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The oblique derivative problem for general elliptic systems in Lipschitz domains

Let $M$ be a smooth, oriented, connected, compact, boundaryless manifold of real dimension $m$, and let $TM$ and $T^*M$ be its tangent and cotangent bundles, respectively. We assume that $M$ is equipped with a Lipschitzian Riemannian metric tensor $g = g_{jk} dx_j \otimes dx_k$, $g_{jk} \in \text{Lip}(M)$. We also denote $\det(g_{jk})$ by $\mathcal{g}$, so that if $d\Vol$ stands for the corresponding volume form, then $d\Vol = \sqrt{\mathcal{g}(x)} \, dx$ locally.

Consider a smooth vector bundle $E \to M$ endowed with Lipschitzian Hermitian structures, and let $L(x, D)$ be a second-order differential operator acting on sections of $E$. Suppose that, about each point, $M$ has a smooth coordinate system and $E$ has a local trivialization with respect to which $L$ takes the form

$$Lu = \sum_{j,k} \partial_j A_{jk}^i(x) \partial_k u + \sum_j B_j^i(x) \partial_j u - V(x)u,$$

where $A_{jk}^i, B_j^i, \text{ and } V$ are matrix-valued functions such that

$$A_{jk}^i \in C^{1+\gamma}, \quad B_j^i \in H^{1,r}, \quad V \in L^r$$

for some $\gamma > 0$ and $r > m = \text{dim } M$. Here and elsewhere, $H^{s,p}$ denotes the usual scale of Sobolev spaces.

Following [6], for a Lipschitz domain $\Omega$ in $M$ (possibly the whole of $M$) we say that $L$ satisfies the non-singularity hypothesis relative to $\Omega$ if

$$u \in H^{1,2}_0(\Omega, E), \quad Lu = 0 \text{ in } \Omega \quad \Rightarrow \quad u = 0 \text{ in } \Omega.$$  \hspace{1cm} (3)

Parenthetically, we observe that if $L$ is strongly elliptic, then $L - \lambda, \lambda \in \mathbb{R}$, satisfies the non-singularity hypothesis (3) relative to any subdomain $\Omega \subseteq M$ provided $\lambda$ is sufficiently large. This is a consequence of Garding’s inequality, which is valid in our setting (even though $V$ may be unbounded). Also, clearly, if $L$ is strongly elliptic and negative semidefinite, then $L - \lambda$ satisfies (3) for any $\lambda > 0$. A concrete example of an elliptic, formally self-adjoint operator satisfying (1)–(3) is the Hodge-Laplacian corresponding to a Riemannian metric with coefficients in $H^{2,r}$, $r > m$.

Let $\Omega$ be a Lipschitz subdomain of $M$, and let $\nu \in T^*M$ be the unit outward conormal to $\partial \Omega$. In order to formalize the partial derivative operator $u \mapsto \nabla_w u + Au$, where $A \in L^\infty(M, \text{Hom}(E, E))$ and $w$ is a vector field on $M$ transversal to $\partial \Omega$ (that is, $\text{essinf } \langle \nu, w \rangle > 0$ on $\partial \Omega$), we work with a first-order differential operator $P = P(x, D)$ on $E$ such that

$$\sigma(P; \nu), \text{ the principal symbol of } P \text{ (at } \nu), \text{ is scalar and } > 0 \text{ on } \partial \Omega.$$  \hspace{1cm} (4)

Before we state the main result, we introduce some notation. With $\Omega$ as before, let $\cdot|_{\partial \Omega}$ be the nontangential boundary trace operator. Specifically, for a section $u : \Omega \to E$ we set

$$u|_{\partial \Omega}(x) := \lim_{y \in \gamma(x)} u(y), \quad x \in \partial \Omega,$$

where $\gamma(x)$ is a curve in $\Omega$ approaching $x$. Note that $u|_{\partial \Omega}$ is a well-defined function on $\partial \Omega$. Now, let $\mathcal{H}(\Omega, E)$ denote the Banach space of $H^{1,2}_0(\Omega, E)$ functions with values in $E$.

Theorem: Let $\Omega$ be a Lipschitz domain in $M$, and let $P$ be a first-order differential operator on $E$ with principal symbol $\sigma(P; \nu) > 0$ on $\partial \Omega$. Then, there exists a constant $C > 0$ such that for any $u \in \mathcal{H}(\Omega, E)$ and $\nu \in T^*M$,

$$\|P \cdot|_{\partial \Omega} - C \| \leq C \|u\|.$$  \hspace{1cm} (5)

Proof: (省略)
where $\gamma(x) \subseteq \Omega$ is an appropriate nontangential approach region. Finally, $\mathcal{N}$ stands for the nontangential maximal operator defined by

$$\mathcal{N}u(x) \coloneqq \sup \{|u(y)|; y \in \gamma(x)\}, \quad x \in \partial \Omega.$$  

(6)

The main result of this paper is the theorem below, which addresses the solvability of the oblique derivative problem for general second-order strongly elliptic PDEs in Lipschitz domains. (For earlier results in the flat Euclidean setting and for constant coefficient PDEs, see [5] and [8].)

