The Initial Dirichlet Boundary Value Problem for General Second Order Parabolic Systems in Nonsmooth Manifolds *

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0 Introduction

In a series of recent papers [1], [2], [3], [4], [5] we have initiated the study of boundary value problems for (variable coefficients, second order, strongly) elliptic PDE's in nonsmooth subdomains of Riemannian manifolds via integral equation methods. Here we take the first steps in the direction of extending this theory to initial boundary value problems (IBVP's) for variable coefficient (strongly) parabolic systems in non-smooth cylinders.

Problems as such have a long history and the field remains a very active area of research. For work in the context of smooth manifolds the reader is referred to [6], [7], [8]. See also [9], [10], [11], [12], [13], [14], [15], [16], for IBVP's associated with PDE's of parabolic type in the smooth Euclidean setting.

With the work of E.B. Fabes, N. Rivière and their collaborators starting in the 1960's, a new direction of research has emerged, emphasizing $L^p$-boundary data and less regular domains and/or coefficients. In this vein see [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27]. The papers cited above deal with domains exhibiting a limited amount of smoothness and new techniques needed to be developed in order to treat the nonsmooth case.

After the breakthrough in the elliptic case, cf. [28], [29], [30], [31], [32] and the references therein, there has been a substantial amount of work in the direction of solving IBVP's for parabolic PDE's in minimally smooth domains. In flat-space, Euclidean Lipschitz cylinders, $\partial_t - \Delta$ was treated via caloric measure estimates in [33] (inspired by the work in [28]) and via integral methods in [34], [35], [36] (after the pioneering work in [22] and by adapting the approach from [30]). The latter work has been further extended to include second-order constant coefficient PDE’s such as the parabolic versions of the Lamé system, the linearized Navier-Stokes system and the Maxwell system in [37], [38], [39]. Higher order, homogeneous, constant coefficient, parabolic PDE’s have been treated in [40], [41], [42], following the work in elliptic case from [43]. The Dirichlet problem for more general, scalar, divergence-form parabolic PDE in Lipschitz cylinders has been considered in [44]. Extensions to time-varying domains have been developed in [45], [46], [47], [48], [49], [50].

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In order to explain the main results of this paper we need some notation. Assume that $M$ is a smooth, compact Riemannian manifold and $E \to M$ is a smooth, Hermitian vector bundle. Let $\text{pr} : M \times \mathbb{R} \to M$ stand for the canonical projection and set $\mathcal{E} := \text{pr}_* E$, the pull-back vector bundle. Next, consider a second order, formally self-adjoint, strongly elliptic differential operator with smooth, real coefficients $L : C^\infty(M, E) \to C^\infty(M, E)$. Extend $L$ naturally as a mapping of $C^\infty(M \times \mathbb{R}, \mathcal{E})$ into itself (by making it independent of time), then set $P := \partial_t - L(x, \partial_x)$. Finally, assume that $\Omega \subseteq M$ is a Lipschitz domain, i.e. a domain whose boundary is given in local coordinates by graphs of Lipschitz functions, and fix $0 < T < \infty$. For $1 < p < \infty$ we consider the Dirichlet initial boundary value problem

\begin{equation}
\begin{cases}
u \in C^\infty(\Omega \times (0, T), \mathcal{E}), \\
P u = 0 \text{ in } \Omega \times (0, T), \\
u(\cdot, 0) = 0 \text{ on } \Omega, \\
u^* \in L^p(\partial \Omega \times (0, T)), \\
u|_{\partial \Omega \times (0, T)} = f \in L^p(\partial \Omega \times (0, T), \mathcal{E}).
\end{cases}
\end{equation}

(0.1)

Here $\nu^*$ stands for the (parabolic) nontangential maximal function; more precise definitions are given in the body of the paper.

Our main result states that, with the above assumptions, there exists $\epsilon = \epsilon(\Omega, L) > 0$ so that for each $2 - \epsilon < p < 2 + \epsilon$, the initial boundary problem (0.1) has a unique solution. This solution satisfies

\begin{equation}
\|\nu^*\|_{L^p(\partial \Omega \times (0, T))} \leq C \|f\|_{L^p(\partial \Omega \times (0, T), \mathcal{E})}
\end{equation}

(0.2)

for some $C = C(\partial \Omega, L, T, p) > 0$. Also, when $p = 2$,

\begin{equation}
u \in H^{1/4}((0, T), L^2(\Omega, E)) \cap L^2((0, T), H^{1/2}(\Omega, E)),
\end{equation}

(0.3)

where $H^s$ is the usual $L^2$-based scale of Sobolev spaces. Furthermore, the solution is more regular if the boundary datum is so. For a complete statement see Theorem 9.1.

Let us point out that the result sketched above encompasses many particular cases of independent interest. Such a list includes the scalar heat operator $\partial_t - \Delta$, where

\begin{equation}
\Delta u = \frac{1}{\sqrt{g}} \sum_{j,k} \frac{\partial}{\partial x_j} \left[ g^{jk} \sqrt{g} \frac{\partial u}{\partial x_k} \right]
\end{equation}

(0.4)

is the Laplace-Beltrami operator associated with the metric tensor $g = \sum g_{jk} dx_j \otimes dx_k$ on $M$ or, more generally, when

\begin{equation}
\Delta = d\delta + \delta d
\end{equation}

(0.5)

is the Hodge-Laplacian on differential forms. Here $d$ and $\delta$ are the exterior derivative operator and its adjoint, respectively. In fact, if $D : C^\infty(M, E) \to C^\infty(M, F)$ is an arbitrary first order elliptic differential operator then $L = DD^*$ will do. Many familiar second order, variable coefficient, operators arise in this fashion: Lamé type operators, symmetric differential operators, Dirac Laplacians, etc.
The strategy for proving the aforementioned result pertaining to the well-posedness of (IBVP) consists of reducing the original problem to an integral equation on the boundary of the Lipschitz cylinder $\Omega \times (0, T)$. In this scenario, as a prerequisite, one has to develop a boundary behavior theory for integral operators of the form

$$\mathbb{J}f(x, t) := \int_0^t \int_{\partial \Omega} \langle k(x, y, t, s), f(y, s) \rangle d\sigma(y) ds,$$

(0.6)

where $k$ is the Schwartz kernel of a classical (casual) parabolic pseudodifferential operator in $OPS_{cl, 2}^{-1, +}$ whose principal symbol $p_{-1}(x, \xi, \tau)$ is odd in $\xi$. Also, $d\sigma$ is the surface measure on $\partial \Omega$ and $\langle \cdot, \cdot \rangle$ refers to the Hermitian structure on the fibers of $E$.

Among the issues of interest for us here are the nontangential maximal function estimate

$$\| (\mathbb{J}f)^* \|_{L^p(\partial \Omega \times (0, T))} \leq C \| f \|_{L^p(\partial \Omega \times (0, T), E)},$$

(0.7)

and the jump-formula

$$\lim_{x \to w, x \in \gamma_+(w)} \mathbb{J}f(x, t) = \mp \frac{1}{2} i p_{-1}(w, \nu(w), 0) f(w, t)$$

$$+ \lim_{\epsilon \to 0} \int_0^{t-\epsilon} \int_{\partial \Omega} \langle k(x, y, t, s), f(y, s) \rangle d\sigma(y) ds,$$

(0.8)

valid at a.e. $(w, t) \in \partial \Omega \times (0, T)$. Here $\nu$ is the outward unit conormal to $\partial \Omega$ and $\gamma_{\pm}(w)$ are appropriate nontangential regions (in $\Omega_+ := \Omega$ and $\Omega_- := M \setminus \overline{\Omega}$, respectively).

It should be mentioned that, as far as the boundedness of the operators involved in this paper is concerned, several techniques can be adapted to the present context. For example, (0.7) can be proved via the powerful methods developed in [49], [50], [46], [47] which can actually handle even more general domains. However, given that we work in a cylinder, we prefer to present an alternative proof, more akin to the original approach in [22]. This makes the presentation somewhat more uniform and self-contained. The approach just alluded to essentially consists of working on the Fourier transform side in time and reducing matters to the elliptic theory. This also sets the stage for the proof of the jump-formula (0.8) which, once again, we treat via elliptic theory.

In the Euclidean setting, the Dirichlet problem for $P = \partial_t - L$ is typically solved by looking for a candidate $u$ in the form of a double layer potential operator. Going further, the kernel of this latter integral operator is obtained by taking the conormal derivative of $k$, a suitable fundamental solution for $P$. Now, in the invariant setting and in the absence of a global ‘product’ structure of the second-order operator $L$, one lacks a canonical choice of a conormal derivative. We overcome this problem by working with the dual of an operator which resembles the one used to solve the oblique derivative problem for $P$. This is an adaptation of an idea which goes back to A.P. Calderón ([51]) in the case of the (flat-space) scalar Laplacian. The rest of the argument in [51] involves a decomposition of the boundary integral operator into an accretive part and a symmetric, compact part. This works well in the elliptic case and yields that the operator in question is bounded from below modulo compacts; cf. [1]. In the parabolic setting, we adapt a somewhat more circuitous approach based on Rellich type estimates to reach essentially the same conclusion.
The organization of the paper is as follows. Sections §1–§2 contain a discussion of anisotropic symbols and pseudodifferential operators, respectively. Layer potential operators of parabolic type on Lipschitz cylinders are introduced and studied in §3. Among other things, here we give a proof of (0.7). The jump-formula (0.8) is proved in §4. In Section 5 we discuss similar issues for fractional time derivatives of parabolic ‘single’ layer potentials. Section 6 deals with square-function estimates, while Section 7 treats Rellich type identities and applications. In §8 we then proceed to prove invertibility results for the relevant boundary integral operators. Finally, in §9, we state and prove the Dirichlet and the Regularity initial boundary problems for second order, strongly parabolic operators of the type \( P = \partial_t - L(x, \partial_x) \) in cylinders of the form \( \Omega \times (0, T) \) where \( \Omega \) is a Lipschitz subdomain of a smooth Riemannian manifold.

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1 Anisotropic Symbols

We consider a class of symbols depending on a parameter. The basic feature is that the parameter is not treated as a lower order perturbation, but rather built into the leading symbol.

Let \( U \subseteq \mathbb{R}^m \) be open, \( k \in \mathbb{R} \) the anisotropy factor, \( \ell \in \mathbb{R} \) the order, \( N \in \mathbb{N} \) the class, and \( 0 \leq 1 - \rho \leq \delta < \rho \leq 1 \). Also, fix \( d_1, d_2 \in \mathbb{N} \). We say that \( p : U \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \text{Hom} (\mathbb{C}^{d_1}, \mathbb{C}^{d_2}) \) belongs to

\[
C^N S_{\rho, \delta, k}^{\ell}(U \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}; \text{Hom} (\mathbb{C}^{d_1}, \mathbb{C}^{d_2}))
\]

if \( p(x, t, \xi, \tau) \) is of class \( C^N \) in \((x, t)\), \( C^\infty \) in \((\xi, \tau)\) and, for each \( K \subset U \times \mathbb{R} \) compact, \( \alpha \in \mathbb{N}^m \), \( \nu \in \mathbb{N} \), \( \beta \in \mathbb{N}^m \), \( \gamma \in \mathbb{N} \) with \(|\alpha| + |\nu| \leq N\),

\[
|\partial_x^\alpha \partial_t^\nu \partial_\xi^\beta \partial_\tau^\gamma p(x, t, \xi, \tau)| \leq C_{\alpha, \beta, \gamma, \nu, K} (1 + |\xi| + |\tau|)^{1/k(\ell + \delta(|\alpha| + |\nu|) - \rho|\beta| - k|\gamma|)},
\]

uniformly for \((x, t) \in K, \xi \in \mathbb{R}^m, \tau \in \mathbb{R}\).

To the point that no ambiguities arise, we shall try to simplify the notation as much as possible. For example, we may write \( C^N S_{\rho, \delta, k}^{\ell} \) (if \( U, d_1, d_2 \) are clear from the context) and, further, \( S_{\rho, \delta, k}^{\ell} \) if \( N = \infty \).

Let us point out that

\[
C^N S_{\rho, 0, k}^{\ell}(U \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}) \subseteq C^N S_{\min \{\frac{\ell}{k}, 1\}}^{\ell/k}(U \times \mathbb{R} \times \mathbb{R}^{m+1}),
\]

where \( C^N S_{\rho, \delta}^{\ell} \) is the ordinary Hörmander’s class of symbols (with a limited amount of smoothness in \((x, t)\)). Also,

\[
p \in C^N S_{\rho, \delta, k}^{\ell} \Rightarrow p(\cdot, \cdot, \cdot, 0) \in C^N S_{\rho, \delta}^{\ell}.
\]

Call \( p \in C^N S_{\rho, \delta, k}^{\ell} \) mixed homogeneous of degree \( d \) in \((\xi, \tau)\) if
\[ p(x, t, \lambda \xi, \lambda^k \tau) = \lambda^d p(x, t, \xi, \tau), \quad \forall \lambda \geq 1. \] (1.5)

Asymptotic sums are defined in the usual way:

\[ p \sim \sum_j p_j \iff \forall M > 0, \exists \ell_M \text{ such that} \]
\[ p - \sum_{j \leq \ell} p_j \in C^N S^M_{p,\delta,k}, \quad \forall \ell \geq \ell_M. \] (1.6)

A symbol \( p \) is called \textit{classical}, and we write \( p \in C^N S_{ho,\delta,k}^\ell \), if \( p \) is independent of \( t \). It is clear that all these classes are stable undertaking asymptotic sums.

Later on, we shall need the following lemma, part of the classical Paley-Wiener Theorem.

**Lemma 1.1.** If \( p : \mathbb{R} \to \mathbb{C} \) extends to \( \mathbb{C}_\mp \) holomorphically and with polynomial growth, i.e. \( |p(z)| \leq c(1 + |z|)^N \) for some \( N \geq 0 \), \( \forall z \in \mathbb{C}_\mp \), then \( \mathcal{F}(p) \), the Fourier transform of \( p \), vanishes on \( \mathbb{R}_\pm \).

## 2 Anisotropic Pseudodifferential Operators

Here we shall introduce some classes of pseudodifferential operators associated with anisotropic symbols.

Let \( U \subseteq \mathbb{R}^m \) be open, \( d_1, d_2 \in \mathbb{N}, k, \ell \in \mathbb{R}, N \in \mathbb{N}, 0 \leq 1 - \rho \leq \delta, \rho \leq 1 \). We say that \( p(x, t, \partial_x, \partial_t) \in OPC^N S^\ell_{p,\delta,k}(U \times \mathbb{R}; \text{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2})) \) if, for any \( u \in C^\infty_{\text{comp}}(U \times \mathbb{R}) \),

\[ p(x, t, \partial_x, \partial_t) u(x, t) \]
\[ = \int_{\mathbb{R} \times \mathbb{R}^m} e^{it\tau + i(x, \xi)} p(x, t, \xi, \tau)(\mathcal{F}_t \mathcal{F}_x u)(\xi, \tau) d\xi d\tau \] (2.1)

for some symbol \( p(x, t, \xi, \tau) \in C^N S^\ell_{p,\delta,k}(U \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}; \text{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2})) \). Note that the integral in (2.1) is absolutely convergent.

**Remark 1.** It is clear that the Schwartz kernel of \( p(x, t, \partial_x, \partial_t) \) in (2.1) is, modulo a normalization constant, \( (\mathcal{F}_x \mathcal{F}_\tau p)(x, t, y - x, s - t) \).
Remark 2. For \( \ell \leq 0, k > 0 \), we have, thanks to (1.3),
\[
OPC^N S^\ell_{\rho,0,k}(U \times \mathbb{R}) \subseteq OPC^N S^{\ell/k}_{\min(\rho/k,1),0}(U \times \mathbb{R})
\]
where the latter class is that of ordinary pseudodifferential operators (with symbols in \( S^{\ell/k}_{\min(\rho/k,1),0} \) and \( C^N \) regularity in \((x,t)\)).

As in the case of anisotropic symbols, we single out several important subclasses of \( OPC^N S^\ell_{\rho,\delta,k} \). First, \( p \in OPC^N S^\ell_{\rho,\delta,k} \) is called of Volterra type, and we write \( p \in V^\pm OPC^N S^\ell_{\rho,\delta,k} \), if for each \( T \in \mathbb{R} \) it satisfies
\[
\text{supp } [p(x,t,\partial_x,\partial_t)u(x,t)] \subseteq U \times [T,\infty) \quad \text{ (or } U \times (-\infty,T], \text{ respectively)}
\]
whenever \( \text{supp } u(x,t) \subseteq U \times [T,\infty) \), (or \( U \times (-\infty,T] \), respectively).

Second, \( p \in V^\pm OPC^N S^\ell_{\rho,\delta,k} \) is called a casual pseudodifferential operator, and we write \( p \in OPC^N S^\ell_{\pm \rho,\delta,k} \), if \( p(x,t,\xi,\tau) \) is independent of \( t \) and \( p(x,t,\partial_x,\partial_t) \) commutes with translations in time.

For the remainder of the paper we shall assume that
\[
\text{all anisotropic symbols are smooth and independent of } t.
\]
(2.3)
Thus, \( N = \infty \) and we shall drop the dependence on the class in the notation for symbols and pseudodifferential operators. It is not hard to relax the smoothness assumption a bit, although the case when only a very limited amount of smoothness is assumed requires further arguments. We hope to return to this problem in a separate paper.

The classes \( OPS^\ell_{\rho,\delta,k}, V^\pm OPS^\ell_{\rho,\delta,k} \), turn out to be stable under composition. Also, taking adjoints intertwines \( V^+ OPS^\ell_{\rho,\delta,k} \) with \( V^- OPS^\ell_{\rho,\delta,k} \) and \( OPS^\ell_{\rho,\delta,k} \) with \( OPS^\ell_{\rho,\delta,k} \). Moreover, if
\[
OPS^\ell_{\rho,\delta,k} \ni p(x,\partial_x,\partial_t) \xrightarrow{\sigma_{\text{princ}}} p(x,\xi,\tau) \in \frac{S^\ell_{\rho,\delta,k}}{S^\ell-(2\rho-1)_{\rho,\delta,k}}
\]
is the principal symbol map, then
\[
\sigma_{\text{princ}}(p_1 \circ p_2) = \sigma_{\text{princ}}(p_1)\sigma_{\text{princ}}(p_2), \quad \forall p_i \in OPS^\ell_{\rho,\delta,k}, \quad i = 1, 2.
\]
(2.5)
In particular,
\[
\sigma_{\text{princ}} : \frac{OPS^\ell_{\rho,\delta,k}}{OPS^\ell-(2\rho-1)_{\rho,\delta,k}} \xrightarrow{\sim} \frac{S^\ell_{\rho,\delta,k}}{S^\ell-(2\rho-1)_{\rho,\delta,k}}
\]
is an isomorphism.

