}{s + 1} - \frac{1}{s + 2}.
   \]
   
   Therefore,
   
   
   \[
   y = L^{-1}\{Y\} = 2e^{-t} - e^{-2t}.
   \]
1. An undamped free vibration is described by the equation:

\[3u'' + 75u = 0.\]

Determine the natural frequency \(\omega_0\) of this vibration.

**ANSWER:**
The characteristic equation is \(3r^2 + 75 = 0\). So its roots are \(r = \pm 5i\), and the general solution is:

\[u = A \cos(5t) + B \sin(5t)\]

Hence \(\omega_0 = 5\).

2. Using the method of variation parameters, solve the initial-value-problem:

\[4y'' - 4y' + y = 16e^t/2, \quad y(0) = 0, \quad y'(0) = 0.\]

**ANSWER:**
Solving \(4y'' - 4y' + y = 0\), we obtain \(y = c_1 e^{t/2} + c_2 t e^{t/2}\). So \(y_1 = e^{t/2}, y_2 = t e^{t/2}\), and the Wronskian \(W(y_1, y_2) = e^t\). Note that \(g = 4e^{t/2}\). By the method of variation parameters, we have \(u_1 = -2t^2\) and \(u_2 = 4t\). Therefore, a particular solution is:

\[y_p = -2t^2 \cdot e^{t/2} + 4t \cdot t e^{t/2} = 2t^2 e^{t/2}\]

So the general solution is

\[y = 2t^2 e^{t/2} + c_1 e^{t/2} + c_2 t e^{t/2}\]

The initial conditions force \(c_1 = c_2 = 0\). So the solution to the initial-value-problem is \(y = 2t^2 e^{t/2}\).
1. Solve the following initial-value-problem:
\[ y'' - 6y' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 4. \]

**ANSWER:**
Solving \( r^2 - 6r + 9 = 0 \), we obtain \( r = 3, 3 \). So the general solution is \( y = (c_1 + c_2 x)e^{3x} \).

Using the initial conditions \( y(0) = 1 \) and \( y'(0) = 4 \), we see that \( c_1 = c_2 = 1 \). So the solution is \( y = (1 + x)e^{3x} \).

2. Using the method of undetermined coefficients, find the general solution to:
\[ y'' + 4y = 3\cos(2x). \]

**ANSWER:**
Solving \( r^2 + 4 = 0 \), we get \( r = \pm 2i \). So the general solution to \( y'' + 4y = 0 \) is \( y = c_1 \cos(2x) + c_2 \sin(2x) \). Now, by the method of undetermined coefficients, we try a particular solution of the form \( y = x^s(A\cos(2x) + B\sin(2x)) \). Note that \( s = 1 \). So \( y = x(A\cos(2x) + B\sin(2x)) \). The differential equation forces \( A = 0 \) and \( B = 3/4 \). So a particular solution is \( y = 3/4 x \sin(2x) \). The general solution is
\[ y = \frac{3}{4} x \sin(2x) + c_1 \cos(2x) + c_2 \sin(2x). \]
1. Solve the following initial-value-problem:

\[ y'' + 64y = 0, \quad y(0) = 1, \quad y'(0) = 8. \]

**ANSWER:**

Solve \( r^2 + 64 = 0 \), we obtain \( r = \pm 8i \). So the general solution is \( y = c_1 \cos(8x) + c_2 \sin(8x) \). Using the initial conditions \( y(0) = 1 \) and \( y'(0) = 8 \), we see that \( c_1 = c_2 = 1 \). So the solution is \( y = \cos(8x) + \sin(8x) \).
QUIZ 3—MATHEMATICS 4100

1. Consider the following initial-value-problem:

\[ y' = 3 + t - y, \quad y(0) = 1. \]

Using Euler’s method with \( n = 2 \), find the approximate value of the solution at \( t = 2 \)
(i.e., estimate \( y(2) \)).

**ANSWER:**

We have \( t_0 = 0 \), \( a = 2 \) (since we want to estimate \( y(2) \)), \( h = (a - t_0)/n = 1. \)
Then, \( t_0 = 0, \quad t_1 = t_0 + h = 1, \quad t_2 = t_0 + 2h = 2. \) Also, \( f(t, y) = 3 + t - y. \) Now
\( y_0 = y(t_0) = y(0) = 1, \quad y_1 = y_0 + hf(t_0, y_0) = 3, \) and \( y_2 = y_1 + hf(t_1, y_1) = 4. \) So the
estimation for \( y(2) \) is \( y_2 = 4. \)

2. Solve the following initial-value-problem:

\[ y'' - 6y' + 8y = 0, \quad y(0) = 2, \quad y'(0) = 6. \]

**ANSWER:**

Solve \( r^2 - 6r + 8 = 0 \), we obtain \( r = 2, 4. \) So the general solution is \( y = c_1 e^{2x} + c_2 e^{4x}. \)
Using the initial conditions \( y(0) = 2 \) and \( y'(0) = 6 \), we see that \( c_1 = c_2 = 1. \) So the
solution is \( y = e^{2x} + e^{4x}. \)
1. Determine whether the following differential equation is exact:

\[(e^x \sin y + 3y)dx - (3x - e^x \sin y)dy = 0.\]

**ANSWER:**

We have \(M = (e^x \sin y + 3y)\) and \(N = -(3x - e^x \sin y)\). So

\[M_y = e^x \cos y + 3, \quad N_x = -3 + e^x \sin y.\]

Since \(M_y \neq N_x\), the differential equation is not exact.
1. Solve the initial-value-problem:

\[ y' = 2xy, \quad y(0) = -4. \]

**ANSWER:**

This is a separable equation:

\[ \frac{dy}{y} = 2x \, dx. \]

Integrating both sides, we obtain \( \ln y = x^2 + c_1 \). So \( y = ce^{x^2} \). Setting \( x = 0 \) and \( y = -4 \), we obtain \( c = -4 \). Hence the solution is

\[ y = -4e^{x^2}. \]