Our arguments, building on an earlier idea of Calderón [2], make essential use of the results devised in [7] and [6]. This approach is constructive, in the sense that it relies on integral equation methods; indeed, we prove global representation formulas for the solutions in terms of boundary layer potentials. The trend of using such layer potentials “for general elliptic systems” in the nonsmooth context was suggested by A. P. Calderón in [1]. For related developments, the interested reader may consult [7], [6] and the references therein.

In order to state our main result, we recall that VMO($\partial \Omega$), the space of functions of vanishing mean oscillation, is the closure of Lip ($\partial \Omega$) in the BMO “norm”; see [3] for a discussion in the context of spaces of homogeneous type.

**Theorem.** Let $\mathcal{E} \to M$ be as above, and let $L$ be a strongly elliptic, (formally) self-adjoint, second-order differential operator acting on sections of $\mathcal{E}$, with coefficients satisfying (2) locally. Suppose that $L$ satisfies the non-singularity hypothesis (3) relative to any Lipschitz subdomain of $M$. Also, let $P$ be a first-order differential operator such that (4) is satisfied, whose coefficients belong to $L^\infty(\partial \Omega)$ and whose top coefficients also belong to VMO($\partial \Omega$).

Then for any fixed Lipschitz domain $\Omega \subset M$ there exists $\varepsilon = \varepsilon(\Omega, L) > 0$ such that for each $p \in (2 - \varepsilon, 2 + \varepsilon)$ the following holds: for any $f \in L^p(\partial \Omega, \mathcal{E})$ satisfying finitely many (necessary) linear conditions, the oblique derivative problem

$$\begin{cases}
u \in C^0_{\text{loc}}(\Omega, \mathcal{E}), \\
Lu = 0 \text{ in } \Omega, \\
N(\nabla u) \in L^p(\partial \Omega), \\
Pu|_{\partial \Omega} = f \text{ on } \partial \Omega,
\end{cases}$$  

(7)

is solvable and its solution is unique modulo a finite-dimensional linear space.

Finally, the index of this problem is zero, that is, the dimension of the space of null-solutions coincides with the number of linearly independent constraints required for the boundary data.

**Proof.** The construction of a suitable parametrix implies that $L : H^{1,2}(M, \mathcal{E}) \to H^{-1,2}(M, \mathcal{E})$ is Fredholm, while the hypotheses of self-adjointness and strong ellipticity allow us to use a deformation argument to imply that $L$ has index zero. Hence, $L : H^{1,2}(M, \mathcal{E}) \to H^{-1,2}(M, \mathcal{E})$ is invertible.

Let $\Gamma(x, y) \in \mathcal{D}'(M \times M, \mathcal{E} \otimes \mathcal{E})$ be the Schwartz kernel of $L^{-1} : H^{\mu - 1,p}(M, \mathcal{E}) \to H^{\mu + 1,p}(M, \mathcal{E})$, $0 < \mu < 1$, $1 < p < r$. It follows that $\Gamma \in C^{1+\gamma}(M \times M \setminus \text{diag}\mathcal{E} \otimes \mathcal{E})$
for some $\gamma > 0$. Next, we introduce the single layer potential acting on sections $f : \partial \Omega \to \mathcal{E}$ by means of the formula

$$Sf(x) := \int_{\partial \Omega} \langle \Gamma(x, y), f(y) \rangle \, dS(y), \quad x \in M \setminus \partial \Omega,$$

where $dS$ is the surface measure on $\partial \Omega$. Also, we set $S_f := Sf|_{\partial \Omega}$. The idea is to look for a solution to the boundary problem (7) expressed in the form

$$u := Sg \quad \text{in } \Omega,$$

for some $g \in L^p(\partial \Omega, \mathcal{E})$ to be chosen later. Clearly, $Lu = 0$ in $\Omega$ and, from [6], we know that $\|N(\nabla u)\|_{L^p(\partial \Omega)} \leq C\|g\|_{L^p(\partial \Omega, \mathcal{E})}$. Going further, it has been shown in [6] that at almost every $x \in \partial \Omega$ we have

$$Pu(x) = -\frac{1}{2} i \sigma(P; \nu(x)) \sigma(L; \nu(x))^{-1} g(x) + Tg(x),$$

where $T$ is the principal-value singular integral operator on $\partial \Omega$ (in the sense of removing geodesic balls) given by

$$Tg(x) = \text{p.v.} \int_{\partial \Omega} \langle (P_x \otimes \text{Id}_y) \Gamma(x, y), g(y) \rangle \, dS(y), \quad x \in \partial \Omega.$$ 

Thus, in order to prove existence for (7), modulo a space of finite dimension, it suffices to show that

$$-\frac{1}{2} i \sigma(P; \nu) \sigma(L; \nu)^{-1} + T : L^p(\partial \Omega, \mathcal{E}) \to L^p(\partial \Omega, \mathcal{E})$$

has a finite codimensional range for $p$ close to 2. What we shall prove is that this operator is Fredholm with index zero.