Everything extends to manifolds via partitions of unity and pull-backing. More specifically, if \( \Phi : U \xrightarrow{\sim} V \) is a smooth diffeomorphism then \( \Phi_* \), acting as composition by \( \Phi \), maps \( S^\ell_{\rho,\delta,k}(V \times \mathbb{R}^m \times \mathbb{R}) \) isomorphically onto \( S^\ell_{\rho,\delta,k}(U \times \mathbb{R}^m \times \mathbb{R}) \). Also, defining \( (\Phi_* p)(u) := [p(u \circ \Phi^{-1})] \circ \Phi \), it follows that
\[
\Phi_* : OPS^\ell_{\rho,\delta,k}(U) \xrightarrow{\sim} OPS^\ell_{\rho,\delta,k}(V)
\]
(2.7)
is an isomorphism and
\[
\sigma_{\text{princ}}(\Phi_* p)(x, \xi, \tau) = \sigma_{\text{princ}}(p)(\Phi^{-1}(x), [d\Phi^{-1}(x)]^t \xi, \tau).
\] (2.8)

In other words, the principal symbol transforms covariantly under a change of coordinates. It should be noted here that in the class of classical pseudodifferential operators $OPS_{cl,k}^\ell$, the principal symbol map is unequivocally defined by
\[
\sigma_{\text{princ}}(p) := p_\ell \circ \Phi^{-1} \quad \text{if the symbol of } \Phi_* p \text{ expands as } p_\ell + p_{\ell-1} + \ldots.
\] (2.9)

Throughout the paper $M$ is going to be a smooth, compact, boundaryless, Riemannian manifold of real dimension $m$ and $E,F \to M$ are two complex vector bundles. We let $\mathcal{E} := \text{pr}_*E$, $\mathcal{F} := \text{pr}_*F$ be the pull-backs of $E$ and $F$, respectively, under the canonical projection $\text{pr} : M \times \mathbb{R} \to M$. We can then define the class of symbols $S^\ell_{\rho,\delta,k}(M \times \mathbb{R}; \mathcal{E}, \mathcal{F})$ and operators $OPS^\ell_{\rho,\delta,k}(M \times \mathbb{R}; \mathcal{E}, \mathcal{F})$ in a natural fashion. For example, the latter is the class of linear operators
\[
p(x, \partial_x, \partial_t) : C^\infty(M \times \mathbb{R}, \mathcal{E}) \to C^\infty(M \times \mathbb{R}, \mathcal{F})
\] (2.10)
such that, when suitably localized and pulled back to an Euclidean domain $U$, they belong to $OPS^\ell_{\rho,\delta,k}(U; \text{Hom}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2}))$, where $d_1 := \text{rank } \mathcal{E}$, $d_2 := \text{rank } \mathcal{F}$. In particular,
\[
\sigma_{\text{princ}}(p)(x, \xi, \tau) \in \text{Hom}(E_x, F_x), \quad x \in M, \quad \xi \in T^*_x M, \quad \tau \in \mathbb{R}.
\] (2.11)

Volterra and casual operators are defined similarly. A simple but useful observation is contained in the proposition below.

**Proposition 2.1.** Let $p \in OPS^\ell_{\rho,\delta,k}(M \times \mathbb{R}; \mathcal{E}, \mathcal{F})$. Then $p$ is of Volterra type if and only if $\sigma_{\text{princ}}(p) \in \frac{V_S^\ell S^\ell_{\rho,\delta,k}}{V_S^\ell S^\ell_{\rho,\delta,k} - 1}$. Moreover, $p$ is casual if and only if $\sigma_{\text{princ}}(p) \in \frac{S^\ell_{\rho,\delta,k}}{S^\ell_{\rho,\delta,k} - 1}$.

We now make the standing assumption that $M$ is Riemannian and $\mathcal{E}, \mathcal{F}$ are Hermitian. A symbol $p \in OPS^\ell_{\rho,\delta,k}(M \times \mathbb{R}; \mathcal{E}, \mathcal{F})$ is called strongly parabolic if $\mathcal{E} = \mathcal{F}$ and
\[
\text{Re} \left( \langle \sigma_{\text{princ}}(p)(x, \xi, \tau) \eta, \bar{\eta} \rangle \right) \geq C(|\xi| + |\tau|^{1/k})^t |\eta|^2,
\] (2.12)
uniformly for $x \in M$, $\xi \in T^*_x M$, $\tau \in \mathbb{R}$ with $(\xi, \tau) \neq 0$ and $\eta \in E_x$, provided that $|\xi| + |\tau|^{1/k}$ is sufficiently large.

**Proposition 2.2.** Let $\Omega \subseteq M$ be a smooth domain, $-m < \ell < -1$ (recall that $m := \dim M$), $p \in OPS^\ell_{\rho,\delta,k}(M \times \mathbb{R}; \mathcal{E}, \mathcal{F})$. Denote by $d\sigma$ the surface measure on $\partial \Omega$.

For each $u \in C^\infty_{\text{comp}}(\partial \Omega \times \mathbb{R}, \mathcal{E})$, regard $u(x, t)\,d\sigma dt$ as a distribution on $M \times \mathbb{R}$ (supported on $\partial \Omega \times \mathbb{R}$). Then the section $[p(x, \partial_x, \partial_t)u\,d\sigma dt]|_{\Omega \times \mathbb{R}}$ extends to an element in $(\Omega \times \mathbb{R}, \mathcal{F})$. Moreover,
\[
(\bar{\rho}u)(\bar{x}, t) := \lim_{x \to \bar{x}} \int_{\Omega} [p(x, \partial_x, \partial_t)u\,d\sigma dt](x, t),
\] (2.13)
\[ \tilde{x} \in \partial \Omega, \ t \in \mathbb{R}, \ \text{has the property that} \ \tilde{p} \in OPS^{\ell+1, \pm}_{\rho, 0,k}(\partial \Omega \times \mathbb{R}; \mathcal{E}|_{\partial \Omega}, \mathcal{F}|_{\partial \Omega}) \ \text{and, in the sense of equivalent classes,} \]

\[
\sigma_{\text{princ}}(\tilde{p})(x, \xi, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sigma_{\text{princ}}(p)(x, \xi + t\nu, \tau) \, dt, \quad (2.14)
\]

for any \( x \in M, \ \xi \in T^* x M, \ \tau \in \mathbb{R} \). Here \( \nu \) is the outward unit conormal to \( \partial \Omega \).

Furthermore, if \( p \) is homogeneous of degree \( d \), then \( \tilde{p} \) is homogeneous of degree \( d + 1 \), and if \( p \) is strongly parabolic, then so is \( \tilde{p} \).

**Proof.** In the class of ordinary pseudodifferential operators, a similar statement is essentially well-known and the proof can be adopted without difficulty to the present anisotropic setting (cf., e.g., [8, Vol. II, p. 34], [52]) to obtain that \( \tilde{p} \in OPS^{\ell+1, \pm}_{\rho, 0,k}(\partial \Omega \times \mathbb{R}; \mathcal{E}|_{\partial \Omega}, \mathcal{F}|_{\partial \Omega}) \) and that (2.14) holds.

Now, the fact that \( \tilde{p} \) is casual follows because \( p \) is so with the aid of Proposition 2.1. Also,

\[
\sigma_{\text{princ}}(\tilde{p})(x, \lambda \xi, \lambda^k \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sigma_{\text{princ}}(p)(x, \lambda \xi + t\nu, \lambda^k \tau) \, dt = \lambda^{d+1} \sigma_{\text{princ}}(\tilde{p})(x, \xi, \tau), \quad (2.15)
\]

by a simple change of variables.

Finally, if \( p \) is strongly parabolic then, for \( |\xi| + |\tau|^{1/k} \) large,

\[
\text{Re} \langle \sigma_{\text{princ}}(\tilde{p})(x, \xi, \tau) \eta, \bar{\eta} \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Re} \langle \sigma_{\text{princ}}(p)(x, \xi + t\nu, \tau) \eta, \bar{\eta} \rangle dt \]

\[
\geq C |\eta|^2 \int_{-\infty}^{+\infty} (|\xi + t\nu| + |\tau|^{1/k}) \, dt \quad (2.16)
\]

\[
\approx C |\eta|^2 (|\xi_{\tan}| + |\tau|^{1/k})^{\ell+1}
\]

\[
\geq C |\eta|^2 (|\xi| + |\tau|^{1/k})^{\ell+1}.
\]

This finishes the proof of the proposition. \( \square \)

Next, call \( p \in OPS^{\ell, \pm}_{\rho, \delta, k}(M \times \mathbb{R}; \mathcal{E}, \mathcal{F}) \) left-parabolic if \( \sigma_{\text{princ}}(p) \) is left-invertible, i.e. there exists \( q \in S^{-(\ell, \pm)}_{\rho, \delta, k} \) such that

\[
q \circ \sigma_{\text{princ}}(p) - I \in S^{-[(2\rho-1), \pm]}_{\rho, \delta, k}. \quad (2.17)
\]

Similarly, one defines right-parabolic (casual) pseudodifferential operators and two-sided (casual) pseudodifferential operators. It follows (from definitions and (2.6)) that a strongly parabolic casual pseudodifferential operator is two-sided parabolic.

A basic result for us in the sequel is as follows.

**Theorem 2.3.** If \( p \in OPS^{\ell, \pm}_{\rho, \delta, k}(M \times \mathbb{R}; \mathcal{E}, \mathcal{F}) \) is left parabolic (right-parabolic, or two sided parabolic, respectively) then there exists \( q \in OPS^{-(\ell, \pm)}_{\rho, \delta, k}(M \times \mathbb{R}; \mathcal{F}, \mathcal{E}) \) such that \( p \circ q = I \) (or \( q \circ p = I, \) or both, respectively).
This is a version of Theorem 30 on p. 90 in [53]. In what follows, we denote \( q \) by \( p^{-1} \), if \( p \) is two-sided parabolic.

Let \( L : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E}) \) be a strongly elliptic, second order differential operator with smooth coefficients on \( M \), and set \( P(x, \partial_x, \partial_t) := \partial_t - L(x, \partial_x) \in OPS^{2, 1}_1(M \times \mathbb{R}; \mathcal{E}, \mathcal{E}). \) Then \( P \) is strongly parabolic and, hence, the inverse \( P^{-1} \in OPS^{-2, 1}_1(M \times \mathbb{R}; \mathcal{E}, \mathcal{E}) \) exists, is strongly parabolic and has a mixed homogeneity of degree \(-2\).

The single layer potential operator associated with \( P \) on \( \partial \Omega \) is defined by

\[
Sf := P^{-1}(fd\sigma dt)|_{\partial \Omega \times \mathbb{R}}, \quad f \in C^\infty_{\text{comp}}(\partial \Omega \times \mathbb{R}, \mathcal{E}).
\] (2.18)

Then, invoking Proposition 2.2 we see that \( S \in OPS^{-1, 1}_{cl, 2}(\partial \Omega \times \mathbb{R}; \mathcal{E}) \) is strongly parabolic and has a mixed homogeneity of degree \(-1\). In particular, thanks to Theorem 2.3, for each \( T > 0 \),

\[
S : C^\infty(\partial \Omega \times (0, T), \mathcal{E}) \to C^\infty(\partial \Omega \times (0, T), \mathcal{E}) \text{ is onto}
\] (2.19)

(actually invertible). This is going to be of importance for us later on.

Next we shall study mapping properties of \( S, \nabla S, D_1^{1/2} S \), etc., in the case when \( S \) is associated with a nonsmooth domain. We take up this task in the next four sections.

### 3 Boundedness Properties of Layer Potentials on Lipschitz Domains

Let \( \mathcal{E}, \mathcal{F} \to M \) be as in §2 and let \( p \in OPS^{-1, 1}_{cl, 2}(M \times \mathbb{R}; \mathcal{E}, \mathcal{F}) \) be such that \( \sigma_{\text{princ}}(p)(x, \xi, \tau) \) is odd in \( \xi \in T^*_x M \). Denote by \( k(p)(x, y, t, s) \) the Schwartz kernel of \( p \) and, for each \( \epsilon > 0 \), set

\[
J_\epsilon f(x, t) := \int_0^t \int_{\partial \Omega} \langle k(p)(x, y, t, s), f(y, s) \rangle_y d\sigma(y) ds,
\] (3.1)

where \( \Omega \subseteq M \) is a Lipschitz domain and \( d\sigma \) is the surface area measure on \( \partial \Omega \). Also, set

\[
J_\ast f(x, t) := \sup_{\epsilon > 0} |J_\epsilon f(x, t)|, \quad (x, t) \in \partial \Omega \times \mathbb{R}_+,
\] (3.2)

and, for \( x \not\in \partial \Omega, t > 0 \),

\[
\mathbb{J} f(x, t) := \int_0^t \int_{\partial \Omega} \langle k(p)(x, y, t, s), f(y, s) \rangle_y d\sigma(y) ds.
\] (3.3)

To state the main result of this section we need some more notation. Set \( \Omega_+ := \Omega, \Omega_- := M \setminus \bar{\Omega} \) and, at each \( x \in \partial \Omega \), consider appropriate nontangential approach regions \( \gamma_{\pm}(x) \subseteq \Omega_{\pm}; \) cf. [30], [2] for more on this. Then, \( \cdot|_{\partial \Omega_{\pm} \times \mathbb{R}} \) will stand for the nontangential boundary trace operators on \( \partial \Omega_{\pm} \times \mathbb{R} \). That is, if \( u : \Omega_{\pm} \times \mathbb{R} \to \mathcal{E} \), then
whenever this exists. Also, the (parabolic) nontangential maximal function $(\cdot)^*$ is defined for sections $u : \Omega_\pm \times \mathbb{R} \to E$ by

$$u^*(x,t) := \sup \{|u(y,t)| ; y \in \gamma_\pm(x)\} \quad x \in \partial \Omega.$$  

**Proposition 3.1.** Let $\Omega \subset M$ be a Lipschitz domain, and fix $0 < T < \infty$, $1 < p < \infty$. Then the following hold.

(i) There exists $C = C(\partial \Omega, T, p) > 0$ such that

$$\|J_* f\|_{L^p(\partial \Omega \times (0,T))} \leq C\|f\|_{L^p(\partial \Omega \times (0,T), E)}.$$  

(ii) For any $f \in L^p(\partial \Omega \times (0,T), E)$ the limit

$$Jf(x,t) := \lim_{\epsilon \to 0} J_\epsilon f(x,t)$$  

exists at almost every $(x,t) \in \partial \Omega \times (0,T)$ and defines a bounded operator

$$J : L^p(\partial \Omega \times (0,T), E) \to L^p(\partial \Omega \times (0,T), F).$$

(iii) There exists $C = C(\partial \Omega, T, p) > 0$ so that

$$\|(Jf)^*\|_{L^p(\partial \Omega \times (0,T))} \leq C\|f\|_{L^p(\partial \Omega \times (0,T), E)}.$$  

The strategy for proving (i) and (ii) is as follows:

**Step 1.** In local coordinates, the total symbol $p(x,\xi,\tau)$ can be decomposed as $p = p_{-1} + p_{-2}$ where $p_{-1}$ is the principal symbol and $p_{-2} \in S^{1,0,2}_{-1}$ is regarded as residual. Accordingly, we have the splitting $k(p) = k_{-1} + k_{-2}$ for the kernel. Now, direct size estimates for $k_{-2}$ give the right bounds for the contribution coming from this part of the kernel, as long as $T > 0$ is finite and $1 < p < \infty$.

**Step 2.** To handle the contribution from $k_{-1}$ we first consider the $L^2$ context. The point is that we can work on the Fourier transform side (assuming $T = \infty$); Plancherel’s theorem allows are to eventually return to the original setting.

In the process, a substantial piece of the operator (containing the delicate cancellations) can be handled via the elliptic theory from [1], where as the remainder can be controlled in terms of maximal functions.

**Step 3.** Passing to the general case $1 < p < \infty$ is then done more or less automatically based as the $L^2$ analysis in Step 2 and Calderón-Zygmund theory in the context of spaces of homogeneous type. For this latter segment we need to check a cancelation condition for the kernel of Hörmander type.
Finally, a Cotlar type inequality (relating $J_*$ and $J$) allows one to transfer the $L^p$-boundedness of $J$ to $J_*$. The case of $\mathbb{J}$ is handled similarly.

**Details of Step 1.** First, it is standard that, locally,

$$k(p)(x, y, t, s) = c_m \int\int p(x, \xi, \tau) e^{i(x-y, \xi)} e^{i(t-s)\tau} \, d\xi \, d\tau,$$

(3.10)

where the integral is interpreted in the oscillatory sense. Estimates for the right side of (3.10) can be derived using the following lemma.

**Lemma 3.2.** Let $p \in S^{\ell}_{1,0,k}$ with $\ell \in \mathbb{R}$ and $k \geq 1$. Then

$$\left| \int\int \xi^\alpha \tau^\beta \partial_x^\gamma p(x, \xi, \tau) e^{i(z, \xi)} e^{it\tau} \, d\xi \, d\tau \right| \leq C_{\alpha, \beta, \gamma} [|z| + |t|^{1/k} - (m + \ell + k + |\alpha| + k|\beta|)]$$

(3.11)

uniformly for $x$ in compacts.

In particular, in the context of Step 1, Lemma 3.2 gives

$$|k_-(x, y, t, s)| \leq C \min \left\{ \frac{1}{|t-s|m/2}, \frac{1}{|x-y|^m} \right\}.$$  

(3.12)

Since the right-side of (3.12) is absolutely integrable on $\partial \Omega \times [0, T]$ uniformly in $(x, t) \in \partial \Omega \times [0, T]$, Schur’s lemma shows that the integral operator with kernel $k_-$ is bounded on $L^p(\partial \Omega \times (0, T))$, as desired.

We now turn to the

**Proof of Lemma 3.2.** Let the double hat $\hat{\cdot}$ stand for $\mathcal{F}_\xi \mathcal{F}_\tau$. With this piece of notation, (3.11) reads

$$\left| \partial_x^\gamma \partial_z^\alpha \partial_t^\beta \hat{p}(x, z, t) \right| \leq C_{\alpha, \beta, \gamma} [|z| + |t|^{1/k} - (m + \ell + k + |\alpha| + k|\beta|)].$$  

(3.13)

Next, observe that $x$ can be treated as a parameter and, hence, it suffices to treat the case when $\gamma = 0$ and $p$ is independent of $x$, which we shall assume in the sequel.

