The method of layer potentials for electromagnetic waves in chiral media∗

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1 Statement of the problem and main results

Let Ω be a bounded domain in \( \mathbb{R}^3 \), with surface measure \( d\sigma \) and outward unit normal \( n \). Throughout the paper we shall assume that \( \Omega \) is Lipschitz (i.e., \( \partial \Omega \) can be locally described by means of graphs of Lipschitz continuous functions in appropriate systems of coordinates) and that \( \partial \Omega \) is connected. Also, we set \( \Omega_+ := \Omega \) and \( \Omega_- := \mathbb{R}^3 \setminus \bar{\Omega} \). The classical Faraday-Amperé-Maxwell equations in \( \Omega_+ \) or \( \Omega_- \) read

\[
\begin{align*}
\text{curl } E &= i\omega B, \\
\text{curl } H &= -i\omega D,
\end{align*}
\]

where \( \omega > 0 \) is the angular frequency, \( E, H \) are the electric and magnetic fields in \( \Omega_+ \) (or \( \Omega_- \)), \( B \) is the magnetic induction and \( D \) is the electric displacement. In this paper we shall assume that the medium is chiral, i.e. it is responding with both magnetic and electric polarization to electric or magnetic excitation. Under this hypothesis, the magnetic induction and the electric displacement are further related by the Drude-Born-Fedorov constitutive relations (cf. [7], [8], [17])

\[
\begin{align*}
D &= \varepsilon(E + \beta \text{curl } E), \\
B &= \mu(H + \beta \text{curl } H),
\end{align*}
\]

where \( \varepsilon > 0 \) is the electric permittivity, \( \mu > 0 \) is the magnetic permeability, and \( \beta \in \mathbb{R} \) is the chirality measure (admittance). We also introduce \( k := \omega(\varepsilon\mu)^{1/2} > 0 \). If \( \beta = 0 \) we recover the classical Maxwell system. An excellent discussion as well as an extensive list of references for the time-harmonic form of Maxwell’s equations can be found in [6]. Combining (1.1) and (1.2) we obtain (either in \( \Omega_+ \) or \( \Omega_- \)) the following system

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\[ \begin{align*}
\text{curl } E &= i\omega \mu \gamma H + k^2 \beta \gamma E, \\
\text{curl } H &= -i\omega \varepsilon \gamma E + k^2 \beta \gamma H,
\end{align*} \quad (1.3) \]

where \( \gamma := \frac{1}{1-k^2 \beta^2} \). Here, and for the rest of the paper we make the assumption \(|k\beta| \neq 1\), which is natural from physical considerations.

In order to continue, we need a couple of definitions. First, for each \( 1 < p < \infty \), define \( L^p_{\tan}(\partial \Omega) \) as the collection of all \( p \)-th power integrable, tangential fields on \( \partial \Omega \) (i.e., \( \langle n, f \rangle = 0 \) a.e. on \( \partial \Omega \)) and recall the Sobolev space \( L^1_{\tan}(\partial \Omega) := \{ f \in L^p(\partial \Omega); \nabla_{\tan} f \in L^p(\partial \Omega) \} \). Here and elsewhere, we shall make no notational distinction between scalar and vector-valued functions; also, \( \nabla_{\tan} = -n \times (n \times \nabla) \) stands for the usual tangential gradient. Next, as in [20], we introduce the surface divergence operator

\[ \text{Div} : L^p_{\tan}(\partial \Omega) \to L^p_{-1}(\partial \Omega) := (L^1_{\tan}(\partial \Omega))^*, \quad 1/p + 1/q = 1, \quad (1.4) \]

by setting

\[ \int_{\partial \Omega} g \text{Div } f \, d\sigma = \int_{\partial \Omega} \langle f, \nabla_{\tan} g \rangle \, d\sigma, \quad (1.5) \]

for each \( f \in L^p_{\tan}(\partial \Omega) \), and \( g \in L^q_{\tan}(\partial \Omega) = (L^p_{\tan}(\partial \Omega))^* \). Also, for \( 1 < p < \infty \), we set

\[ L^p_{\tan}(\partial \Omega) := \{ f \in L^p_{\tan}(\partial \Omega); \text{Div } f \in L^p(\partial \Omega) \}, \quad (1.6) \]

which we equip with the norm

\[ \| f \|_{L^p_{\tan}(\partial \Omega)} := \| f \|_{L^p(\partial \Omega)} + \| \text{Div } f \|_{L^p(\partial \Omega)}. \quad (1.7) \]

Second, \( \mathcal{N}(\cdot) \) will stand for the non-tangential maximal operator acting on a function \( u : \Omega \to \mathbb{R} \) given at each boundary point \( x \) by

\[ \mathcal{N}(u)(x) := \sup \{|u(y)|; y \in \Omega, |x - y| \leq \text{dist}(y, \partial \Omega)\}. \quad (1.8) \]

Moreover, \( u|_{\partial \Omega} \) will denote the restriction of \( u \) to the boundary in the (pointwise) non-tangential limit sense.

Then, in this notation, the assumptions made on the boundary behavior for \( E \) and \( H \) are as follows:

\[ \mathcal{N}(E), \mathcal{N}(H) \in L^p(\partial \Omega) \quad \text{and} \quad n \times E|_{\partial \Omega} = f \in L^p_{\tan}(\partial \Omega), \quad (1.9) \]

where \( f \) is a given field. In this paper we solve the boundary value problem \(\{(1.3), (1.9)\}\), together with the corresponding version for \( \Omega_- \). In this latter case one must add the so-called radiation conditions:
\[
\begin{cases}
\frac{x}{|x|} \times H + \sqrt{\frac{\varepsilon}{\mu}} E = o\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty, \\
\frac{x}{|x|} \times E + \sqrt{\frac{\mu}{\varepsilon}} H = o\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty.
\end{cases}
\] (1.10)

In passing, let us point out that, much as in [6], one can show that the two conditions above are, in fact, equivalent to each other.