To begin with, it has been shown in [6] that the operator (11) is bounded for any $1 < p < \infty$. Hence, since the quality of being Fredholm and the index are stable on complex interpolation scales (cf. the discussion in [4]), it is enough to prove this only for $p = 2$. Rewriting the operator (12) in the form

$$\left[ -\frac{1}{2} \sigma(P; \nu) i \sigma(L; \nu)^{-1} + \frac{1}{2}(T - T^*) \right] + \frac{1}{2}(T + T^*) =: T_1 + T_2,$$

we see that, by (4) and the strong ellipticity assumption on $L$, $T_1$ is a strictly accretive operator on $L^2(\partial \Omega, \mathcal{E})$ and, hence, invertible. Moreover, we claim that

$$T_2 = \frac{1}{2}(T + T^*)$$

is a compact operator on $L^p(\partial \Omega, \mathcal{E}), \quad 1 < p < \infty.$ (14)

We prove this claim first under the stronger assumption that the top coefficients of $P$ are Lipschitz continuous on $\partial \Omega$. In this case, the desired conclusion is going to be a consequence of the weak singularity of the kernel of $T_2$ along the diagonal in $\partial \Omega \times \partial \Omega$. 

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To see this, we work in local coordinates in a small, open neighbourhood $U$ of a boundary point $x_0 \in \partial \Omega$. Let $\{\theta_\alpha\}$ be a local orthonormal frame of $\mathcal{E}$ over $U$. In local coordinates, the fundamental solution $\Gamma$ has the asymptotic expansion

$$\Gamma(x, y) = \frac{1}{\sqrt{g(y)}} \Gamma_0(y, x - y) + R(x, y),$$

(15)

where, for each $\varepsilon > 0$, the matrix-valued residual term $R$ satisfies

$$|R(x, y)| + |x - y| \cdot |\nabla_x R(x, y)| + |x - y| \cdot |\nabla_y R(x, y)| = \mathcal{O}(|x - y|^{-m+3-\varepsilon}).$$

(16)

Also, the symmetric matrix-valued function $\Gamma_0(y, z)$ is even in $z$ and satisfies

$$\Gamma_0(\rho z, y) = |\rho|^{-(m-2)} \Gamma_0(z, y)$$

for any $\rho \in \mathbb{R}$ and $z \in \mathbb{R}^m$, (17)

as well as the estimates

$$(\nabla_y \Gamma_0)(z, y) = \mathcal{O}(|z|^{-(m-2)}) \text{ as } z \to 0, \text{ uniformly in } y,$$

(18)

and

$$(\nabla_z \Gamma_0)(z, y), \ (\nabla_z \nabla_y \Gamma_0)(z, y) = \mathcal{O}(|z|^{-(m-1)}) \text{ as } z \to 0, \text{ uniformly in } y.$$

(19)

Identifying $U$ with an open subset of $\mathbb{R}^n$ and $P$ with a matrix of first-order differential operators $P \in \text{Diff}_1(U \times \mathbb{R}^n, U \times \mathbb{R}^n)$ (where $n := \text{rank } \mathcal{E}$), we deduce that the main singularity in the kernel of $T + T^*$ is contained in

$$P(x, D_x)g(x)^{-1/2} \Gamma_0(y, x - y) + (P(y, D_y)g(x)^{-1/2} \Gamma_0(x, y - x))^*,$$

(20)

where the asterisk denotes matrix transposition. Now, since the principal part of $P$ is scalar and $\Gamma_0$ is symmetric, the expression

$$(P(y, D_y)g(x)^{-1/2} \Gamma_0(x, y - x))^* - P(y, D_y)g(x)^{-1/2} \Gamma_0(x, y - x)$$

(21)

is actually $\mathcal{O}(|x - y|^{-m+2})$; therefore, it suffices to examine

$$P(x, D_x)g(y)^{-1/2} \Gamma_0(y, x - y) + P(y, D_y)g(x)^{-1/2} \Gamma_0(x, y - x)$$

$$= P(x, D_x)g(y)^{-1/2} \Gamma_0(y, x - y) + P(y, D_y)g(x)^{-1/2} \Gamma_0(x, y - x).$$

(22)