In order to estimate $\partial_x^\gamma \partial_z^\alpha \partial_t^\beta \hat{p}$ we make a parabolic Littlewood-Paley decomposition. Fix $\eta \in C^\infty_{\text{comp}}(\mathbb{R}^m \times \mathbb{R})$ such that $\eta(\xi, \tau) = 1$ for $|\xi| + |\tau|^{1/k} \leq 1$ and $\eta \equiv 0$ for $|\xi| + |\tau|^{1/k} \geq 2$. Also, set

$$\delta(\xi, \tau) := \eta(\xi, \tau) - \eta(2\xi, 2^k \tau),$$

(3.14)

i.e. $\delta := \eta - \eta \circ D$ where $D$ is an “anisotropic” dilation operator. It follows that $\delta$ is supported where $\frac{1}{2} \leq |\xi| + |\tau|^{1/k} \leq 2$ and

$$1 = \eta(\xi, \tau) + \sum_{j=1}^{\infty} \delta(2^{-j} \xi, 2^{-kj} \tau).$$

(3.15)

Accordingly, write
\[ p(\xi, \tau) = p_0(\xi, \tau) + \sum_{j=1}^{\infty} p_j(\xi, \tau), \]  

where

\[ p_0(\xi, \tau) := p(\xi, \tau) \eta(\xi, \tau) \quad \text{and} \quad p_j(\xi, \tau) := p(\xi, \tau) \delta(2^{-j} \xi, 2^{-kj} \tau). \]  

Next, define

\[ q_j(\xi, \tau) := p(2^j \xi, 2^{kj} \tau) \delta(\xi, \tau), \quad j \geq 1, \]  

so that

\[ p_j(\xi, \tau) = q_j(2^{-j} \xi, 2^{-kj} \tau), \]  

and, consequently,

\[ p(\xi, \tau) = p_0(\xi, \tau) + \sum_{j=1}^{\infty} q_j(2^{-j} \xi, 2^{-kj} \tau). \]  

Now we make a general claim to the effect that

if \( p \in S_1^{\ell, 0, k} \) and \( p_r(\xi, \tau) := r^{-\ell} p(r \xi, r^k \tau) \) then

\[ \{p_r\}_{r \geq 1} \]  

is bounded in \( C^\infty(2^{-1} \leq |\xi| + |\tau|^{1/k} \leq 2). \)  

To see this, simply note that

\[ |\partial_\alpha^\ell \partial_\beta^\beta p_r(\xi, \tau)| = r^{\ell + |\alpha| + k|\beta|} |(\partial_\alpha^\ell \partial_\beta^\beta p)(r \xi, r^k \tau)| \]
\[ \leq C_{\alpha, \beta} r^{-\ell + |\alpha| + k|\beta|} (1 + r|\xi| + r|\tau|^{1/k})^{\ell - |\alpha| - k|\beta|} \]
\[ \approx C_{\alpha, \beta}, \]  

uniformly for \( r \geq 1, \) as long as \( 2^{-1} \leq |\xi| + |\tau|^{1/k} \leq 2. \) In particular, (3.20) gives that

\[ \{2^{-j\ell} q_j(\xi, \tau)\}_{j \geq 1} \]  

is bounded in \( S(\mathbb{R}^m \times \mathbb{R}), \)  

since \( \delta(\xi, \tau) \) has the effect of a cut off function in (3.18). In turn, (3.22) implies that for all \( \alpha, \beta \) and \( N > 0, \)

\[ |\partial_\alpha^\ell \partial_\beta^\beta q_j(z, t)| \leq C_{N, \alpha, \beta} 2^{j\ell}(1 + |z| + |t|^{1/k})^{-N}, \]  

since \( 1 + |z| + |t|^{1/k} \leq C(1 + |z| + |t|). \) Now, from (3.19),

\[ \hat{p}(z, t) = \hat{p}_0(z, t) + \sum_{j \geq 1} 2^{jm} \hat{q}_j(z, 2^j t), \]  

so that

\[ \partial_\alpha^\ell \partial_\beta^\beta \hat{p}(z, t) = \partial_\alpha^\ell \partial_\beta^\beta \hat{p}_0(z, t) \]
\[ + \sum_{j \geq 1} 2^{jm+jk+|\alpha|+jk|\beta|} (\partial_\alpha^\ell \partial_\beta^\beta \hat{q}_j)(2^j z, 2^j t). \]  

\[ (3.25) \]
Introduce \( \theta := \log_2(|z| + |t|^{1/k}) \) and \( L := m + k + \ell + |\alpha| + k|\beta| \). Then, by (3.23), the sum in (3.25) is bounded by

\[
C_{N,\alpha,\beta} \sum_{j \geq 1} 2^{jL} (1 + 2^{\theta+j})^{-N} = C_{N,\alpha,\beta} \sum_{j' \geq \theta+1} 2^{(j' - \theta)L} (1 + 2^j)^{-N} \leq C_{L,\alpha,\beta} 2^{-\theta L} \sum_{j' \geq 1} 2^{j'L} (1 + 2^{j'})^{-N} \leq C_{N,\alpha,\beta} 2^{\theta L} \left[ |z| + |t|^{1/k} \right]^{-1 - (m+k+\ell+|\alpha|+k|\beta|)},
\]

where the last inequality follows by choosing \( N \) sufficiently large. Clearly, this bound is of the right order. Finally, since \( p_0 \in C^\infty_{\text{comp}}(\mathbb{R}^m \times \mathbb{R}) \) it follows that \( \hat{p}(z, t) \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}) \) and (3.11) is proved.

This completes Step 1 and now we provide the

**Details of Step 2.** Consider \( p \in S_{-1}^{-1} \) such that \( p \) is odd in \( \xi \) and homogeneous of degree \(-1\) in \( (\xi, \tau) \). From (3.10) we know that locally

\[
k(p)(x, y, t, s) = c_m \hat{p}(x, y - x, s - t),
\]

where, recall that the double hat \( \hat{\hat{\cdot}} \) stands for the iterated Fourier transform \( \mathcal{F}_\xi \mathcal{F}_\tau \). Thus,

\[
J_\epsilon f(x, t) = c_m \int_{\partial \Omega} \int_{\partial \Omega} \left\langle \hat{\hat{p}}(x, y - x, s - t) \chi_{(-\infty, -\epsilon)}(x - t), f(y, s) \right\rangle d\sigma(y) ds.
\]

Since \( J_\epsilon \) is of convolution type in time, denoting \( \hat{\hat{\cdot}} := \mathcal{F}_t \), we have

\[
\hat{J}_\epsilon f(x, \tau) = c_m \int_{\partial \Omega} \left\langle \int_{\epsilon}^{\infty} \hat{\hat{p}}(x, y - x, s) e^{-ist} ds, \hat{f}(y, \tau) \right\rangle d\sigma(y)
=: I + II + III + IV + V,
\]

where
\[ I := c_m \int \frac{1}{|x - y|} \geq |x - y| \geq \sqrt{\epsilon} \left( \int_0^{\infty} \hat{p}(x, y - x, s) \, ds \right) \, \hat{f}(y, \tau) \, d\sigma(y) \]

\[ II := c_m \int \frac{1}{|x - y|} \leq \sqrt{\epsilon} \left( \int_0^{\infty} \hat{p}(x, y - x, s) e^{-ist} \, ds \right) \, \hat{f}(y, \tau) \, d\sigma(y) \]

\[ III := -c_m \int \frac{1}{|x - y|} \geq \sqrt{\epsilon} \left( \int_0^{\epsilon} \hat{p}(x, y - x, s) e^{-ist} \, ds \right) \, \hat{f}(y, \tau) \, d\sigma(y) \]

\[ IV := c_m \int \frac{1}{|x - y|} \geq \max\left(\frac{1}{\sqrt{\epsilon}}, \frac{1}{|x - y|}\right) \left( \int_0^{\infty} \hat{p}(x, y - x, s) e^{-ist} \, ds \right) \, \hat{f}(y, \tau) \, d\sigma(y) \]

\[ V := c_m \int \frac{1}{|x - y|} \geq \sqrt{\epsilon} \left( \int_0^{\epsilon} \hat{p}(x, y - x, s) (e^{-ist} - 1) \, ds \right) \, \hat{f}(y, \tau) \, d\sigma(y). \]

The claim is that

\[ \left\| \sup_{|\tau| > \epsilon > 0} |I| \right\|_{L^2(\partial \Omega)} \leq C \left\| \hat{f}(\cdot, \tau) \right\|_{L^2(\partial \Omega)}, \] (3.31)

uniformly for \( \tau \in \mathbb{R} \),

and

\[ |II| + |III| + |IV| \leq C(M_{\partial \Omega} \hat{f}(\cdot, \tau))(x), \] (3.32)

uniformly for \( x \in \partial \Omega \) and \( \tau \in \mathbb{R} \),

where \( M_{\partial \Omega} \) stands for the Hardy-Littlewood maximal function on \( \partial \Omega \). Of course, granted (3.31)–(3.32), the conclusion in Step 2 follows from the boundedness of \( M_{\partial \Omega} \) on \( L^2(\partial \Omega) \) and Plancherel’s Theorem.

We now check (3.31)–(3.32) starting with the first. To this end, let

\[ b(x, z) := \int_0^{\infty} \hat{p}(x, z, s) \, ds. \] (3.33)

The fact that \( p(x, \xi, \tau) \) is odd in \( \xi \) entails that \( \hat{p}(x, z, s) \) is odd in \( z \) and, further, that \( b(x, z) \) is odd in \( z \). Also, since \( p \) is mixed homogeneous of degree \(-1\), a simple calculation shows that

\[ b(x, \lambda z) = \lambda^{m-1} b(x, z), \] (3.34)

i.e. \( b \) is homogeneous of degree \((m - 1)\) in \( z \). Hence, (3.31) follows from the elliptic theory in [2], [1].

In order to check (3.32), we shall use the estimates
\[
\begin{align*}
|\hat{p}(x, y - x, s)| & \leq C(|x - y| + s^{1/2})^{-(m+1)}, \\
|\partial_s \hat{p}(x, y - x, s)| & \leq C(|x - y| + s^{1/2})^{-(m+3)}.
\end{align*}
\]

They both follow from Lemma 3.2 since \( p \in S_{1,0,2}^{-1} \). Based on these, we may write

\[
|II| \leq C \int_{\{y : |x - y| \leq \sqrt{\epsilon}\}} \left( \int_{y \in \partial \Omega} \frac{1}{s^{1/2}(m+1)} ds \right) \hat{f}(y, \tau) |d\sigma(y)| \\
\leq C \frac{1}{(\sqrt{\epsilon})^{m-1}} \int_{\{y : |x - y| \leq \sqrt{\epsilon}\}} \hat{f}(y, \tau) |d\sigma(y)| \leq C(M_{\partial \Omega} \hat{f}(\cdot, \tau))(x). \tag{3.36}
\]

Also,

\[
|III| \leq C \sum_{k=0}^{\infty} \left( \int_{2^k \sqrt{\epsilon} \leq |x - y| \leq 2^{k+1} \sqrt{\epsilon}} \int_{0}^{\sqrt{\epsilon}} \frac{ds}{|x - y|^{m+1}} \hat{f}(y, \tau) |d\sigma(y)| \right) \\
\leq C \sum_{k=0}^{\infty} 2^{-2k+m-1} \frac{1}{(2^{k+1} \sqrt{\epsilon})^{m-1}} \int_{|x - y| \leq 2^{k+1} \sqrt{\epsilon}} \hat{f}(y, \tau) |d\sigma(y)| \\
\leq C \left( \sum_{k=0}^{\infty} 2^{-2k+m-1} \right) (M_{\partial \Omega} \hat{f}(\cdot, \tau))(x) \\
\leq C (M_{\partial \Omega} \hat{f}(\cdot, \tau))(x). \tag{3.37}
\]

To estimate \(|IV|\), introduce

\[
F_{x,y}(\tau) := \int_{0}^{\infty} \hat{p}(x, y - x, s) e^{-i \tau s} ds. \tag{3.38}
\]

The decay of \( F_{x,y}(\tau) \) as \(|\tau| \to \infty\) can be justified via the classical Lebesgue-Riemann lemma. Specifically,

\[
\tau F_{x,y}(\tau) = -i \hat{p}(x, y - x, 0) + \int_{0}^{\infty} \partial_s \hat{p}(x, y - x, s) e^{-i \tau s} ds, \tag{3.39}
\]

via an integration by parts and the fact that \( \lim_{s \to \infty} \hat{p}(x, y - x, s) = 0 \); cf. (3.35). Then, it follows from (3.39), (3.35) and some algebra that

\[
|F_{x,y}(\tau)| \leq C |\tau|^{-1} |x - y|^{-(m+1)}. \tag{3.40}
\]

Consequently,
\[|IV| \leq C \sum_{k=0}^{\infty} \int_{|x-y| \leq \frac{|\tau|}{\sqrt{|\tau|}}} \int_{|x-y| \leq \frac{2k+1}{\sqrt{|\tau|}}} |F_{x,y}(\tau)||\hat{f}(y, \tau)| \, d\sigma(y) \]
\[ \leq C \sum_{k=0}^{\infty} \int_{|x-y| \leq \frac{|\tau|}{\sqrt{|\tau|}}} \left( \frac{2^k}{\sqrt{|\tau|}} \right)^{-(m+1)} |\hat{f}(y, \tau)| \, d\sigma(y) \]
\[ \leq C \left[ \sum_{k=0}^{\infty} \frac{1}{|\tau|} \left( \frac{2^k}{\sqrt{|\tau|}} \right)^{-(m+1)} \left( \frac{2^k}{\sqrt{|\tau|}} \right)^{m-1} \right] (M_{\partial\Omega}|\hat{f}(\cdot, \tau)|)(x) \]
\[ \leq C(M_{\partial\Omega}|\hat{f}(\cdot, \tau)|)(x). \tag{3.41} \]

Finally, to estimate \(|V|\), we use \(|e^{ix\tau} - 1| \leq C|\tau|\) and (3.35) to write

\[|V| \leq C \int_{|x-y| \leq \frac{|\tau|}{\sqrt{|\tau|}}} |\tau| \left( \int_{0}^{\infty} \frac{s}{(|x-y| + s^{1/2})^{m+1}} ds \right) |\hat{f}(y, \tau)| \, d\sigma(y) \]
\[ \approx C \int_{|x-y| \leq \frac{|\tau|}{\sqrt{|\tau|}}} \left( \frac{|\tau|}{|x-y|^{m-3}} \right) |\hat{f}(y, \tau)| \, d\sigma(y) \]
\[ \leq C \sum_{k=0}^{\infty} \int_{\frac{2^{k-1}}{\sqrt{|\tau|}} \leq |x-y| \leq \frac{2^{k}}{\sqrt{|\tau|}}} \frac{|\tau|}{|x-y|^{m-3}} |\hat{f}(y, \tau)| \, d\sigma(y) \]
\[ \leq C \left[ \sum_{k=0}^{\infty} |\tau| \left( \frac{2^{-k}}{\sqrt{|\tau|}} \right)^{-(m-3)} \left( \frac{2^{-k}}{\sqrt{|\tau|}} \right)^{m-1} \right] (M_{\partial\Omega}|\hat{f}(\cdot, \tau)|)(x) \]
\[ \leq C(M_{\partial\Omega}|\hat{f}(\cdot, \tau)|)(x). \tag{3.42} \]

This concludes the detailed presentation of Step 2. \(\square\)

Next we give the

**Details of Step 3.** First we concentrate on the Hörmander type estimate

\[ \iint_{\partial\Omega \times \mathbb{R} \setminus [S_{2r}(y) \times (s-4r^{2}, s+4r^{2})]} |\hat{\rho}(x, y - x, s - t) - \hat{\rho}(x, y' - x, s' - t)| \, d\sigma(x) \, dt \leq C, \tag{3.43} \]

which, so we claim, is valid whenever

\[ |y' - y| + |s' - s|^{1/2} \leq r, \tag{3.44} \]

uniformly for \(r > 0, x, y, y' \in \partial\Omega, s, s', t \in \mathbb{R}\). Hereafter, \(S_{r}(y) \subseteq \partial\Omega, y \in \partial\Omega, r > 0\), will denote the surface ball of radius \(r\) centered at \(y\).

The domain of integration in (3.43) is covered by \(D_{1} \cup D_{2}\) where
\[ D_1 := S_{2r}(y) \times [\mathbb{R} \setminus (s - 4r^2, s + 4r^2)], \]
\[ D_2 := [\partial \Omega \setminus S_{2r}(y)] \times \mathbb{R}. \]

In \( D_1 \) we shall simply use size estimates for \( \hat{p} \), whereas in \( D_2 \) we shall apply the Mean-Value Theorem. More concretely, granted (3.44) it is easy to check that
\[ |s' - t| \geq c |s - t|, \quad \text{uniformly for } (x, t) \in D_1. \]  

(3.46)

This and (3.35) then give
\[ |\hat{p}(x, y - x, s - t)|, |\hat{p}(x, y' - x, s' - t)| \leq \frac{C}{|s - t|^{\frac{m+1}{2}}} \]
on the domains of integration, provided (3.44) holds. Thus,
\[
\int \int_{D_1} |\hat{p}(x, y - x, s - t) - \hat{p}(x, y' - x, s' - t)| \, d\sigma(x) \, dt \\
\leq C \int \int_{D_1} \frac{d\sigma(x) \, dt}{|s - t|^{\frac{m+1}{2}}} \leq C r^{m-1} \int_4^\infty \frac{dt}{t^{\frac{m+1}{2}}} = C_m, 
\]

(3.48)

as desired. There remains the contribution from integrating over \( D_2 \). First, by the Mean-Value Theorem and (3.13),
\[
|\hat{p}(x, y - x, s - t) - \hat{p}(x, y' - x, s' - t)| \\
\leq |\hat{p}(x, y - x, s - t) - \hat{p}(x, y - x, s' - t)| \\
+ |\hat{p}(x, y - x, s' - t) - \hat{p}(x, y' - x, s' - t)| \\
\leq |s - s'| \sup_{\theta \in [s, s']} |\partial_3 \hat{p}(x, y - x, \theta - t)| \\
+ |y - y'| \sup_{z \in [y, y']} |\partial_2 \hat{p}(x, z - y, s' - t)| \\
\leq C \sup_{\theta \in [s, s']} \left( |x - y| + |\theta - t|^{\frac{1}{2}} \right)^{m+3} \\
+ C r \sup_{z \in [y, y']} \left( |x - z| + |s' - t|^{\frac{1}{2}} \right)^{m+2}, 
\]

(3.49)

where (3.44) has also been used in the last step.

Going further, note that,
\[
\int_{\partial \Omega \setminus S_2(y)} \left( \int_{-\infty}^{+\infty} \frac{r^2}{||x - y| + |\theta - t|^{\frac{1}{2}}|^{m+3}} dt \right) d\sigma(x)
\leq Cr^2 \int_{x \in \mathbb{R}^{m-1}} \left( \int_{-\infty}^{+\infty} \frac{dt}{||x| + |t|^{\frac{1}{2}}|^{m+3}} \right) dx
\]

which has the right form. Also, granted (3.44), the pointwise existence of \( \lim_{\epsilon \to 0} \) can be dualized, so that \( J[2], [1] \) for a more detailed discussion in the elliptic case). The bottom line is that our result

symbols are amenable to virtually an identical treatment (we refer the interested reader to

adjoints of operators like bounded from \( L^p \) on [54, Theorem 3, p. 19] that

homogeneous type (equipped with the quasi-distance arguments. Here we only outline the main steps. First, regarding

This finishes the proof of (3.43).

Given what we have proved so far, all the conclusions in Proposition 3.1 follow by routine arguments. Here we only outline the main steps. First, regarding \( \partial \Omega \times \mathbb{R} \) as a space of homogeneous type (equipped with the quasi-distance \( |x - y| + |t - s|^{\frac{1}{2}} \)) one concludes, based on [54, Theorem 3, p. 19] that \( J_{-1} \), the integral operator corresponding to the kernel \( k_{-1} \), is bounded from \( L^p(\partial \Omega \times \mathbb{R}, E) \) into \( L^p(\partial \Omega \times \mathbb{R}, F) \) for each \( 1 < p \leq 2 \). The class of formal adjoints of operators like \( J_{-1} \) can also be handled along similar lines since the corresponding symbols are amenable to virtually an identical treatment (we refer the interested reader to [2], [1] for a more detailed discussion in the elliptic case). The bottom line is that our result can be dualized, so that \( J_{-1} \) turns out to be bounded on \( L^p \) for \( 2 \leq p < \infty \) also.

Now, Cotlar’s inequality (cf. [34, Theorem 4.5, p. 70], for a relevant discussion) yields that \( J_{-} : L^p(\partial \Omega \times \mathbb{R}, E) \to L^p(\partial \Omega \times \mathbb{R}) \) is bounded for each \( 1 < p < \infty \).

Finally, the boundedness of this maximal operator together with the existence of the a.e. pointwise limit for a dense subclass of \( L^p(\partial \Omega \times \mathbb{R}, E) \) (e.g., \( C^\infty_{\text{comp}}(\partial \Omega \times \mathbb{R}, E) \) will do) entails the pointwise existence of \( \lim_{\epsilon \to 0} J_{-} f \) a.e. on \( \partial \Omega \times \mathbb{R} \), for each \( f \in L^p(\partial \Omega \times \mathbb{R}, E) \), \( 1 < p < \infty \).

This proves (i) – (ii) in Proposition 3.1.

