QUIZ 7—MATHEMATICS 2300

1. Compute the double integral \( \int \int_{R^*} \sqrt{x^2 + y^2} \, dx \, dy \) where \( R^* \) is the region:

\[ x^2 + y^2 \leq 9, \quad x \geq 0, \quad y \geq 0. \]

**SOLUTION:**

Use polar coordinates. We have \( R = [0, 3] \times [0, \pi/2] \). Since \( x^2 + y^2 = r^2 \), we have

\[
\int \int_{R^*} \sqrt{x^2 + y^2} \, dx \, dy = \int \int_{R} \sqrt{r^2} \cdot r \, dr \, d\theta = \frac{9\pi}{2}.
\]
1. Compute the double integral \( \int_R f \) where \( f(x, y) = x^2 + y^2 \) and \( R = [0, 1] \times [0, 6] \).

SOLUTION:

\[
\int_R f = \int_0^1 \left[ \int_0^6 (x^2 + y^2) \, dy \right] \, dx
\]
\[
= \int_0^1 \left[ x^2 y + \frac{y^3}{3} \right]_0^6 \, dx
\]
\[
= \int_0^1 (6x^2 + 72) \, dx
\]
\[
= (2x^3 + 72x) \bigg|_0^1
\]
\[
= 74.
\]

2. Find all the local extremal values of \( f(x, y) = x^2 + y^2 - 2018 \).

SOLUTION:

\textbf{Step 1.} Find all the critical points.

We have \( \frac{\partial f}{\partial x} = 2x \) and \( \frac{\partial f}{\partial y} = 2y \). So \( \frac{\partial f}{\partial x} = 0 \) and \( \frac{\partial f}{\partial y} = 0 \) forces \( x = 0 \) and \( y = 0 \). This says that the only critical point of \( f \) is the origin \((0, 0)\).

\textbf{Step 2.} Compute the Hessian.

We have \( \frac{\partial^2 f}{\partial x^2} = 2 \), \( \frac{\partial^2 f}{\partial y^2} = 2 \), and \( \frac{\partial^2 f}{\partial x \partial y} = 0 \). So \( H_f = 4 \).

\textbf{Step 3.} Use the 2nd-order derivative test.

We have \( H_f(0, 0) = 4 > 0 \) and \( \frac{\partial^2 f}{\partial x^2}(0, 0) = 2 > 0 \). So \( f \) has a local minimum at \((0, 0)\), and the local minimum is \( f(0, 0) = -2018 \).
1. Compute the gradient of $f(x, y, z) = x^2 \cos(8y^3z)$.

**SOLUTION:**

$$\nabla f = (2x \cos(8y^3z), -24x^2y^2z \sin(8y^3z), -8x^2y^3 \sin(8y^3z)).$$
1. Let $f(x, y, z) = x^3z^5 + x^2\sin(8y^3z)$. Compute $\partial f/\partial y$.

**SOLUTION:**

$\partial f/\partial y = 24x^2y^2z\cos(8y^3z)$. 
1. Find the arc length of the following curve:

\[ \mathbf{r}(t) = (3t, 2\sin(2t), -2\cos(2t)), \quad t \in [-\pi, \pi]. \]

**SOLUTION:**

Note that \( \mathbf{r}'(t) = (3, 4\cos(2t), 4\sin(2t)). \) So \( |\mathbf{r}'(t)| = 5. \) We have

\[ \ell(\mathbf{r}) = \int_{-\pi}^{\pi} |\mathbf{r}'(t)| \, dt = \int_{-\pi}^{\pi} 5 \, dt = 10\pi. \]
1. Let $X = (1, 2, -3)$ and $Y = (-2, -4, 6)$. Compute $X \times Y$.

SOLUTION:
We have $X \times Y = (12 - 12)i - (6 - 6)j + (-4 + 4)k = 0$. 

1. Let \( \mathbf{X} = (1, -1, -2) \) and \( \mathbf{Y} = (1, -2, 1) \) be two vectors. Compute: 
\[ (-2)\mathbf{X} - 4\mathbf{Y}. \]

**SOLUTION:**
We have 
\[ (-2)\mathbf{X} - 4\mathbf{Y} = (-2, 2, 4) - (4, -8, 4) = (-6, 10, 0). \]