To this end, since $g \in C^1$ and the commutator $[P, g^{-1/2}]$ is bounded, the above expression can be rewritten as $g^{-1/2}(I + II + III) + \{\text{less singular terms}\}$, where

$$I := P(x, D_x)\Gamma_0(y, x - y) + P(x, D_y)\Gamma_0(y, x - y),$$

$$II := P(y, D_y)\Gamma_0(y, x - y) - P(x, D_y)\Gamma_0(y, x - y),$$

$$III := P(y, D_y)\Gamma_0(x, x - y) - P(y, D_y)\Gamma_0(y, x - y).$$

(23)
On account of the smoothness of the coefficients of \( P \) and the estimates (18) and (19), we find that each of the quantities above and, hence, the entire expression (21), is \( O(|x-y|^{-m+2-\varepsilon}) \) for any \( \varepsilon > 0 \). Consequently, the proof of the fact that \( T + T^* \) is weakly singular is finished by invoking (16). This concludes the proof of (14) when the (top) coefficients of \( P \) are Lipschitz on \( \partial \Omega \).

Returning to the general case when the top coefficients of \( P \) are just VMO functions, we select a sequence of operators \( \{P_j\}_j \) whose top coefficients are Lipschitzian and approximate the top coefficients of \( P \) in the BMO-sense arbitrarily well. The idea is that, while the term \( II \) in (23) is no longer \( O(|x-y|^{-m+2-\varepsilon}) \), this kernel still gives rise to an operator \( K_{II} \) that is compact on \( L^p(\partial \Omega, E) \), \( 1 < p < \infty \). This is because \( K_{II} \) can be expressed as a limit of commutators between some Calderón-Zygmund type operator \( K_j \) (whose kernels are of the form \( \partial_y \Gamma(y, x-y) \)) and the operators \( M_{b_j} \) of multiplication by Lipschitz functions \( b_j \) (arising as coefficients of \( P_j \)). By construction, the sequence \( \{b_j\}_j \) converges to some \( b \) in the BMO-sense. In this case,

\[
\|K_{II} - [K_j, M_{b_j}]\|_{\mathcal{L}(L^p)} \leq C\|b_j - b\|_{\text{BMO}} \to 0, \tag{24}
\]

where \( \mathcal{L}(L^p) \) is the space of all bounded operators on \( L^p \); see [7] for a discussion. Since, by the previous piece of reasoning, each \( [K_j, M_{b_j}] \) is compact on \( L^p \), it follows that \( K_{II} \) is also compact on \( L^p \). Thus, (14) holds. This completes the proof of (14) for operators \( P \) with \( L^\infty \) coefficients and whose top coefficients are also in VMO.

So far we have proved that, under the assumptions in the theorem, there exists \( \varepsilon > 0 \) so that the operator (12) is Fredholm with index zero for each \( p \in (2-\varepsilon, 2+\varepsilon) \).

It follows that the problem (7) is solvable whenever the boundary datum \( f \) satisfies

\[
f \in \text{Im} \left( -\frac{1}{2} i\sigma(P; \nu)\sigma(L; \nu)^{-1} + T : L^p(\partial \Omega, E) \right) \tag{25}
\]

and the right side (that is, the range of the operator in (12)) is a closed subspace of finite codimension in \( L^p(\partial \Omega, E) \) for each \( |2-p| < \varepsilon \).

To see that (25) is also a necessary condition for the solvability of (7), let us observe that, by the results in [6], any section \( u \) satisfying \( Lu = 0 \) in \( \Omega \) and \( \mathcal{N}(\nabla u) \in L^p(\partial \Omega) \), \( |2-p| < \varepsilon \), is representable in the form \( u = Sg \) in \( \Omega \) for some \( g \in L^p(\partial \Omega, E) \). Consequently,

\[
Pu|_{\partial \Omega} = (-\frac{1}{2} i\sigma(P; \nu)\sigma(L; \nu)^{-1} + T)g \in \text{Im} \left( -\frac{1}{2} i\sigma(P; \nu)\sigma(L; \nu)^{-1} + T : L^p(\partial \Omega, E) \right). \tag{26}
\]

Finally, by essentially the same token, any solution \( u \) of the homogeneous version of (7) has the form \( u = Sh \) for some section \( h \) belonging to \( \text{Ker} \left( -\frac{1}{2} i\sigma(P; \nu)\sigma(L; \nu)^{-1} + T : L^p(\partial \Omega, E) \right) \). Consequently, the dimension of the null-space of the oblique derivative problem (7) is

\[
\dim \text{Ker} \left( -\frac{1}{2} i\sigma(P; \nu)\sigma(L; \nu)^{-1} + T : L^p(\partial \Omega, E) \right) = \dim \text{Im} \left( -\frac{1}{2} i\sigma(P; \nu)\sigma(L; \nu)^{-1} + T : L^p(\partial \Omega, E) \right).
\]

This completes the proof of the theorem.

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References


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