We now tackle the

Proof of (iii) in Proposition 3.1. Again, decompose the total symbol \( p = p_{-1} + p_{-2} \) with \( p_{-1} \in S^{-1}_{1,0,2}, \ p_{-2} \in S^{2}_{1,0,2} \), so that \( k(p) = k_{-1} + k_{-2} \) and \( J = J_{-1} + J_{-2} \).
Next, for each $0 < T < \infty$, $1 < p < \infty$, the estimate

$$\| (\mathcal{J}_x f)^* \|_{L^p(\partial\Omega \times (0,T))} \leq C(\partial\Omega, T, p) \| f \|_{L^p(\partial\Omega \times (0,T), \mathcal{E})}$$

(3.52)
can be established directly, by reducing matters to the analysis of a convolution type operator with an absolutely integrable kernel. There remains the contribution from $\mathcal{J}_{-1}$. Below we assume that $p \in S_{1,0,2}^1$ drop the index $-1$. Fix $1 < p < \infty$, $f \in L^p(\partial\Omega \times \mathbb{R}, \mathcal{E})$, $x_0 \in \partial\Omega$, $x \in \gamma(x_0)$ (the nontangential approach region with “vertex” at $x_0$), $\epsilon := |x - x_0|$ and decompose

$$\mathcal{J} f(x,t) = (-J_{2x} f(x_0, t) + \mathcal{J} f(x, t)) + J_{2x} f(x_0, t).$$

(3.53)

Now $|J_{2x} f(x_0, t)| \leq J_x f(x_0, t)$ and, thus, by $(i)$ in Proposition 3.1, the contribution from this term has the appropriate control. As for the first term in the right side of (3.53), we first estimate the contribution of the piece of $\mathcal{J} f(x, t)$ corresponding to integrating near $x_0$. More concretely, consider

$$\int_{-\infty}^{t} \int_{|y-x|=\epsilon} |k(p)(x,y,t,s)||f(y,s)|\,d\sigma(y)\,ds.$$ (3.54)

Note that $|x-y| \geq C \text{dist}(x, \partial\Omega) \approx |x-x_0| = \epsilon$. Thus, since $|k(p)(x,y,t,s)| \leq C(|x-y| + |t-s|^\frac{1}{2})^{-(m+1)}$, it suffices to bound

$$\int_{y \in \mathbb{R}^m-1} \int_{-\infty}^{+\infty} \chi_{B_2(0)} \left( \frac{x_0-y}{\epsilon} \right) \times \chi(0,\infty) \left( \frac{t-s}{\epsilon^2} \right) \epsilon^{-(m+1)} \left( 1 \right) \left. \frac{|(t-s)^\frac{1}{2}}{|(t-s)|^{(m+1)}} \right)^{-(m+1)} |f(y,s)| \, dyds.$$ (3.55)

To this end, we shall employ the following lemma

**Lemma 3.3.** Let $\Phi \in L^1(\mathbb{R}^{m-1} \times \mathbb{R})$ be so that

$$\overline{\Phi}(x,t) := \sup \{|\Phi(\tilde{x},\tilde{t})|; \tilde{x} \in \mathbb{R}^{m-1}, \tilde{t} \in \mathbb{R}, |\tilde{x}| \geq |x|, |\tilde{t}| \geq |t|\}$$
is integrable on $\mathbb{R}^{m-1} \times \mathbb{R}$. Also, introduce the maximal function

$$\mathcal{M} f(x,t) := \sup_{r>0, \mu>0} \frac{1}{\mu} \int_{\mathbb{R}} \int_{|x-y| \leq r} \frac{1}{\mu} \int_{|t-s| \leq \mu} |f(y,s)| \, dyds.$$ (3.56)

Then there exists $C > 0$ so that, for each $x, t$,

$$\sup_{r>0, \mu>0} \left| \frac{1}{\mu} \int_{\mathbb{R}} \int_{\mathbb{R}^{m-1}} \Phi \left( \frac{x-y}{r}, \frac{t-s}{\mu} \right) f(y,s) \, dyds \right| \leq C \| \overline{\Phi} \|_{L^1} \mathcal{M} f(x,t).$$ (3.57)
The proof of lemma is an exercise; cf. [55, Theorem 2, p. 62]. For a related version see also [34, Lemma 4.4, p. 70].

Returning to the analysis of (3.55) and taking
\[ \Phi(x,s) := \chi_{B_2(0)}(x) \chi_s(1 + |s|^\frac{1}{2} - (m+1)) \] and \( r := \epsilon, \mu := \epsilon^2 \).

Lemma 3.3 gives that the integral in (3.55) is \( \leq C M f(x_0, t) \), uniformly in \( \epsilon \). Again, since \( \mathcal{M} \) is bounded on \( L^p(\mathbb{R}^{m-1} \times \mathbb{R}) \), this leads to a bound of the right order. At this point, we are left with estimating the contribution of
\[ \int_0^t \int_{|y-x_0| \geq 2\epsilon} |k(p)(x,y,t,s) - k(p)(x_0,y,t,s)| |f(y,s)| d\sigma(y) ds. \tag{3.58} \]

Now
\[ |k(p)(x,y,t,s) - k(p)(x_0,y,t,s)| \leq |\hat{p}(x,y-x,s-t) - \hat{p}(x_0,y-s,s-t)| + |\hat{p}(x_0,y-x,s-t) - \hat{p}(x_0,y-x_0,s-t)| \tag{3.59} \]
\[ =: A + B. \]

We regard part \( A \) as residual (here we use \( 0 < t < T \)). Note that
\[ |A| \leq C |x - x_0| \sup_{w \in [x,x_0]} |\partial_1 \hat{p}(w,y-x,s-t)| \]
\[ \leq C |x - x_0| \frac{1}{(|x-y| + |s+t|^\frac{1}{2})^{m+1}}, \tag{3.60} \]
by (3.13). Observe that on the domain of integration in (3.58), we have \( |x - y| \approx |y - x_0| \), due to the definition of \( \epsilon \). Also, \( |x - x_0| = \epsilon \leq \frac{1}{2} |y - x_0| \). Thus,
\[ \int_0^t \int_{|y-x_0| \geq 2\epsilon} |A| |f(y,s)| d\sigma(y) ds \]
\[ \leq \int_0^T \int_{\partial\Omega} \frac{|y-x_0|}{(|y-x_0| - |s-t|^\frac{1}{2})^{m+1}} |f(y,s)| d\sigma(y) ds \tag{3.61} \]

since the last double integral above is a convolution type operator with an absolutely integrable kernel (evaluated at \( (x_0, t) \)), it follows that the contribution from \( A \) in \( \| (\mathcal{J}f)^* \|_{L^p(\partial\Omega \times (0,T))} \) is \( \leq C \| f \|_{L^p(\partial\Omega \times (0,T), \xi)} \).

As for the contribution from \( B \), first note that, if \( |y - x_0| \geq 2\epsilon \), then
\[ |B| \leq C |x - x_0| \sup_{w \in [x,x_0]} |\partial_2 \hat{p}(x_0,y-w,s-t)| \]
\[ \leq C \frac{\epsilon}{(|y-x_0| + |t-s|^\frac{1}{2})^{m+2}}, \tag{3.62} \]
where we have used the fact that $|x - x_0| = \epsilon$ and $|w - y| \approx |x_0 - y|$ uniformly for $w \in [x, x_0]$ and $|y - x_0| \geq 2\epsilon$. Thus,

$$
\int_{-\infty}^{t} \int_{|y-x_0| \geq 2\epsilon} |B||f(y, s)| d\sigma(y)ds \\
\leq C \int_{y \in \mathbb{R}^{m-1}} \int_{s \in \mathbb{R}} \frac{1}{\epsilon^{m-1}} \frac{1}{\epsilon^2} \chi_{\mathbb{R}^{m-1} \setminus B_2(0)}(y-x_0) \chi(0, \infty) \left(\frac{t-s}{\epsilon^2}\right) \times \\
\times \frac{1}{\left(|y-x_0| + \left|\frac{t-s}{\epsilon^2}\right|^{\frac{1}{2}}\right)^{m+2}} |f(y, s)| dyds \\
\leq C \left( \int_{\mathbb{R}^{m-1}} \chi_{\mathbb{R}^{m-1} \setminus B_2(0)}(y) \chi(0, \infty)(s) \frac{ds dy}{\left(|y| + |s|^{\frac{1}{2}}\right)^{m+2}} \right) \mathcal{M} f(x_0, t) \\
\leq C M f(x_0, t),
$$

(3.63)

where the second inequality utilizes Lemma 3.3. Since $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^{m-1} \times \mathbb{R})$, the last bound has the right size.

This completes the proof of ($iii$) in Proposition 3.1.

4 Jump Relations

Consider $P \in S^{-1, 1+}_{\text{cl}}(M \times \mathbb{R}; \mathcal{E}, \mathcal{F})$ whose principal symbol $p$ is odd in $\xi$, and let $k = k(p)$ be the Schwartz kernel of $p$. Recall that this implies

$$
k(x, y, s, t) = c_m \widehat{p}(x, y - x, s - t).
$$

(4.1)

Also, denote by $d(x, y)$ the geodesic distance between $x, y \in M$.

**Lemma 4.1.** Let $\Omega$ be a Lipschitz domain, $\Omega \subseteq M$, $0 < T < \infty$, $1 < p < \infty$, $f \in L^p(\partial \Omega \times (0, T), \mathcal{E})$. Then for a.e. $x \in \partial \Omega$, $t \in (0, T)$,

$$
\lim_{\epsilon \to 0} \left| \int_{0}^{t-\epsilon} \int_{\partial \Omega} \langle k(x, y, s, t), f(y, s) \rangle d\sigma(y)ds \\
- \int_{0}^{t} \int_{d(x, y) \geq \sqrt{\epsilon}} \langle k(x, y, s, t), f(y, s) \rangle d\sigma(y)ds \right| = 0.
$$

(4.2)

**Proof.** Let

$$
A_\epsilon f(x, t) := \int_{0}^{t-\epsilon} \int_{d(x, y) \leq \sqrt{\epsilon}} \langle k(x, y, s, t), f(y, s) \rangle d\sigma(y)ds,
$$

(4.3)

$$
B_\epsilon f(x, t) := \int_{t-\epsilon}^{t} \int_{d(x, y) \geq \sqrt{\epsilon}} \langle k(x, y, s, t), f(y, s) \rangle d\sigma(y)ds.
$$

(4.4)
We shall prove that:

\[
\lim_{\epsilon \to 0} A_{\epsilon} f(x, t) = 0 \quad \forall f \text{ in a dense subclass of } L^p(\partial \Omega \times (0, T), \mathcal{E}),
\]

\[
\lim_{\epsilon \to 0} B_{\epsilon} f(x, t) = 0 \quad \forall f \text{ in a dense subclass of } L^p(\partial \Omega \times (0, T), \mathcal{E}),
\]  

(4.5)  

(4.6)

\[
\| \sup_{\epsilon > 0} |A_{\epsilon} f| \|_{L^p(\partial \Omega \times (0, T))} \leq C \| f \|_{L^p(\partial \Omega \times (0, T), \mathcal{E})},
\]

\[
\| \sup_{\epsilon > 0} |B_{\epsilon} f| \|_{L^p(\partial \Omega \times (0, T))} \leq C \| f \|_{L^p(\partial \Omega \times (0, T), \mathcal{E})}.
\]  

(4.7)  

(4.8)

Then the lemma follows from (4.5)-(4.8) in the usual fashion.

**Proof of (4.5). Step I.** Work in local coordinates and decompose \( p = p_{-1} + p_{-2} \) where \( p_{-1} \in S^{-1,1}_{1,0.2} \), \( p_{-2} \in S^{-2,2}_{1,0.2} \). Accordingly, \( k = k_{-1} + k_{-2} \) and, further, \( A_{\epsilon} = A_{\epsilon_{-1}} + A_{\epsilon_{-2}} \), with some self-explanatory notation. Since \( k_{-2} \) has an absolutely integrable singularity, \( A_{\epsilon_{-2}} f(x, t) \to 0 \) by Lebesgue’s Dominated Convergence Theorem. Hence, it is enough to consider \( A_{\epsilon_{-1}} \); in what follows, we drop the subindex \(-1\).

**Step II.** As in [2, Appendix B],

\[
\lim_{\epsilon \to 0} A_{\epsilon} f(x, t) = \lim_{\epsilon \to 0} \int_0^{t-\epsilon} \int_{|x-y| \leq \sqrt{\epsilon}} \langle k(x, y, t, s), f(y, s) \rangle \, d\sigma(y) ds,
\]

(4.9)
i.e. \( d(x, y) \) can be replaced by \( |x-y| \), the Euclidean distance.

**Step III.** Assume that \( x = (0, 0) \in \mathbb{R}^{m-1} \times \mathbb{R} \), \( \partial \Omega \) is the graph of a Lipschitz function \( \varphi : \mathbb{R}^{m-1} \to \mathbb{R} \) such that \( \varphi(0) = 0 \), \( \nabla \varphi(0) = 0 \). In particular, there exists \( \omega \in L^\infty \), \( \omega \geq 0 \) such that

\[
|\varphi(y)| \leq |y| \omega(|y|), \quad \| \omega \|_{L^\infty} \leq \| \nabla \varphi \|_{L^\infty}, \quad \lim_{\lambda \to 0^+} \omega(\lambda) = 0.
\]  

(4.10)

Change coordinates so that we work on \( \mathbb{R}^{m-1} \) in place of \( \partial \Omega \), absorb the Jacobian of the transformation (\( \in L^\infty \)) into \( f \) and, at that stage, restrict attention to functions \( f \) of the type \( u(x)v(t) \) with \( u, v \in C^\infty_{\text{comp}} \) (note that the linear span of this class is a dense subspace of \( L^p \)). Adding and subtracting \( u(0) \) in the integral (and utilizing the smoothness of \( u \)), matters can finally be reduced to showing that

\[
\lim_{\epsilon \to 0} \int_0^{t-\epsilon} \int_{|y| \leq \sqrt{\epsilon}} k((0, 0), (y, \varphi(y)), s, t) \, dy ds = 0.
\]

(4.11)

What is crucial is the fact that \( k((0, 0), (y, 0), s, t) = \hat{p}((0, 0), (y, 0), s-t) \) is odd in \( y \). In particular,

\[
\int_0^{t-\epsilon} \int_{|y| \leq \sqrt{\epsilon}} k((0, 0), (y, 0), s, t) \, dy ds = 0.
\]

(4.12)
Thus, it suffices to analyze
\[\int_{t-\epsilon}^{t} \int_{y \in \mathbb{R}^{m-1}} |y| \leq \sqrt{\epsilon} \, \left| k((0,0),(y,\varphi(y)),s,t) - k((0,0),(y,0),s,t) \right| dyds. \tag{4.13} \]

The integrand is, by the Mean-Value Theorem,
\[\leq C|\varphi(y)| \sup_{\theta \in [0,\varphi(y)]} |\partial_{2}k((0,0),(y,\theta),s,t)| \]
\[\leq C|y|\omega(|y|) \frac{1}{(|y| + |s-t|^{\frac{1}{2}})^{m+2}}, \tag{4.14} \]

where the last inequality above utilizes (4.10), (4.1) and (3.13). Thus, the integral in (4.13) is dominated by
\[\int_{t-\epsilon}^{t} \int_{y \in \mathbb{R}^{m-1}} |y| \omega(|y|) \frac{1}{(|y| + |s-t|^{\frac{1}{2}})^{m+2}} dyds \leq C \int_{0}^{\infty} \frac{1}{s^{\frac{m+2}{2}}} ds \right) \left( \int_{|y| \leq 1} \omega(\sqrt{\epsilon}|y'|) dy' \right), \tag{4.16} \]

i.e. \(o(1)\) as \(\epsilon \to 0\) by Lebesgue’s Dominated Convergence Theorem. This proves (4.5).

\textbf{Proof of (4.6).} This is pretty similar to that of (4.5). Once again, following the same reduction procedure as before, the crux of the matter is showing that
\[\lim_{\epsilon \to 0} \int_{t-\epsilon}^{t} \int_{y \in \mathbb{R}^{m-1}} |y| \geq \sqrt{\epsilon} \, k((0,0),(y,\varphi(y)),t,s) dyds = 0. \tag{4.17} \]

Again, it suffices to show that actually
\[\lim_{\epsilon \to 0} \int_{t-\epsilon}^{t} \int_{y \in \mathbb{R}^{m-1}} \left| k((0,0),(y,\varphi(y)),t,s) - k((0,0),(y,0),t,s) \right| dyds = 0. \tag{4.18} \]

As in (4.14), the above integrand is \(\leq C|y|\omega(|y|)|y| + |s-t|^{\frac{1}{2}}\right)^{(m+2)}\) so that we are led to consider
\[\int_{0}^{\epsilon} \int_{y \in \mathbb{R}^{m-1}} \frac{|y|\omega(|y|)}{(|y| + s^{\frac{1}{2}})^{m+2}} dyds. \tag{4.19} \]

Making the change of variables \(y = \sqrt{\epsilon}y', s = \epsilon s'\), this becomes
\[
\int_0^1 \int_{|y'| \geq 1} \frac{|y'| \omega(\sqrt{\epsilon}|y'|)}{(|y'| + |s'|^{1/2})^{m+2}} dy'ds' \\
\leq C \int_{|y'| \geq 1} \frac{\omega(\sqrt{\epsilon}|y'|)}{|y'|^{m+1}} dy' = o(1) \text{ as } \epsilon \to 0,
\]

by Lebesgue’s Dominated Converge Theorem. This takes care of (4.6).

**Proof of (4.7).** Working in local coordinates and pull-backing \(\partial \Omega\) to \(\mathbb{R}^{m-1}\) a direct estimate on \(A_\epsilon\) gives

\[
|A_\epsilon f(x, t)| \\
\leq C \frac{1}{\epsilon} \left(\frac{1}{\epsilon}\right)^{m-1} \int_{\mathbb{R}} \int_{\mathbb{R}^{m-1}} \chi_{B_1(0)} \left(\frac{x - y}{\sqrt{\epsilon}} \right) \frac{\chi(1, \infty)}{\epsilon} \left|f(y, s)\right| \left[\left|\frac{x - y}{\sqrt{\epsilon}}\right| + \left|\frac{s - t}{\sqrt{\epsilon}}\right|^{1/2}\right] \left[m + 1\right] dy ds. \tag{4.21}
\]

Hence Lemma 3.3 with

\[
\Phi(y, s) := \chi_{B_1(0)}(y) \chi_{(1, \infty)}(s) [||y| + |s|^{1/2}]^{-(m+1)} \in L^1(\mathbb{R}^{m-1} \times \mathbb{R})
\]

applies and gives that

\[
\sup_{\epsilon > 0} |A_\epsilon f(x, t)| \leq C M f(x, t). \tag{4.22}
\]

From this, (4.7) follows.

**Proof of (4.8).** This is virtually identical to that of (4.7) except that, this time, we need to take

\[
\Phi(y, s) := \chi_{\mathbb{R}^{m-1} \setminus B_1(0)}(y) \chi_{(0, 1)}(s) [||y| + |s|^{1/2}]^{-(m+1)} \in L^1(\mathbb{R}^{m-1} \times \mathbb{R}). \tag{4.23}
\]

Thus, by the same token,

\[
\sup_{\epsilon > 0} |B_\epsilon f(x, t)| \leq C M f(x, t). \tag{4.24}
\]

Now (4.8) follows from this, thanks to the boundedness of \(M\) on \(L^p\). This finishes the proof of (4.8) and, with it, the proof of Lemma 4.1.

After these preparations, we are ready to discuss the main result of this section.

**Theorem 4.2.** Let \(\Omega \subseteq M\) be a Lipschitz domain with outward unit conormal \(\nu\), \(0 < T < \infty\), \(1 < p < \infty\). Also, let \(p \in S^{-1,1}_{\alpha, 2} (M \times \mathbb{R}; \mathcal{E}, \mathcal{F})\) have principal symbol \(p_{-1}\) which we assume is odd in \(\xi\). Let \(J\) be associated with \(p\) and \(\Omega\) as in (3.3). Then for each \(f \in L^p(\partial \Omega \times (0, T), \mathcal{E})\) and at a.e. \(w \in \partial \Omega, t \in (0, T),\)

\[
\lim_{x \to w} \mathbb{J} f(x, t) = \pm \frac{1}{2} i p_{-1}(w, \nu(w), 0) f(w, t) + \lim_{\epsilon \to 0} J_\epsilon f(w, t). \tag{4.25}
\]
Proof. Given the results in §3, it suffices to assume that \( f \in C^\infty_{\text{comp}}(\partial \Omega \times (0,T); \mathcal{E}) \). In this case matters can be reduced to proving (4.25) on the Fourier transform side in time, i.e. to show that for a.e. \( w \in \partial \Omega, \tau \in \mathbb{R}, \)

\[
\lim_{x \to w} \mathcal{J} f(x, \tau) = \mp \frac{1}{2} i p_{-1}(w, \nu(w), 0) \hat{f}(w, \tau) + \lim_{\epsilon \to 0} \mathcal{J}_\epsilon f(w, \tau),
\]

(4.26)

where hat denotes \( \mathcal{F}_t \). Indeed, integrating (4.26) against \( \hat{\varphi}(\tau) d\tau \), for \( \tau \in \mathbb{R} \), (where \( \varphi \in C^\infty_{\text{comp}}(\mathbb{R}) \) is arbitrary) and using Plancherel’s formula allows us to return to (4.25). Moreover, it is clear that we can assume that \( \mathcal{J} \) is associated with \( p_{-1} \); in the sequel, we drop the subscript \(-1\). Now,

\[
\hat{\mathcal{J} f}(x, \tau) = \int_{\partial \Omega} \left< \int_0^\infty \hat{p}(x, y - x, s) e^{-ist} ds, \hat{f}(y, \tau) \right> d\sigma(y)
\]

\[
= \int_{\partial \Omega} \left< \int_0^\infty \hat{p}(x, y - x, s)[e^{-ist} - 1] ds, \hat{f}(y, \tau) \right> d\sigma(y)
\]

\[
+ \int_{\partial \Omega} \left< \int_0^\infty \hat{p}(x, y - x, s) ds, \hat{f}(y, \tau) \right> d\sigma(y)
\]

\[
=: A(x, \tau) + B(x, \tau).
\]

Next, as \( x \to w \),

\[
A(x, \tau) \to \int_{\partial \Omega} \left< \int_0^\infty \hat{p}(w, y - w, s)[e^{-ist} - 1] ds, \hat{f}(y, \tau) \right> d\sigma(y),
\]

(4.28)

by Lebesgue’s Dominated Convergence Theorem. To justify this, note that by the elementary estimate

\[
|e^{ist} - 1| \leq C \min\{|\tau||s|, 1\}
\]

and the fact that \(|x - y| \geq C|y - w|\) uniformly for \( w, y \in \partial \Omega, x \in \gamma_{\pm}(w) \), we have

\[
|\hat{p}(x, y - x, s)[e^{-ist} - 1]| \leq C_r \frac{|s|^\frac{1}{2}}{|w - y| + |s|^\frac{1}{2}m+1}
\]

(4.30)

and

\[
\int_{\partial \Omega} \int_0^\infty \frac{|s|^\frac{1}{2}}{(|w - y| + |s|^\frac{1}{2})m+1} ds d\sigma(y)
\]

\[
\leq \left( \int_{\partial \Omega} \frac{1}{|w - y|m-2} d\sigma(y) \right) \left( \int_0^\infty \frac{|s'|^\frac{1}{2}}{(1 + |s'|^\frac{1}{2})m+1} ds' \right) < +\infty.
\]

(4.31)

Going further, if \( b(x, z) := \int_0^\infty \hat{p}(x, z, s) ds \) then \( b(x, z) \) is odd and homogeneous of degree \((-m + 1)\) in \( z \). The elliptic theory from [2],[1] applies in this case to give
\[
\lim_{x \to w} \quad B(x, \tau) = \pm \frac{1}{2} i (F_z b)(w, \nu(w)) \tilde{f}(w, \tau)
\]
\[
+ \lim_{\epsilon \to 0} \int_{d(w, y) \geq \sqrt{\epsilon}} d\sigma(y) \left\langle \int_0^\infty \hat{p}(w, y - w, s) \, ds, \, \tilde{f}(y, \tau) \right\rangle d\sigma(y).
\]

(4.32)

Now, by assumptions and Lemma 1.1, \( \hat{p} \) vanishes for \( s < 0 \). In particular, \( b(x, z) = \int_{-\infty}^{+\infty} \hat{p}(x, z, s) \, ds = (F_\xi p)(x, z, 0) \) so that \( (F_z b)(x, \xi) = p(x, \xi, 0) \). Thus, the jump term in (4.32) agrees with that of (4.26). Combining the right side of (4.28) with the limit in the right side of (4.32) we obtain

\[
\lim_{\epsilon \to 0} \int_{d(w, y) \geq \sqrt{\epsilon}} d\sigma(y) \left\langle \int_0^\infty \hat{p}(w, y - w, s) e^{-is\tau} \, ds, \, \tilde{f}(y, \tau) \right\rangle d\sigma(y)
\]

(4.33)

as desired. Note that, in the second equality, the support conditions on \( \hat{p} \) and \( f \) have been used. Also, Lemma 4.1 is invoked in the third equality. Finally commuting \( F_t \) with \( \lim_{\epsilon \to 0} \) can be done due to, e.g., the continuity of \( F_t \) in \( L^2 \).

This finishes the proof of Theorem 4.2.

5 Fractional Time-Derivative Layer Potentials

Let \( f \in L^1(-\infty, T) \) which decays fast enough at \( -\infty \). For \( 0 < \sigma \leq 1 \), introduce the fractional integral operator of order \( \sigma \) (i.e. the one-dimensional Riemann-Liouville integral; cf. [56, p. 217]), by

\[
I_\sigma f(t) := \frac{1}{\Gamma(\sigma)} \int_{-\infty}^t \frac{f(s)}{(t-s)^{1-\sigma}} \, ds,
\]

(5.1)

where \( \Gamma \) is the usual Euler’s gamma function. Hence,

\[
I_\sigma f = \frac{1}{\Gamma(\sigma)} (t^{\sigma-1} \chi_{(0,\infty)} * f).
\]

(5.2)
Finally, for \( \sigma = 0 \), take \( I_0 \) to be the identity operator.

Next, define the family of fractional time-derivative operators by

\[
D^\sigma_t := \frac{\partial}{\partial t} I_{1-\sigma}, \quad 0 \leq \sigma \leq 1.
\]  

(5.3)

In particular, \( D^1_t = \frac{\partial}{\partial t} \), the ordinary time-derivative. Some of the immediate properties of the operators \( I_\sigma, D^\sigma_t \) are summarized below.

**Lemma 5.1.** For each \( 0 < \sigma < 1 \), the following hold:

(i) the operators \( I_\sigma, D^\sigma_t \) are casual, in the sense that they commute with translations and if \( f \equiv 0 \) for \( t \leq t_0 \) then \( D^\sigma_t f, I_\sigma f \equiv 0 \) for \( t \leq t_0 \);

(ii) \( D^\sigma_t [f(\lambda t)] = \lambda^\sigma (D^\sigma_t f)(\lambda t) \) and \( I_\sigma [f(\lambda t)] = \lambda^{-\sigma} (I_\sigma f)(\lambda t) \);

(iii) \( D^\sigma_t I_\sigma = D^\sigma_t I_\sigma = D^{\sigma_1+\sigma_2} t \) and \( I_{\sigma_1} \cdot I_{\sigma_2} = I_{\sigma_1+\sigma_2} \) for \( 0 \leq \sigma_1, \sigma_2 \leq 1, \sigma_1 + \sigma_2 \leq 1 \);

(iv) \( D^\sigma_t (f * g) = (D^\sigma_t f) * g = f * (D^\sigma_t g) \) and \( I_\sigma (f * g) = (I_\sigma f) * g = f * (I_\sigma g) \);

(v) \( \widehat{D^\sigma_t f}(\tau) = (2\pi)^{\sigma}|\tau|^{\sigma-1} (1 + i \text{sign}(\tau)) \hat{f}(\tau) \);

(vi) \( \widehat{I_\sigma f}(\tau) = -i(2\pi)^{1-\sigma} |\tau|^{1-\sigma} \sqrt{2} (1 + i \text{sign}(\tau)) \hat{f}(\tau) \).

For proofs see, e.g., [56, p. 217], [34, p. 35], and [41, p. 40-41].

To state our next result recall that, as before, \( \hat{\cdot} \) denotes the iterated Fourier transform \( \mathcal{F}_\xi \mathcal{F}_\tau \).

**Proposition 5.2.** Assume that \( m + \ell + k > 1 \). If \( p \in S_{1,0,k}^{\ell-} \) then, for each \( 0 < \sigma < 1 \) and \( \alpha, \beta, \gamma, \) there holds

\[
|I_\sigma \partial_x^\alpha \partial_z^\beta \hat{p}(x, z, t)| \leq C_{\alpha,\beta,\gamma,\sigma} \left\{ \min \left\{ 1, \frac{|z|^k}{t} \right\} \right\}^{\beta+1} \left| \frac{t^\sigma}{\Gamma(\sigma)} \right|, \quad \forall z, \forall t > 0, \text{ uniformly for } x \text{ in compacts.}
\]  

\[
(5.4)
\]

Also,

\[
|D^\sigma_t D^\sigma_z \partial_x^\alpha \partial_z^\beta \hat{p}(x, z, t)| \leq C_{\alpha,\beta,\gamma,\sigma} \left\{ \min \left\{ 1, \frac{|z|^k}{t} \right\} \right\}^{\beta+2} \left| \frac{t^{1-\sigma}}{\Gamma(1-\sigma)} \right|, \quad \forall z, \forall t > 0, \text{ uniformly for } x \text{ in compacts.}
\]  

\[
(5.5)
\]

**Proof.** Invoke Lemma 1.1 and decompose

\[
I_\sigma \partial_x^\alpha \partial_z^\beta \hat{p}(x, z, t) = \frac{1}{\Gamma(\sigma)} \int_0^t \partial_x^\alpha \partial_z^\beta \hat{p}(x, z, s) \frac{1}{(t-s)^{1-\sigma}} ds
\]

\[
= \frac{1}{\Gamma(\sigma)} \int_0^t (\cdots) ds + \frac{1}{\Gamma(\sigma)} \int_t^\infty (\cdots) ds =: I + II.
\]  

(5.6)

To estimate \( I \), we integrate by parts \( \nu \) times, \( 0 \leq \nu \leq \beta \), and write
\[ I = \sum_{j=1}^{\nu} c_{j,\sigma} \frac{(-1)^{j+1}}{(t)^{j-\sigma}} \partial_x^j \partial_s \partial_y^{\beta-j} \tilde{p}(x, z, t) \]
\[ + \tilde{c}_{\nu,\sigma} \int_0^{\frac{t}{2}} \partial_x^\nu \partial_y^{\beta-\nu} \tilde{p}(x, s) \frac{1}{(t-s)^{\nu-\sigma+1}} ds. \]  

Estimating each \( \partial_x^\nu \partial_y^{\beta-\nu} \tilde{p} \) according to (3.13) then gives

\[ |I| \leq C \sum_{j=1}^{\nu} \frac{1}{t^{j-\sigma}} \frac{1}{[|z| + |t|^\frac{1}{2}]^{m+\ell+|\alpha|+k(|\beta|-j)}} \]
\[ + C \frac{1}{t^{\nu-\sigma+1}} \int_0^t \frac{ds}{[|z| + |s|^\frac{1}{2}]^{m+\ell+|\alpha|+k(|\beta|-\nu)}} \]
\[ \leq C \frac{1}{t^{\nu-\sigma+1}} \int_0^t \frac{ds}{[|z| + |s|^\frac{1}{2}]^{m+\ell+|\alpha|+k(|\beta|-\nu)}}. \]

where the last inequality follows by observing that a multiple of the last expression above dominates each term of the sum in (5.8). Going further, we have

\[ |II| \leq \left[ \sup_{t/2 \leq s \leq t} \left| \partial_x^\nu \partial_y^{\beta-\nu} \tilde{p}(x, z, s) \right| \right] \left( \int_0^t \frac{ds}{s^{1-r}} \right). \]

Once again, (a multiple of) the last expression in (5.8) dominates the last expression above. Thus, at this point, we have shown that the left-hand side of (5.4) is dominated by the last expression in (5.8).

In order to continue, we need an elementary result to the effect that, for \( a, b > 0, \theta \in (0, 1) \) and \( N > 1/\theta \), there holds

\[ \int_0^a \frac{ds}{(b + s^\theta)^N} \approx \min \left\{ \frac{a}{b^N}, \frac{1}{b^{N-1}/\theta} \right\}. \]

Returning to the task of further estimating the last expression in (5.8), we utilize (5.10) to write

\[ \frac{1}{t^{\nu-\sigma+1}} \int_0^t \frac{ds}{[|z| + |s|^\frac{1}{2}]^{m+\ell+|\alpha|+k(|\beta|-\nu)}} \]
\[ \approx C \frac{1}{t^{\nu-\sigma+1}} \min \left\{ \frac{t}{\left[|z|^{m+\ell+|\alpha|+k(|\beta|-\nu)}\right]}, \frac{1}{\left[|z|^{m+\ell+|\alpha|+k(|\beta|-\nu)}\right]} \right\} \]
\[ = C \frac{1}{t^{\nu-\sigma+1}} \frac{1}{|z|^{m+\ell+|\alpha|+k(|\beta|-\nu)}} \min \left\{ \frac{t}{|z|^k}, 1 \right\} \]
\[ = C \frac{1}{t^{1-\sigma}} \frac{1}{|z|^{m+\ell+|\alpha|+k(|\beta|-\nu)}} \left( \frac{|z|^k}{t} \right)^\nu \min \left\{ \frac{t}{|z|^k}, 1 \right\}. \]
When $\frac{|z|^k}{t} \leq 1$, we make the choice $\nu = \beta$ in which case the last expression in (5.11) becomes $C \frac{1}{|z|^m + |t| + \epsilon} \left( \frac{|z|^k}{t} \right)^\beta$, in agreement with (5.4). If, on the other hand, $\frac{|z|^k}{t} \geq 1$, then we select $\nu = 0$ and, once again, the bound provided by (5.11) agrees with (5.4). This concludes the proof of (5.4). The proof of (5.5) is similar and we omit it.

**Proposition 5.3.** Assume that $P \in OPS_{c,2,+}^{-2}(M \times \mathbb{R}; E, F)$ and let $k(x, y, t, s)$ denote the Schwartz kernel of $P$. Also, for $\Omega \subseteq M$ Lipschitz, introduce the associated single layer potential operator, i.e.

$$J f(x, t) := \int_{-\infty}^{t} \int_{\partial \Omega} \langle k(x, y, t, s), f(y, s) \rangle d\sigma(y) ds, \quad x \not\in \partial \Omega, \ t \in \mathbb{R},$$

and fix some $1 < p < \infty$, $0 < T < \infty$.

(i) For each $\epsilon > 0$ there holds

$$\| (Jf)^* \|_{L^p(\partial \Omega \times (0, T))} \leq C_{\epsilon} \max \{ T^{\frac{1}{2} - \epsilon}, 1 \} \| f \|_{L^p(\partial \Omega \times (0, T), E)}$$

uniformly in $f$.

(ii) For each $f \in L^p(\partial \Omega \times (0, T), E)$,

$$\lim_{x \to w \atop x \in \gamma_{\pm}(w)} J f(x, t) = \int_{0}^{t} \int_{\partial \Omega} \langle k(w, y, t, s), f(y, s) \rangle d\sigma(y) ds$$

at a.e. $w \in \partial \Omega, \ t \in (0, T)$.

(iii) There holds

$$\| (D_{t}^{\frac{1}{2}} Jf)^* \|_{L^p(\partial \Omega \times (0, T))} \leq C \| f \|_{L^p(\partial \Omega \times (0, T), E)},$$

uniformly in $f$.

(iv) For each $f \in L^p(\partial \Omega \times (0, T), E)$,

$$\lim_{x \to w \atop x \in \gamma_{\pm}(w)} D_{t}^{\frac{1}{2}} J f(x, t) = \lim_{\epsilon \to 0} \int_{0}^{t-\epsilon} \int_{\partial \Omega} \langle (D_{t}^{\frac{1}{2}} k)(w, y, t, s), f(y, s) \rangle d\sigma(y) ds$$

$$= \lim_{\epsilon \to 0} \int_{0}^{t} \int_{d(x, y) \geq \sqrt{\epsilon}} \langle (D_{t}^{\frac{1}{2}} k)(w, y, t, s), f(y, s) \rangle d\sigma(y) ds$$

at a.e. $w \in \partial \Omega, \ t \in (0, T)$. 

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Let $\hat{f}$ denote the Fourier transform in time:

$$\hat{f}(x, y, t, s) = \int_{\mathbb{R}} f(x, y, t, s) e^{-i 2\pi k y} dk \int_{\mathbb{R}} f(x, y, t, s) e^{-i 2\pi k x} dk$$

We define the convolution operator with kernel $K_{\tau}(x) := |\tau|^{\frac{1}{2}} \frac{1}{|x|^{m-2}} \max\{|\tau|^{N}|x|^{2N}, 1\}$ as

$$K_{\tau}(x) := |\tau|^{\frac{1}{2}} \frac{1}{|x|^{m-2}} \max\{|\tau|^{N}|x|^{2N}, 1\}$$

This operator is bounded in $L^2(\mathbb{R}^{m-1})$ uniformly in $\tau$. However,
\[
\int_{\mathbb{R}^{m-1}} |K_\tau(x)| \, dx = \int_{\mathbb{R}^{m-1}} \frac{|\tau|^\frac{1}{2}}{|x|^{m-2}} \max\{[|\tau||x|^2]^N, 1\} \, dx \\
= \int_{\mathbb{R}^{m-1}} \frac{1}{|x|^{m-2}} \max\{[x]^{2N}, 1\} \, dx \leq C_m < +\infty
\] (5.22)

if \( N = 1 \). Thus, \( \sup_\tau \|K_\tau\|_{L^1} \leq C_m < +\infty \), so that \( \sup_\tau \|T_\epsilon\|_{L^2 \to L^2} < +\infty \). Similar considerations also show that

\[
\lim_{\eta, \epsilon \to 0} \|T_\epsilon f - T_\eta f\|_{L^2} = 0; 
\] (5.23)

cf. also [34, p. 67]. Hence

\[
T f := \lim_{\epsilon \to 0} T_\epsilon f \text{ exists in } L^2 \\
T : L^2 \to L^2 \text{ is a bounded operator.}
\] (5.24)

As before (cf. §3) proving that \( T \) extends to a bounded operator in \( L^p \), \( 1 < p < \infty \), depends on duality and a Hörmander type cancelation property for the kernel. Since \( T \) is of convolution type in the time variable, one can show that its formal adjoint satisfies similar properties as \( T \) itself. In particular, it is amenable to the same analysis as above.

There remains to show that

\[
\iint_{\partial \Omega \times \mathbb{R} \setminus [S_{2r}(y) \times (s-4r^2,s+4r^2)]} \left| D^\frac{1}{2}_t \left( \hat{p}(x, y - x, s - t) \right) \right| \, d\sigma(x) \, dt \leq C < +\infty,
\] (5.25)

whenever \(|y' - y| + |s' - s|^\frac{1}{2} \leq r\).

However, given the estimates (5.4)-(5.5) this closely parallels the calculation done in §3, Step III. For example,

\[
\iint_{S_{2r}(y) \times \mathbb{R} \setminus (s-4r^2,s+4r^2)} \left| D^\frac{1}{2}_t \left( \hat{p}(x, y - x, s - t) \right) \right| \, dx \, dt \\
\leq C \int_{|x-y| \leq 2r} \left( \int_{|t| \geq 4r^2} \frac{t^{\frac{1}{2}}}{|x-y|^{m+2}} \left( \min\left\{ 1, \frac{|x-y|^2}{t} \right\} \right)^2 \, dt \right) \, dx \\
\leq C \left( \int_{|x-y| \leq 2r} \frac{|x-y|^4}{|x-y|^{m+2}} \, dx \right) \left( \int_{|t| \geq 4r^2} \frac{dt}{t^{\frac{3}{2}}} \right) \approx C \cdot r \cdot \frac{1}{r} = C < +\infty.
\] (5.26)

We omit the remaining details.

At this stage we have proved the \( L^p \)-boundedness of the operator (5.17), uniformly in \( \epsilon \). Next, semi-standard arguments entail the boundedness of the maximal operator.
This step is based on a Cotlar type inequality which, in turn, can be deduced from the estimates (5.4), (5.5). We refer the interested reader to [34, Theorem A.5, p. 70] for details in similar circumstances. Next, much as in [34, Proposition 1.7, p. 74], we can prove that

\[ (D^2_{t} Jf)^* \leq C(T_* f + \mathcal{M} f) \text{ pointwise on } \partial \Omega \times (0,T), \]

where, recall that \( \mathcal{M} \) stands for the (parabolic) Hardy-Littlewood maximal function on \( \partial \Omega \times (0,T) \) from Lemma 3.3. This allows us to finally conclude that (5.15) holds.

Now we turn our attention to (5.16). First, for each \( f \in C^\infty_{\text{comp}}(\partial \Omega \times (0,T)) \) we have \( D^1_{t} Jf = J(D^1_{t} f) \); cf. Lemma 5.1. Granted (5.14), the nontangential trace of the latter function is

\[
\lim_{\epsilon \to 0} \int_{d(x,y) \geq \sqrt{\epsilon}} \left( \int_{\mathbb{R}} \langle k(x,y,t,s), (D^1_{s} f)(y,s) \rangle \, ds \right) \, d\sigma(y) = \lim_{\epsilon \to 0} \int_{0}^{t} \int_{d(x,y) \geq \sqrt{\epsilon}} \langle (D^1_{t} k)(x,y,t,s), f(y,s) \rangle \, d\sigma(y) \, ds.
\]

Hence, we are left with proving that for each \( f \in L^p \) and a.e. \( (x,t) \in \partial \Omega \times (0,T) \),

\[
\lim_{\epsilon \to 0} \int_{0}^{t} \int_{d(x,y) \geq \sqrt{\epsilon}} \langle (D^1_{t} k)(x,y,t,s), f(y,s) \rangle \, d\sigma(y) \, ds = \lim_{\epsilon \to 0} \int_{0}^{t} \int_{d(x,y) \geq \sqrt{\epsilon}} \langle (D^1_{t} k)(x,y,t,s), f(y,s) \rangle \, d\sigma(y) \, ds.
\]

The strategy is to show that: (I) the two limits coincide for a dense subclass of \( L^p \), and (II) the corresponding maximal operators are bounded in \( L^p \). We now tackle these two points starting with the first. To this effect, fix \( f \in C^\infty_{\text{comp}}(\partial \Omega \times (0,T), \mathcal{E}) \) so that, via an integration by parts,

\[
\int_{0}^{t-\epsilon} \int_{\partial \Omega} \left\langle -\frac{\partial}{\partial s} \left[ I_{\frac{1}{2}} \tilde{p}(x,y,-x,\cdot)(t-s) \right], f(y,s) \right\rangle \, d\sigma(y) \, ds = \int_{0}^{t-\epsilon} \int_{\partial \Omega} \left\langle I_{\frac{1}{2}} \tilde{p}(x,y,-x,\cdot)(t-s), \frac{\partial f}{\partial s}(y,s) \right\rangle \, d\sigma(y) \, ds + \int_{\partial \Omega} \left\langle I_{\frac{1}{2}} \tilde{p}(x,y,-x,\cdot)(\epsilon), f(y,t-\epsilon) \right\rangle \, d\sigma(y).
\]

Now, the last integral converges to zero as \( \epsilon \to 0^+ \) by Lebesgue’s theorem since \( \tilde{p} \) vanishes for \( \tau > 0 \). As for the first integral in the right side of (5.31), the kernel has an integrable
singularity so passing to the limit \( \epsilon \to 0^+ \) is straightforward. All in all, the left side of (5.31) can be written in the form

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}} \int_{y \in \partial \Omega} \left( I_1 \left( \hat{p}(x, y - x, \cdot) \right)(t - s), \partial_s f(y, s) \right) d\sigma(y) ds
= \lim_{\epsilon \to 0} \int_{\mathbb{R}} \int_{y \in \partial \Omega} \left( \hat{p}(x, y - x, s - t), \left( I_1 \frac{\partial}{\partial s} \right) f(y, s) \right) d\sigma(y) ds
= \lim_{\epsilon \to 0} \int_{\mathbb{R}} \int_{y \in \partial \Omega} \left( \hat{p}(x, y - x, s - t), (D^1_s f)(y, s) \right) d\sigma(y) ds
= \lim_{\epsilon \to 0} \int_{\mathbb{R}} \int_{y \in \partial \Omega} \left( D^1_t \left( \left[ \hat{p}(x, y - x, s - t) \right], f(y, s) \right) \right) d\sigma(y) ds,
\]

as desired.

Consider next (II) in the strategy outlined above. As usual, it suffices to prove that for \( x \in \partial \Omega, t \in (0, T) \),

\[
\sup_{\epsilon > 0} \left| \int_{0}^{t-\epsilon} \int_{y \in \partial \Omega} \left( D^1_t k(x, y, t, s), f(y, s) \right) d\sigma(y) ds \right| \leq C M f(x, t)
\]

and

\[
\sup_{\epsilon > 0} \left| \int_{t-\epsilon}^{t} \int_{y \in \partial \Omega} \left( D^1_t k(x, y, t, s), f(y, s) \right) d\sigma(y) ds \right| \leq C M f(x, t).
\]

Passing to local coordinates and pull-backing \( \partial \Omega \) to \( \mathbb{R}^{m-1} \), we see that the left-side of (5.33) is bounded by

\[
C \int_{-\infty}^{t-\epsilon} \int_{y \in \mathbb{R}^{m-1}} \frac{|s - t|^{1/2}}{|x - y|^{m+2}} \left( \min\left\{ 1, \frac{|x - y|^2}{|s - t|} \right\} \right)^2 |f(y, s)| dy ds
\leq C \frac{1}{\epsilon} \left( \frac{1}{\sqrt{\epsilon}} \right)^{m-1} \int_{\mathbb{R}^{m-1}} \chi(1,\infty) \left( \frac{s - t}{\epsilon} \right) \chi_B(0) \left( \frac{x - y}{\sqrt{\epsilon}} \right) \frac{|s - t|^{1/2}}{|x - y|^{m+2}}
\times \left( \min\left\{ 1, \frac{|x - y|^2}{|s - t|} \right\} \right)^2 |f(y, s)| dy ds.
\]

Now (5.33) follows from (5.35) with the aid of Lemma 3.3 in which we choose

\[
\Phi(y, s) := \chi(1,\infty)(s) \chi_B(0)(y) \frac{s^{1/2}}{|y|^{m+2}} \left( \min\left\{ 1, \frac{|y|^2}{s} \right\} \right)^2 \in L^1(\mathbb{R}^{m-1} \times \mathbb{R}).
\]

The estimate (5.34) is proved in a similar fashion; we omit the details. This completes the proof of (iii) and (iv) in Proposition 5.3.

Granted (5.4)–(5.5), the estimates (i) and (ii) are elementary and we leave them to the interested reader. The proof of Proposition 5.3 is finished. \( \Box \)
6 Square-Function Estimates

The main aim of this section is to prove the following.

**Proposition 6.1.** Let $P \in \text{OPS}_{\text{cl}}^{2,+}(M \times \mathbb{R}; \mathcal{E}, \mathcal{F})$ have an even principal symbol (in $x$), $\Omega \subseteq M$ Lipschitz, and denote by $J$ the single layer potential operator associated with $P$ (and $\Omega$) as in (5.12).

Then, for each $0 < T < \infty$ there exists $C = C(\Omega, P, T) > 0$ so that if $u := Jf$ with $f \in L^2(\partial \Omega \times (0, T), \mathcal{E})$, we have

\[
\int_0^T \int_{\Omega} |\nabla^2 u|^2 \text{dist}(\cdot, \partial \Omega) \, d\text{Vol} \, dt \leq C \|f\|^2_{L^2(\partial \Omega \times (0, T), \mathcal{E})},
\]

\[
\int_0^T \int_{\Omega} |D^{1/4}_t \nabla u|^2 \, d\text{Vol} \, dt \leq C \|f\|^2_{L^2(\partial \Omega \times (0, T), \mathcal{E})}.
\]

**Proof.** This follows from the work of S. Hofmann and J. Lewis [46], where very general results of this type are proved; we shall only sketch the main steps.

Dealing first with (6.1), work in local coordinates and decompose the symbol $\sigma_{\text{princ}}(P)$ as $p_{-2} + p_{-3}$ where $p_{-2} \in S_{1,0,2}^{-2,+}$, $p_{-2}(x, \xi, \tau)$ is even in $\xi$, and $p_{-3} \in S_{1,0,2}^{-3,+}$. Thus, $J$ splits accordingly, as $J_{-2} + J_{-3}$. Now, the contribution from $J_{-3}$ is easily controlled, due to the weak singularity in the kernel. In fact this leads to an estimate like (6.1) with $C(\Omega, P, T) \to 0$ as $T \to 0$.

Next, extend $f$ to $\partial \Omega \times \mathbb{R}$ by setting $f \equiv 0$ outside $\partial \Omega \times (0, T)$ and concentrate on the contribution from

\[
J_{-2} f(x, t) := \int_{-\infty}^{+\infty} \int_{\partial \Omega} \left( \hat{p}_{-2}(x, y - x, t - s), f(y, s) \right) \, d\sigma(y) \, ds.
\]

Note that by localizing the problem there is no loss of generality in assuming that $f$ is scalar-valued, has small support and $\Omega$ is Euclidean so that $\partial \Omega$ is the graph of a Lipschitz function $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$. To proceed, for $z \in \mathbb{R}^m$, $r \in \mathbb{R}$, let $\rho = \rho(z, r) > 0$ solve the equation $|z|^2 + r^2 = 1$. In particular, $\rho(\lambda z, \lambda^2 r) = \lambda \rho(z, r)$ and $(z/\rho, r/\rho^2)$ belongs to $S^m$, the unit sphere in $\mathbb{R}^m \times \mathbb{R}$.

Let now $\{\psi_j : j \geq 1\}$ be an orthonormal basis of $L^2(S^m)$ consisting of eigenfunctions of the Laplace-Beltrami operator on $S^m$ and, for each fixed $x$, expand $\hat{p}_{-2} \left( x, \frac{z}{\rho}, \frac{r}{\rho^2} \right)$ in spherical harmonics:

\[
\hat{p}_{-2} \left( x, \frac{z}{\rho}, \frac{r}{\rho^2} \right) = \sum_{j \geq 1} a_j(x) \psi_j \left( \frac{z}{\rho}, \frac{r}{\rho^2} \right),
\]

\[
\text{with } \|a_j\|_{C^1} \cdot \|\psi_j\|_{C^N} \leq C_N j^{-2}, \quad \forall j.
\]

Given that $\hat{p}_{-2}$ is even in the second variable, we may replace each $\psi_j$ above by $\psi_j^+$, where $\psi_j^+(z, r) := \frac{1}{2}(\psi_j(z, r) + \psi_j(-z, r))$. It follows that
\[
\hat{p}_{-2}(x, z, r) = \sum_{j \geq 1} a_j(x) b_j(z, r),
\]
where \( b_j(z, r) := \rho^{-m} \psi_j^\pm \left( \frac{z}{\rho}, \frac{r}{\rho^2} \right) \).

In particular,

1. Each \( b_j \) is even in \( z \) for fixed \( r \)
2. And \( b_j(\lambda z, \lambda^2 r) = \lambda^{\pm m} b_j(z, r) \).

At this stage, the arguments in [46] (cf. especially Lemma 5.11 and Theorem 4.1) apply to each

\[
u_j(x, t) := \int_{-\infty}^{+\infty} \int_{\partial \Omega} b_j(x - y, t - s) f(y, s) \, d\sigma(y) \, ds \tag{6.7}
\]
to yield

\[
\int_{-\infty}^{+\infty} \int_{\Omega} |\nabla^2 u_j|^2 \text{dist}(\cdot, \partial \Omega) \, dx \, dt \leq C \|\psi_j\|_{C^M} \cdot \|f\|_{L^2(\partial \Omega \times \mathbb{R}, E)}^2 \tag{6.8}
\]
for some fixed \( M > 0 \), uniformly in \( j \). Clearly, since \( u = \sum_j a_j u_j \), (6.8) and (6.4) yield (6.1).

The reader should be aware that the estimate (6.8) was obtained in [46] in the particular case when \( P = (\partial_t - \Delta)^{-1} \). Three proofs of this were given, one of which has a purely real-variable nature. Inspection of the arguments reveals that the techniques developed in the course of this latter proof are in fact powerful enough to handle (6.7). There is only one aspect which we would like to emphasize here. Specifically, if \( \theta \) is a nice bump function and \( x = (x', \phi(x') + \lambda) \in \Omega, \ y = (y', \phi(y')) \in \partial \Omega \) then, for \( b(x, t) \) as in (6.6),

\[
\int_{\mathbb{R}^m} (\partial_{x_j} \partial_{x_k} b)(x' - y', (x' - y', \nabla(\theta \lambda \ast \phi)(x'))) + \lambda, t - s \, dy' \, ds = 0, \tag{6.9}
\]
for all \( \lambda, x', t \), as long as \( 1 \leq j, k \leq m \).

Indeed, if we set \( F(x) := \int_{\mathbb{R}} (\partial_{x_j} \partial_{x_k} b)(x, t) \, dt \), then \( F \) becomes an even function which is (positive) homogeneous of degree \(-m\) in \( \mathbb{R}^m \). In this scenario, (6.9) is going to be a simple consequence of the elementary identity

\[
\int_{\mathbb{R}^{m-1}} F(y', (a, y') + \lambda) \, dy' = \frac{1}{2\lambda} \int_{S^{m-2}} \int_{\mathbb{R}} F(\omega, s) \, ds \, d\omega = \int_{\mathbb{R}^{m-1}} F(y', \lambda) \, dy'
\]
valid for each \( \lambda > 0 \) and \( a \in \mathbb{R}^{m-1} \); see Lemma 1.3 in [1] for a proof.

Finally, (6.2) is proved much as in [46] based on (6.1), this finishes the proof of the proposition.  

\[\square\]
In the sequel, we shall not utilize the full force of (6.2). The case of interest to us here is when $P = (\partial_t - L)^{-1}$ with $L : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$ formally self-adjoint, real, second order, strongly elliptic differential operator. At least for some common, special cases, other alternative approaches to (6.2) can be developed. Below we briefly elaborate on this idea.

First, anticipating notation, terminology and results which are actually developed in §7, let us assume that $L$ is locally given by (7.1) and that the coefficients of the principal part can be chosen so that for all $\zeta = (\zeta^\alpha_j^k)$

$$a_{jk}^{\alpha\beta}(x)\zeta^\alpha_j^k \zeta^\beta_k \geq C|\zeta|^2, \quad \text{uniformly in } x. \quad (6.11)$$

This is always possible when rank $\mathcal{E} = 2$, i.e. when $1 \leq \alpha, \beta \leq 2$. To see this, introduce $A := (a_{jk}^{11})_{j,k}$, $B := (a_{jk}^{12})_{j,k}$, $C := (a_{jk}^{22})_{j,k}$ so that

$$(a_{jk}^{\alpha\beta})_{\alpha,\beta,j,k} = \begin{bmatrix} A & B \\ B^t & C \end{bmatrix}. \quad (6.12)$$

Also, note that $A, C$ are symmetric and $\lambda^2 A + \lambda (B + B^t) + C$ is positive semi-definite for each $\lambda \in \mathbb{R}$. Then there exists an anti-symmetric $n \times n$ matrix $D = -D^t$ so that

$$\begin{bmatrix} A & B + D \\ B^t - D & C \end{bmatrix} \quad (6.13)$$

is positive semi-definite. This result may be deduced from [57, Theorem 5.51]. A direct proof is given in [58]. The interested reader is also referred to [59] which contains a survey of work on necessary and sufficient conditions for a pair of quadratic forms to admit a positive definite linear combination. Clearly, applying this result to $(a_{jk}^{\alpha\beta}) - \epsilon I$ for $\epsilon > 0$ small it follows that in the case when $1 \leq \alpha, \beta \leq 2$ matters can always be arranged so that (6.11) is true.

Another important situation in which (6.11) holds is when $L = L_0 - DD^* + \text{lower order terms}$, where $D$ is a first-order differential operator, $D^*$ is its formal adjoint, and $L_0$ is a second-order differential operator for which (6.11) is valid. This is easily seen by a direct calculation: if the principal part of $D$ is locally given by $A_j^{\alpha\beta} \partial_j$, then $L$ is locally given by (7.1) with $a_{jk}^{\alpha\beta} = A_j^{\alpha\gamma} A_k^{\beta\gamma}$. Thus,

$$\sum a_{jk}^{\alpha\beta} \zeta_j^\alpha \zeta_k^\beta = \sum_{\beta} \left( \sum_{\alpha,j} a_{jk}^{\alpha\beta} \zeta_j^\alpha \right)^2 \geq 0. \quad (6.14)$$

Returning to the main-stream discussion and assuming that (6.11) holds, we may use (7.13) with $v := w := D_t^{\frac{1}{2}} u$ to get, by virtue of (6.11) and (7.12), that
\[
\int_0^T \int_\Omega |D_t^{1/2} \nabla u|^2 \leq C \int_0^T \int_\Omega \sum_j \partial_j (D_t^{1/2} u^\alpha) a^\alpha_{jk} \partial_k (D_t^{1/2} u^\beta) \\
\leq -\frac{C}{2} \int_0^T \int_\Omega \frac{\partial}{\partial t} |D_t^{1/2} u|^2 \\
+ \mathcal{O}(\|D_t^{1/2} u\|_{L^2(\partial\Omega \times (0,T))} \|\nabla u\|_{L^2(\partial\Omega \times (0,T))}) \\
+ C \int_0^T \int_\Omega |D_t^{1/2} u|^2.
\]

The first term after the second inequality sign is \(-\frac{1}{2} \int_\Omega |D_t^{1/2} u| |D_t^{1/2} u|^2\) and can be dropped based on, e.g., sign considerations. Now all the remaining terms are \(\leq C \|f\|_{L^2(\partial\Omega \times (0,T),E)}\) by the results in §§4-5. This yields (6.2).

Another approach to (6.2) in the case when \(P = (\partial_t - L)^{-1}\), is based on Rellich identities and duality. An approach in the Fourier transform side, i.e. for a single layer potential associated with \((\lambda - L)^{-1}\) where \(\lambda \in \mathbb{C}\) is regarded as a parameter has been developed in [60]. This also can be adapted to the situation under discussion to give an alternative proof of (6.2). We leave the details to the interested reader.

### 7 Parabolic Rellich Estimates

Let \(L : C^\infty(M,E) \to C^\infty(M,F)\) be a second order differential operator with smooth, real-valued coefficients. We assume that, in local coordinates over which \(E,F\) can be trivialized, \(L\) takes the form

\[
(Lu)^\alpha = \sum_{\beta,j,k} \partial_j (a^\alpha_{jk} (x) \partial_k u^\beta) + \sum_{\beta,j} b^\alpha_j (x) \partial_j u^\beta + \sum_{\beta} c^\alpha (x) u^\beta.
\]

Recall the canonical projection \(pr : M \times \mathbb{R} \to M\) and the pull-backs \(\mathcal{E} := pr_* \mathcal{E}, \mathcal{F} := pr_* \mathcal{F} \to M \times \mathbb{R}\). Then \(L\) extends in a natural fashion as a mapping

\[
L : C^\infty(M \times \mathbb{R}, \mathcal{E}) \to C^\infty(M \times \mathbb{R}, \mathcal{F}).
\]

The same applies to \(P := \partial_t - L \in OPS_{cl,2}^+ (M \times \mathbb{R}; \mathcal{E}, \mathcal{F})\). Then, in local coordinates,

\[
p(x,\xi,\tau) := \sigma_{\text{princ}}(P)(x,\xi,\tau) = i\tau + \sum_{j,k} a^\alpha_{jk} (x) \xi_j \xi_k.
\]

This clearly satisfies the hypothesis in Lemma 1.1 (relative for \(\mathbb{C}_-\)) and it accounts for the fact that \(p\) is casual.

Assume next that \(L\) is strongly elliptic, i.e. \(E = F\) and

\[
C |\eta|^2_x \leq \text{Re} \left( -\sigma_{\text{princ}}(L)(x,\xi) \eta, \bar{\eta} \right)_x = \text{Re} \left[ \sum_{j,k,\alpha,\beta} a^\alpha_{jk} (x) \xi_j \xi_k \eta^\alpha \bar{\eta}_\beta \right],
\]

\[
\forall x \in M, \quad \xi \in T_x^* M, \quad |\xi|_x = 1, \quad \eta \in E_x.
\]
Then, it follows that $P$ is strongly parabolic, i.e.

$$\text{Re} \langle p(x, \xi, \tau) \eta, \eta \rangle \geq C(|\xi| + |\tau|^{\frac{1}{2}})|\eta|^2,$$

(7.5) uniformly for $x \in M$, $\xi \in T^*_x M \setminus \{0\}$, $\tau \in \mathbb{R}_+$, $\eta \in E_x$. In particular, by Theorem 2.3, $P^{-1} \in OPS^{2,+}_{cl,2}(M \times \mathbb{R}; \mathcal{E})$ exists and, locally,

$$\sigma_{\text{princ}}(P^{-1})(x, \xi, \tau) = \left[i\tau + \sum_{j,k} a_{jk}^\alpha(x) \xi_j \xi_k \right]^{-1}.$$

(7.6)

Next consider $k(x, y, t, s)$, the Schwartz kernel of $P^{-1}$, and for a Lipschitz domain $\Omega \subseteq M$ introduce the single layer potential operator (associated with $P := \partial_t - L$), i.e.

$$Sf(x, t) := \int_{-\infty}^t \int_{\partial \Omega} \langle k(x, y, t, s), f(y, s) \rangle d\sigma(y) dt;$$

(7.7) recall that $d\sigma$ is the surface measure on $\partial \Omega$.

Finally, fix $0 < T < \infty$, and for arbitrary $f \in L^2(\partial \Omega \times (0, T), \mathcal{E})$ set

$$u := Sf \text{ in } \Omega \times (0, T).$$

(7.8)

Our aim is to show the following.

**Proposition 7.1.** For each $\epsilon > 0$ there holds

$$\int_0^T \int_{\partial \Omega} |\nabla \tan u|^2 d\sigma dt + \int_0^T \int_{\partial \Omega} |D_t^{\frac{1}{2}} u|^2 d\sigma dt \approx \int_0^T \int_{\partial \Omega} |\nabla u|^2 d\sigma dt$$

modulo $\epsilon\|f\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})}^2$ and terms which are small with $T$.

(7.9)

Generally speaking, writing ‘$A(s) \approx B(s)$ modulo $C$’ indicates that $A \leq \kappa B + C$ and $B \leq \kappa A + C$ for some constant $\kappa > 0$ independent of the relevant parameters in $A$, $B$, $C$. In (7.9), the equivalence constant may depend on $\epsilon$ but not on $f$. Also, a quantity is called “small with $T$” if its absolute value can be bounded by $C(T)\|f\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})}$ with $C(T) = o(1)$ as $T \to 0^+$.

With an eye on (7.9), observe that it suffices to show a local version of this estimate, i.e. when $\partial \Omega$ is contained in a coordinate patch over which $\mathcal{E}$ trivializes. Indeed, there is no loss of generality assuming that $f$ has small support and we may also truncate $u$, i.e. work with $\psi u$ in place of $u$, where $\psi \in C^\infty_{\text{comp}}(M)$ has a suitable support. This latter reduction is relatively harmless: the only thing we loose is the quality of $u$ of being a null solution for $\partial_t - L$ in $\Omega$. However, as an a posteriori inspection of the proof shows, $(\partial_t - L)u = O(|\nabla u|)$ is just as good.

Assuming the above reductions and working in local coordinates where $L$ has the form (7.1) we now proceed to present the

**Proof of Proposition 7.1.** Let $h = (h_j)_j$ be a smooth, arbitrary vector field. Then the following Rellich type identity is known to hold in the case under discussion
\[
\sum_{\alpha,\beta,j,k,\ell} \frac{\partial}{\partial x_\ell} \left[ (h_\ell a^\alpha_{jk} - h_j a^\alpha_{k\ell} - h_k a^\alpha_{j\ell}) \frac{\partial u^\alpha}{\partial x_j} \frac{\partial u^\beta}{\partial x_k} \right]
\]
\[= -2 \sum_{\ell,\alpha} h_\ell \frac{\partial u^\alpha}{\partial x_\ell} \left( \sum_{j,k} \frac{\partial}{\partial x_j} (a^\alpha_{jk} \frac{\partial u^\beta}{\partial x_k}) \right) + O(|\nabla u|^2 + |u|^2). \tag{7.10}\]

See, e.g., [61]. Under the current assumption that \((\partial_t - L)u = O(|\nabla u|)\), the entire right side of (7.10) becomes \(O(|\nabla u|^2 + |u|^2)\). Consequently, much as in the elliptic case (cf. [1])

\[
\int_0^T \int_{\partial \Omega} |\nabla u|^2 \, d\sigma dt \leq C \int_0^T \int_{\partial \Omega} |\nabla \tan u|^2 \, d\sigma dt + C \int_0^T \int_{\Omega} \sum_{\alpha} \frac{\partial u^\alpha}{\partial t} \cdot \frac{\partial u^\alpha}{\partial t} \, d\text{Vol} \, dt \tag{7.11}\]

In the sequel, we shall keep \(A\). Also, in the applications that we have in mind, \(C\) is going to be small with \(T\), which suits our purposes. There remains to estimate \(B\).

In order to continue, we need two auxiliary facts. First, if \(f, g\) are sufficiently smooth and decay fast enough at \(-\infty\), then

\[
\left| \int_{-\infty}^T (D^\frac{1}{2}_t f) g \right| \leq C \left( \int_{-\infty}^T |f|^2 \right)^{\frac{1}{2}} \left( \int_{-\infty}^T |D^\frac{1}{2}_t g|^2 \right)^{\frac{1}{2}}, \tag{7.12}\]

with \(C > 0\) independent of \(T\). See [34] for a proof. Second, we need an identity to the effect that for each \(v, w\), there holds

\[
\int_0^T \int_{\Omega} \sum_{\alpha,\beta,j,k} \frac{\partial w^\alpha}{\partial x_j} a^\alpha_{jk} \frac{\partial v^\beta}{\partial x_k} \, d\text{Vol} \, dt + \int_0^T \int_{\Omega} \sum_{\alpha} w^\alpha \frac{\partial v^\alpha}{\partial t} \, d\text{Vol} \, dt
\]
\[= \int_0^T \int_{\partial \Omega} \sum_{\alpha,\beta,j,k} w^\alpha n_j a^\alpha_{jk} \frac{\partial v^\beta}{\partial x_k} \, d\sigma \, dt + \int_0^T \int_{\Omega} \sum_{\alpha} w^\alpha [(\partial_t - L) v]^\alpha \, d\text{Vol} \, dt \tag{7.13}\]
\[+ \int_0^T \int_{\Omega} O(|w||\nabla v|) \, d\text{Vol} \, dt, \]

where \(n = (n_j)_j\) and \(d\sigma\) are, respectively, the Euclidean unit normal and surface measure on \(\partial \Omega\) (the latter differs from the surface measure induced by the Riemannian structure by a factor \(\rho\) with \(\rho, \rho^{-1} \in L^\infty\)). Indeed,
\[
\iint_{\Omega} \sum_{\alpha, \beta, j, k} \frac{\partial w^\alpha}{\partial x_j} a^\alpha_{jk} \frac{\partial v^\beta}{\partial x_k} = \iint_{\Omega} \sum_{\alpha} \frac{\partial}{\partial x_j} \left( w^\alpha a^\alpha_{jk} \frac{\partial v^\beta}{\partial x_k} \right) - \iint_{\Omega} \sum_{\alpha} w^\alpha \frac{\partial}{\partial x_j} \left( a^\alpha_{jk} \frac{\partial v^\beta}{\partial x_k} \right) =: I + II. \tag{7.14}
\]

An integration by parts gives
\[
I = \int_{\partial \Omega} w^\alpha n_j a^\alpha_{jk} \frac{\partial v^\beta}{\partial x_k} \, d\sigma, \tag{7.15}
\]
whereas for each \(\alpha\),
\[
\sum_{\alpha} \frac{\partial}{\partial x_j} \left( a^\alpha_{jk} \frac{\partial v^\beta}{\partial x_k} \right) = \frac{\partial v^\alpha}{\partial t} - [(\partial_t - L)v]^\alpha + O(|\nabla u|). \tag{7.16}
\]
Using these back in (7.14) and integrating in \(t\) on \((0, T)\) yields (7.13).

Returning now to the task of estimating \(B\) in (7.11) we write \(\partial_t = D_t^\frac{1}{4} D_t^\frac{3}{4}\) so that, by (7.12),
\[
|B| \leq C \left( \int_0^T \int_{\Omega} |D_t^\frac{3}{4} \nabla u|^2 \, d\text{Vol} \, dt \right)^{\frac{1}{2}} \cdot \left( \int_0^T \int_{\Omega} |D_t^\frac{3}{4} u|^2 \, d\text{Vol} \, dt \right)^{\frac{1}{2}} \tag{7.17}
\]

To estimate \(B_2\) we set \(v := I_{\frac{1}{4}} u\), \(w := \partial_t(I_{\frac{1}{4}} u) = D_t^\frac{3}{4} u\) so that, availing ourselves of (7.13), we get \(\sum_{\alpha} w^\alpha \frac{\partial w^\alpha}{\partial t} = |D_t^\frac{3}{4} u|^2\) and
\[
\int_0^T \int_{\Omega} \sum_{\alpha} \frac{\partial w^\alpha}{\partial x_j} a^\alpha_{jk} \frac{\partial v^\beta}{\partial x_k} \, d\text{Vol} \, dt = \frac{1}{2} \int_0^T \int_{\Omega} \sum_{\alpha} \frac{\partial}{\partial t} \left[ \frac{\partial(I_{\frac{1}{4}} u^\alpha)}{\partial x_j} a^\alpha_{jk} \frac{\partial(I_{\frac{1}{4}} u^\beta)}{\partial x_k} \right] \, d\text{Vol} \, dt \tag{7.18}
\]
\[
= \frac{1}{2} \int_0^T \int_{\Omega} \sum_{\alpha} \frac{\partial(I_{\frac{1}{4}} u^\alpha)}{\partial x_j} (\cdot, T) a^\alpha_{jk} \frac{\partial(I_{\frac{1}{4}} u^\beta)}{\partial x_k} (\cdot, T) \, d\text{Vol}
\leq C \int_\Omega |I_{\frac{1}{4}} \nabla u(\cdot, T)|^2 \, d\text{Vol} = \text{small with } T.
\]

The last equality is easy to check based on the estimates established in §5. Thus,
\[
|B_2|^2 = \int_0^T \int_{\Omega} |D_t^\frac{3}{4} u|^2 \, d\text{Vol} \, dt = O \left( \|\nabla u\|_{L^2(\partial \Omega \times (0, T))} \cdot \|D_t^\frac{1}{4} u\|_{L^2(\partial \Omega \times (0, T))} \right) + \text{terms which are small with } T. \tag{7.19}
\]
As for $B_1$ in (7.17), we may invoke Proposition 6.1 to write
\[ |B_1| \leq C||f||_{L^2(\partial\Omega \times (0,T),\mathcal{E})}. \]  
In particular, since for each $\epsilon > 0$ there holds $|B| \leq B_1 \cdot B_2 \leq \epsilon B_1^2 + C\epsilon B_2^2$, (7.11), (7.19) and (7.20) give the inequality “≥” in (7.9).

To see the opposite inequality, fix a smooth field $h = (h_j)_j$ which is transversal to $\partial \Omega$ and write (recall that $n$ is the outward unit normal to $\partial \Omega$)
\[
\int_0^T \int_{\partial \Omega} |D_t^{\frac{1}{2}} u|^2 d\sigma dt \leq C \int_0^T \int_{\partial \Omega} \langle h, n \rangle |D_t^{\frac{1}{2}} u|^2 d\sigma dt \\
+ C \int_0^T \int_{\Omega} |\nabla u|^2 d\sigma dt \leq C \left( \int_0^T \int_{\Omega} |D_t^{\frac{1}{2}} \nabla u|^2 d\sigma dt \right)^{\frac{1}{2}} \\
\cdot \left( \int_0^T \int_{\Omega} |D_t^{\frac{3}{2}} u|^2 d\sigma dt \right)^{\frac{1}{2}} + \text{small terms with } T,
\]  
via the divergence theorem and (7.12). Note that the last product above is the familiar $B_1 \cdot B_2$ (cf. (7.17)). Thus, granted (7.19) and (7.20), for each given $\epsilon > 0$, (7.21) leads to
\[
\int_0^T \int_{\partial \Omega} |D_t^{\frac{1}{2}} u|^2 d\sigma dt \leq C \epsilon \int_0^T \int_{\partial \Omega} |\nabla u|^2 d\sigma dt + \epsilon ||f||_{L^2(\partial\Omega \times (0,T),\mathcal{E})}^2 + \text{small terms with } T.
\]  
With this at hand, the inequality “≤” in (7.9) readily follows. This finishes the proof of Proposition 6.1.

8 Inverting Parabolic Layer Potentials

Retain the hypotheses made on the differential operator $L$ from §7. Also, fix an arbitrary Lipschitz domain $\Omega \subseteq M$ and let $\{w_j\}_{1 \leq j \leq m}$ be a family of $C^1$-vector fields on $M$ which are linearly independent in a neighborhood of $\partial \Omega$. Hence, in local coordinates $w_j = \sum_k c_{jk} \frac{\partial}{\partial x_k}$, where $C := (c_{jk})_{jk}$ has $C^1$ entries and is invertible near $\partial \Omega$.

Next, equip $\mathcal{E}$ with a smooth connection $\nabla$, so that locally
\[
(\nabla_w u)^\alpha = \sum_k c_{jk} \partial_k u^\alpha + O(|u|), \quad \forall j,
\]  
(8.1)
for any smooth section $u$. It follows that, locally,

$$|\nabla u| \approx \sum_j |\nabla w_j u| \text{ modulo } \mathcal{O}(|u|), \text{ pointwise near } \partial \Omega,$$

where $\nabla u$ in the left side stands for the Euclidean gradient acting componentwise on $u$.

Denote now by $\Gamma(x, y, t, s)$ the Schwartz kernel of the operator $P := (\partial_t - L)^{-1} \in OPS^{-2, +}_{cl, 2}(M \times \mathbb{R}; \mathcal{E})$, so that $(\partial_t - L_x)\Gamma(x, y, t, s) = \delta_y(x)\delta_s(t)$, and introduce the single layer potential $J$ associated with $P$ and $\Omega$. Also, denote by $S$ the trace of $S$ on $\partial \Omega \times (0, T)$. Going further we also introduce

$$T : \bigoplus_{1 \leq j \leq m} L^p(\partial \Omega \times (0, T), \mathcal{E}) \to L^p(\partial \Omega \times (0, T), \mathcal{E})$$

by setting

$$T(g_1, g_2, \ldots, g_m) := \lim_{\epsilon \to 0} \int_0^{-\epsilon} \int_{\partial \Omega} \sum_{j=1}^m \langle \nabla w_{j, y} \Gamma(x, y, t, s), g_j(y, s) \rangle d\sigma(y) ds. \quad (8.4)$$

Here $\nabla w_{j, y}$ indicates that $\nabla w_j$ acts in the $y$-variable. Denote by $\nu \in T^* M$ the outward unit conormal to $\partial \Omega$.

**Theorem 8.1.** For each $0 < T < \infty$, $1 < p < \infty$, the operator $T$ in $(8.3)$ is well defined and bounded. Moreover, there exists $\epsilon = \epsilon(\partial \Omega, L) > 0$ such that for each $2 - \epsilon < p < 2 + \epsilon$ and $T > 0$, the assignment

$$(g_1, \ldots, g_m) \mapsto \frac{1}{2} \left[ \sigma_{\text{princ}}(L)(\cdot, \nu) \right]^{-1} \left( \sum_j \nu(w_j) g_j \right) + T(g_1, \ldots, g_m)$$

maps $\bigoplus_{1 \leq j \leq m} L^p(\partial \Omega \times (0, T), \mathcal{E})$ onto $L^p(\partial \Omega \times (0, T), \mathcal{E})$.

A remark is in order here. As the proof will show the operator in $(8.5)$ satisfies a slightly stronger property. More specifically, there exists $0 < C = C(\partial \Omega, L, T)$ such that

$$\forall f \in L^p(\partial \Omega \times (0, T), \mathcal{E}), \; \exists (g_j)_j \in \bigoplus_j L^p(\partial \Omega \times (0, T), \mathcal{E}) \text{ with}$$

$$\frac{1}{2} \sum_j \nu(w_j) \left[ \sigma_{\text{princ}}(L)(\cdot, \nu) \right]^{-1} g_j + T(g_1, \ldots, g_m) = f \quad (8.6)$$

and

$$\sum_j \|g_j\|_{L^p(\partial \Omega \times (0, T), \mathcal{E})} \leq C \|f\|_{L^p(\partial \Omega \times (0, T), \mathcal{E})}.$$

**Proof.** Let us consider the operator

$$T' : L^p(\partial \Omega \times (0, T), \mathcal{E}) \to \bigoplus_{1 \leq j \leq m} L^p(\partial \Omega \times (0, T), \mathcal{E})$$

defined by

$$\mathcal{T} : \bigoplus_{1 \leq j \leq m} L^p(\partial \Omega \times (0, T), \mathcal{E}) \to L^p(\partial \Omega \times (0, T), \mathcal{E})$$

by setting

$$\mathcal{T}(g_1, g_2, \ldots, g_m) := \lim_{\epsilon \to 0} \int_{-\epsilon}^0 \int_{\partial \Omega} \sum_{j=1}^m \langle \nabla w_{j, y} \Gamma(x, y, t, s), g_j(y, s) \rangle d\sigma(y) ds. \quad (8.7)$$
\[
(T'f)_j := \lim_{\epsilon \to 0} \int_0^{t-\epsilon} \int_{\partial \Omega} \langle \nabla w_j, \Gamma(x, y, t, s), f(y, s) \rangle \, d\sigma(y) \, ds
\]  
(8.8)
from the results in §5, \( T' \) is well-defined and bounded for each \( 1 < p < \infty \) and \( 0 < T < \infty \).

We now make two basic observations regarding the operator (8.7)-(8.8). First, for \( f \in L^p(\partial \Omega \times (0, T), \mathcal{E}) \) let \( u := \mathcal{S}f \) in \( \Omega_\pm \times (0, T) \). Then, for each \( j \), so we claim,

\[
(\nabla w_j u)|_{\partial \Omega_\pm \times (0, T)} = \pm \frac{1}{2} \nu(w_j)[\sigma_{\text{princ}}(L)(\cdot, \nu)]^{-1} f + (T'f)_j.
\]  
(8.9)
Our second claim is that

\[
R \circ (T')^* \circ R = T, \quad \text{modulo operators whose norms are small with } T.
\]  
(8.10)
Here \((Rf)(u, t) := f(x, T-t)\) is a time-reflection and the asterisk indicates the adjoint.

To prove (8.9), note that \( \nabla w_j \circ P \in OPS_{cl_2}^{-1} \) has the principal symbol (cf. the discussion in §7)

\[
\sigma_{\text{princ}}(\nabla w_j \circ P)(x, \xi, \tau) = \sigma_{\text{princ}}(\nabla w_j)(x, \xi, \tau) \sigma_{\text{princ}}(P)(x, \xi, \tau) \\
= iw_j(\xi)[i\tau + \sum a_{jk}^\alpha(x)\xi_j\xi_k]^{-1}.
\]  
(8.11)
Also, the Schwartz kernel of \( \nabla w_j \circ P \) is \( \nabla w_j, \Gamma(x, y, t, s) \). Thus, the jump formula (8.9) is a consequence of (4.25).

As for (8.10) we first observe that, like \( \partial_t - L \), the operator \( P = (\partial_t - L)^{-1} \) is invariant under changing \( t \) into \( -t \) and taking the adjoint. At the level of Schwartz kernels this property reads

\[
[\Gamma(x, y, t, s)]^t = \Gamma(x, y, t, s),
\]  
(8.12)
where the superscript \( t \) indicates adjunction (in \( \text{Hom}(\mathcal{E}, \mathcal{E}) \)). Also, as is well known,

\[
(\nabla w_j)^* = -\nabla w_j + \text{zero order terms}.
\]  
(8.13)
Now (8.10) follows from (8.12), (8.13) and (5.13). Armed with (8.9)-(8.10) we are now ready to prove the ontoness of the operators (8.5).

First, we consider the case \( p = 2 \). Let \( f \in L^2(\partial \Omega \times (0, T), \mathcal{E}) \) be arbitrary and set \( u := \mathcal{S}f \) in \( \Omega_\pm \times (0, T) \). Then Proposition 7.1 and (8.2) give that for each \( \epsilon > 0 \) there holds

\[
\int_0^T \int_{\partial \Omega} |\nabla_{\text{tan}} u|^2 \, d\sigma dt + \int_0^T \int_{\partial \Omega} |D_{\text{tan}}^2 u|^2 \, d\sigma dt \\
\quad \approx \sum_j \int_0^T \int_{\partial \Omega} |\nabla w_j u|_{\partial \Omega_\pm \times (0, T)}|^2 \, d\sigma dt
\]  
(8.14)
modulo \( \epsilon \|f\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})}^2 \) and terms which are small with \( T \).
Also, given that \( \{w_j\}_j \) are linearly independent near \( \partial \Omega \) and that \( L \) is strongly elliptic, we get from (8.9) and the triangle inequality that

\[
\| f \|_{L^2(\partial \Omega \times (0,T), \mathcal{E})} \leq C \sum_{j=1}^{m} \| (\nabla w_j u) \|_{\partial \Omega_+ \times (0,T)} L^2(\partial \Omega \times (0,T), \mathcal{E})
\]

\[
+ C \sum_{j=1}^{m} \| (\nabla w_j u) \|_{\partial \Omega_- \times (0,T)} L^2(\partial \Omega \times (0,T), \mathcal{E})
\]

(8.15)

where \( C \) is independent of \( T \). In order to continue, we note that, for each tangent field \( h \), \( \nabla h u \) does not jump, i.e.

\[
(\nabla h u) \big|_{\partial \Omega_+ \times (0,T)} = (\nabla h u) \big|_{\partial \Omega_- \times (0,T)}.
\]

(8.16)

This is a consequence of (8.9) given that \( \nu(h) = 0 \). A similar comment applies to \( D_t^{1/2} u \); cf. (5.16). These observations in concert with (8.14) give that

\[
\| f \|_{L^2(\partial \Omega \times (0,T), \mathcal{E})} \leq C \sum_{j=1}^{m} \| (\nabla w_j u) \|_{\partial \Omega \times (0,T)} L^2(\partial \Omega \times (0,T), \mathcal{E})
\]

\[
+ \epsilon \| f \|_{L^2(\partial \Omega \times (0,T), \mathcal{E})} + C(T) \| f \|_{L^2(\partial \Omega \times (0,T), \mathcal{E})},
\]

(8.17)

where \( C(T) \to 0 \) as \( T \to 0 \). Choosing first \( \epsilon \) sufficiently small and then making \( T \) small enough, it follows that

\[
\| f \|_{L^2(\partial \Omega \times (0,T), \mathcal{E})} \leq C \left( \left\{ \frac{1}{2} \nu(w_j) [\sigma_{princ}(L)(\cdot, \nu)]^{-1} \right\}_{1 \leq j \leq m} + \mathbb{T} \right) \| f \|_{L^2(\partial \Omega \times (0,T), \mathcal{E})},
\]

(8.18)

for some \( C = C(\partial \Omega, T) > 0 \) independent of \( f \), granted that \( T \) is small enough.

At this stage we invoke a simple functional analysis lemma to the effect that if a linear operator \( A \) maps a Hilbert space \( H \) into itself so that, for some \( \delta > 0 \),

\[
\delta \| x \|_H \leq \| Ax \|_H, \quad \forall x \in H,
\]

(8.19)

then

\[
\forall y \in H, \quad \exists x \in H \text{ with } \| x \| \leq \delta^{-1} \| y \| \text{ and } A^* x = y.
\]

(8.20)

This follows easily from [62, Theorem 4.13, p. 100]. When applied to (8.18) this gives, in the light of (8.10) and the fact that \( R \) is an isomorphic isometry, that

\[
\frac{1}{2} \sum_{j} \nu(w_j) [\sigma_{princ}(L)(\cdot, \nu)]^{-1} \pi_j + \mathbb{T} \text{ maps}
\]

\[
\bigoplus_{1 \leq j \leq m} L^p(\partial \Omega \times (0,T), \mathcal{E}) \text{ onto } L^p(\partial \Omega \times (0,T), \mathcal{E})
\]

(8.21)

when \( p = 2 \) and \( T > 0 \) is sufficiently small.
Here \( \pi_j : \bigoplus_{1 \leq j \leq m} L^p(\partial\Omega \times (0, T), \mathcal{E}) \to L^p(\partial\Omega \times (0, T), \mathcal{E}) \) is the canonical projection on the \( j \)-th component.

Now, since the operator in (8.21) is well-defined and bounded for each \( 1 < p < \infty \) and since the property of being onto is stable on complex interpolation scales (cf. [63]), we get that the condition \( p = 2 \) in (8.21) can be relaxed to \( 2 - \epsilon < p < 2 + \epsilon \) for some \( \epsilon = \epsilon(\partial\Omega, L) > 0 \).

Finally, the smallness assumption on \( T \) in (8.21) can be eliminated thanks to an abstract argument which, for the convenience of the reader, we formulate below. Modulo this, Theorem 8.1 is proved.

Here is the lemma which finishes the proof of Theorem 8.1. This is taken from [39]. Similar results have been used in [22], [34], [37].

**Lemma 8.2.** Let \( \mathcal{X} \) be an arbitrary, fixed set, and let \( V \) be a certain vector space of functions defined on \( \mathcal{X} \times \mathbb{R} \). We assume that \( f(\cdot, \cdot + h) \in V \) for any \( f \in V \) and any \( h \in \mathbb{R} \). Set \( V_0 := \{ f \in V; f|_{\mathcal{X} \times (-\infty, 0]} \equiv 0 \} \) and, for each \( T > 0 \), \( V_T := \{ f|_{\mathcal{X} \times (0, T)}; f \in V_0 \} \). Suppose that \( B : V \to V \) is a linear operator so that

1. \( B(f(\cdot, \cdot + h)) = B(f)(\cdot, \cdot + h) \) for any \( f \in V \) and for any \( h \in \mathbb{R} \);
2. \( V_0 \) is an invariant subspace of \( B \);
3. there exists \( T_0 > 0 \) such that \( B : V_T \to V_T \) is a bijection for any \( T_0 > T > 0 \).

Then \( B : V_T \to V_T \) is a bijection for each \( T > 0 \).

**Remark.** If \( F \in V_T \) for \( (x, t) \in \mathcal{X} \times (0, T) \), we define \( B(f)(x, t) := B(F)(x, t) \), where \( F \in V_0 \) is such that \( F|_{\mathcal{X} \times (0, T)} = f \). The properties of \( B \) ensure that this definition does not depend on the particular extension \( F \) of \( f \).

To state our next result we introduce the space \( L^p_{1, \frac{1}{2}}(\partial\Omega \times (0, T), \mathcal{E}) \) as the collection of all sections \( f : \partial\Omega \times (0, T) \to \mathcal{E} \) such that \( |\nabla_{\text{tan}} f| \) and \( |D^\frac{1}{2}_t f| \) belong to \( L^p(\partial\Omega \times (0, T)) \).

**Theorem 8.3.** Let \( L \) be a second order, strongly elliptic, formally self-adjoint differential operator with smooth, real coefficients and fix \( \Omega \subseteq M \), arbitrary Lipschitz domain. Set \( P := (\partial_t - L)^{-1} \) and denote by \( S \) the (nontangential trace on \( \partial\Omega \times (0, T) \) of the) single layer potential associated with \( P \) and \( \partial\Omega \).

Then there exists \( \epsilon = \epsilon(\Omega, L) > 0 \) such that for each \( 2 - \epsilon < p < 2 + \epsilon \) and \( 0 < T < \infty \), the operator

\[
S : L^p(\partial\Omega \times (0, T), \mathcal{E}) \to L^p_{1, \frac{1}{2}}(\partial\Omega \times (0, T), \mathcal{E})
\]

is an isomorphism.

Let us point out that, as the proof will show, one also has

\[
\|S^{-1}\|_{L^p_{1, \frac{1}{2}} \to L^p} \leq C(\partial\Omega, L, T, p).
\]
Proof. For an arbitrary \( f \in L^2(\partial \Omega \times (0, T), \mathcal{E}) \), set \( u := Sf \) in \( \Omega_\pm \times (0, T) \). Select now the vector fields \( \{w_j\}_j \) as in the proof of Theorem 8.1 so that, much as before,

\[
\|f\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})} \leq C \sum_j \|\nabla w_j u\|_{\partial \Omega \times (0, T)} \|f\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})}
\]

\[
+ C \sum_j \|\nabla w_j u\|_{\partial \Omega_+ \times (0, T)} \|f\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})}
\]

\[
+ \text{terms which are small with } T
\]

\[
\leq C \|\nabla \tan u\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})} + C \|D^\frac{1}{2} u\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})}
\]

\[
+ \text{terms which are small with } T
\]

\[
\leq C \|\nabla \tan Sf\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})} + C \|D^\frac{1}{2} Sf\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})}
\]

\[
+ \text{terms which are small with } T
\]

Therefore,

\[
\|f\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})} \leq C \|\nabla \tan Sf\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})}
\]

\[
+ C \|D^\frac{1}{2} Sf\|_{L^2(\partial \Omega \times (0, T), \mathcal{E})},
\]

if \( T > 0 \) is sufficiently small. Now, \( \nabla \tan S, D^\frac{1}{2} S : L^p(\partial \Omega \times (0, T), \mathcal{E}) \rightarrow L^p(\partial \Omega \times (0, T), \mathcal{E}) \) are bounded for each \( 1 < p < \infty \) and \( 0 < T < \infty \). Thus, utilizing the fact that estimates for below like (8.25) on complex interpolation scales are stable under small perturbations of the scale parameter ([63]), we finally obtain that there exists \( \epsilon > 0 \) so that

\[
\|f\|_{L^p(\partial \Omega \times (0, T), \mathcal{E})} \leq C \|\nabla \tan Sf\|_{L^p(\partial \Omega \times (0, T), \mathcal{E})} + C \|D^\frac{1}{2} Sf\|_{L^p(\partial \Omega \times (0, T), \mathcal{E})}
\]

\[
= C \|Sf\|_{L^p_{1, \frac{1}{2}}(\partial \Omega \times (0, T), \mathcal{E})},
\]

uniformly in \( f \) for each \( 2 - \epsilon < p < 2 + \epsilon \), provided \( T > 0 \) is sufficiently small.

It is important to point out that the constant \( C \) in (8.26) depends exclusively on \( p, T, L \) and the Lipschitz character of \( \partial \Omega \). Thus, \( S \) in (8.22) is injective and with closed range if \( |2 - p| \) and \( T \) are small, which we shall assume for now.

If we now take \( \Omega_j \not\subseteq \Omega, \partial \Omega_j \in C^\infty \), a suitable approximating sequence and denote by \( S_j \) the corresponding single layer potential on \( \partial \Omega_j \), it follows from (2.19) that \( S_j : L^p(\partial \Omega_j \times (0, T), \mathcal{E}) \rightarrow L^p_{1, \frac{1}{2}}(\partial \Omega \times (0, T), \mathcal{E}) \) is invertible. This, the fact that sup \( \|S_j^{-1}\| < +\infty \), and a standard limiting argument (cf. [30], [2] for details in similar circumstances) then imply that \( S \) in (8.22) also has dense range.

Summarizing, at this point we have proved that \( S \) in (8.22) is invertible provided \( 2 - \epsilon < p < 2 + \epsilon \) and \( T > 0 \) is sufficiently small. Finally, the latter restriction can be lifted by invoking Lemma 8.2. This completes the proof of the theorem. \( \square \)
9 Initial Boundary Value Problems

Let \( L : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E}) \) be as in §8, i.e. a second order, formally self-adjoint, strongly elliptic differential operator with smooth, real coefficients.

For \( \Omega \subseteq M \) Lipschitz domain, \( 1 < p < \infty \) and \( 0 < T < \infty \) we consider the Dirichlet initial boundary value problem

\[
\begin{aligned}
(u, t - L)u &= 0 \text{ in } \Omega \times (0, T), \\
\left. u \right|_{t=0} &= 0 \text{ on } \Omega, \\
\left. u^* \right|_{\partial \Omega \times (0, T)} &= f \in L^p(\partial \Omega \times (0, T), \mathcal{E}).
\end{aligned}
\]  

(9.1)

**Theorem 9.1.** With the above assumptions, there exists \( \epsilon = \epsilon(\Omega, L) > 0 \) so that for each \( 2 - \epsilon < p < 2 + \epsilon, \ 0 < T < \infty \), the initial boundary problem (9.1) has a unique solution which also satisfies

\[
\|u^*\|_{L^p(\partial \Omega \times (0, T))} \leq C\|f\|_{L^p(\partial \Omega \times (0, T))}
\]  

(9.2)

for some \( C = C(\partial \Omega, L, T, p) > 0 \). Also, when \( p = 2 \) there holds

\[
u \in H^{1/4}((0, T), L^2(\Omega, E)) \cap L^2((0, T), H^{1/2}(\Omega, E)),
\]

(9.3)

where \( H^s \) is the usual \( L^2 \)-based scale of Sobolev spaces.

Moreover, the following regularity statement is true:

\[
(\nabla u)^*, (D^3_t u)^* \in L^p(\partial \Omega \times (0, T)) \Leftrightarrow f \in L^p_{1,2}(\partial \Omega \times (0, T), \mathcal{E}).
\]  

(9.4)

In this case, one also has

\[
\| (\nabla u)^* \|_{L^p(\partial \Omega \times (0, T))} + \| (D^3_t u)^* \|_{L^p(\partial \Omega \times (0, T))} \leq C\|f\|_{L^p_{1,2}(\partial \Omega \times (0, T), \mathcal{E})},
\]

(9.5)

for some \( C = C(\partial \Omega, L, T, p) > 0 \). If, in addition, \( p = 2 \) then

\[
u \in H^{3/4}((0, T), L^2(\Omega, E)) \cap L^2((0, T), H^{3/2}(\Omega, E)).
\]

(9.6)

Finally, in each case, there is an integral representation formula for the solution.

**Proof.** Let \( \epsilon > 0 \) be such that the invertibility results of §8 hold. Also, select \( \{ w_j \} \) a family of smooth vector fields as in §8. Assume that \( 2 - \epsilon < p < 2 + \epsilon \).

For a system of functions \( (g_j)_{j=1,...,m} \in \bigoplus_{1 \leq j \leq m} L^p(\partial \Omega \times (0, T), \mathcal{E}) \) yet to be determined, we look for a solution \( u \) of (9.1) in the form

\[
u(x, t) := \int_0^t \int_{\partial \Omega} \sum_j (\nabla w_{j,y} \Gamma(x, y, t, s), g_j(y, s)) \, d\sigma(y) \, ds, \quad (x, t) \in \Omega \times (0, T).
\]

(9.7)
Here, as before, $\Gamma$ is the Schwartz kernel of $P := (\partial_t - L)^{-1}$. Clearly, the function $u$ in (9.7) satisfies the first four conditions in (9.1), while the fifth condition amounts to

$$\frac{1}{2} \sum_j \nu(w_j)[\sigma_{princ}(L)(\cdot, \nu)]^{-1}g_j + \mathcal{T}(g_1, \ldots, g_m) = f. \quad (9.8)$$

By Theorem 8.1 this system has a solution $(g_1 \ldots g_m)$ satisfying $\sum \|g_j\|_{L^p} \leq C\|f\|_{L^p}$. Thus, (9.1) admits a solution for which (9.2) is verified uniformly in $f$.

Next, the global regularity statement (9.3) follows from the layer potential representation of the solution, used in concert with the square-function estimates from §6.

There remains uniqueness for the $L^p$-Dirichlet initial boundary problem, $2 - \epsilon < p < 2 + \epsilon$. Since this proceeds much as in the elliptic case (cf. [1]; compare also with [22, Theorem 2.3, p. 188] and [37, p. 332]), we only sketch the main steps. Let $\Omega_j \not\to \Omega$, $\partial \Omega_j \in C^\infty$, and fix $x_0 \in \Omega$, $t_0 \in (0, T)$. Using the invertibility of the single layer potential $S_j$ on $\partial \Omega_j$ (cf. Theorem 8.3) we can construct, for each $j$, a Green function $G_j$ with pole at $(x_0, t_0) \in \Omega_j$ for the problem (9.1) written for $\Omega_j$. The important feature is that

$$\sup_j \|\nabla G_j^*\|_{L^q(\partial \Omega_j \times (0, T))} \leq C < +\infty, \quad (9.9)$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Indeed, this is more or less a direct consequence of (8.23).

Let now $u$ be a null solution for (9.1). Having constructed $G_j$ we then produce a Poisson type integral representation formula for $u|_{\Omega_j \times (0, T)}$ for each $j$. In turn, this leads to the estimate

$$|u(x_0, t_0)| \leq C(x_0, t_0)\|\nabla G_j^*\|_{L^q(\partial \Omega_j \times (0, T))} \cdot \|u\|_{L^p(\partial \Omega_j \times (0, T), \mathcal{E})}. \quad (9.10)$$

Passing to the limit in (9.10) and utilizing (9.9) plus the fact that $u|_{\partial \Omega_j \times (0, T)} \to 0$ as $j \to \infty$, Lebesgue’s Dominated Convergence Theorem allows us to conclude that $u(x_0, t_0) = 0$. Since $(x_0, t_0) \in \Omega \times (0, T)$ was arbitrary, the desired conclusion follows.

Turning our attention to the regularity statement, we only need to observe that, in this case, the solution of (9.1) is given by

$$u := S(S^{-1}f) \text{ in } \Omega \times (0, T). \quad (9.11)$$

The rest is a consequence of this integral representation formula and the results in §4–§6.

**References**


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