In the case of a perfectly conducting obstacle, the boundary datum in \((BVP_−)\) has the form

\[f = -n \times E_{\text{inc}} \quad \text{on} \quad \partial \Omega,\] (1.11)

where the incident wave \(E_{\text{inc}}\) has the form

\[E_{\text{inc}}(x) = q_1 \exp \left(ik\langle p_1, x \rangle/(1-k\beta)\right) + q_2 \exp \left(ik\langle p_2, x \rangle/(1+k\beta)\right), \quad x \in \mathbb{R}^3.\] (1.12)

Here, the direction vectors \(q_1, q_2\) and the polarization vectors \(p_1, p_2\) are related by

\[\langle p_j, q_j \rangle = 0, \quad p_j \times q_j = (-1)^j iq_j, \quad j = 1, 2.\] (1.13)

For a related discussion see also pp. 30-33 in [16]. It is visible from (1.11)-(1.13) that the smoothness of the domain influences, via the unit normal \(n\), the smoothness of the boundary datum \(f\). In particular, since \(n \in L^\infty\) only if \(\partial \Omega\) is Lipschitz, it is natural to consider \(f\) in (suitable) \(L^p\)-classes.

It is customary to translate the aforementioned boundary value problem in the language of Beltrami fields, an issue on which we now elaborate. Essentially, \(Q\) (defined either in \(\Omega_+\) or \(\Omega_-\)) is called Beltrami, if it is an eigenfield of the curl operator ([16]). That is, we shall call \(Q\) \(\lambda\)-Beltrami, for some \(\lambda \in \mathbb{R} \setminus \{0\}\), if \(\text{curl} Q = \lambda Q\). In the case of the unbounded domain \(\Omega_-\), one also requires that \(Q\) radiates, i.e.

\[\frac{x}{|x|} \times Q \pm iQ = o\left(\frac{1}{|x|}\right) \quad \text{as} \quad |x| \to \infty,\] (1.14)

where the ± sign is that of \(\lambda\). If we now set

\[Q_1 := \frac{1}{2}(E + i\eta H),\]
\[Q_2 := \frac{1}{2}(-E + i\eta H),\]

then \(\eta\) can be chosen so that \(Q_1, Q_2\) become Beltrami fields. Indeed, for \(\eta = \sqrt{\varepsilon}\), the so-called intrinsic impedance of the chiral medium, \(Q_j\) is \(\lambda_j\)-Beltrami, \(j = 1, 2\), with

\[\lambda_1 := \frac{k}{1-k\beta} > 0, \quad \lambda_2 := \frac{-k}{1+k\beta} < 0.\] (1.16)
It is important to point out that \( Q_1 \) and \( Q_2 \) are, respectively, left- and right-circularly polarized waves. Also, as opposed to the electric and magnetic waves, they propagate independently; cf. [15]. The formulas (1.15) are occasionally referred to as Bohren’s decomposition ([4]). In terms of Beltrami fields in \( \Omega_+ \), the system \{(1.3), (1.9)\} becomes

\[
\begin{cases}
\text{curl} \, Q_1 = \lambda_1 Q_1 \text{ in } \Omega_+, \\
\text{curl} \, Q_2 = \lambda_2 Q_2 \text{ in } \Omega_+, \\
\mathcal{N}(Q_1), \mathcal{N}(Q_2) \in L^p(\partial \Omega), \\
\left. n \times Q_1 \right|_{\partial \Omega} - \left. n \times Q_2 \right|_{\partial \Omega} = f \in L^{p, \text{Div}}_{\text{tan}}(\partial \Omega),
\end{cases}
\]

while the corresponding version in \( \Omega_- \) reads

\[
\begin{cases}
\text{curl} \, Q_1 = \lambda_1 Q_1 \text{ in } \Omega_-, \\
\text{curl} \, Q_2 = \lambda_2 Q_2 \text{ in } \Omega_-, \\
\mathcal{N}(Q_1), \mathcal{N}(Q_2) \in L^p(\partial \Omega), \\
\left. n \times Q_1 \right|_{\partial \Omega} - \left. n \times Q_2 \right|_{\partial \Omega} = f \in L^{p, \text{Div}}_{\text{tan}}(\partial \Omega), \\
\frac{x}{|x|} \times Q_1 + iQ_1 = o\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty, \\
\frac{x}{|x|} \times Q_2 - iQ_2 = o\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty.
\end{cases}
\]

Note that, as opposed to the original problem for the Drude-Born-Fedorov system, the problems \((BVP_\pm)\) are decoupled inside the domain but coupled on the boundary.

It is immediate that treatment of the original (interior/exterior) boundary problems for the Drude-Born-Fedorov system comes down to the solvability of \((BVP_\pm)\), since the electromagnetic waves can be recovered from the Beltrami fields as

\[
\left\{ \begin{array}{l}
E = Q_1 - Q_2, \\
H = \frac{1}{i\eta}(Q_1 + Q_2).
\end{array} \right.
\]

Thus, in the sequel, we shall focus exclusively on \((BVP_\pm)\).

Preparatory to stating our main results regarding these boundary problems, we need a couple of definitions. Recall from [13] that \((A, B)\) is called a Fredholm pair for the Banach space \(X\) if \(A, B\) are closed subspaces of \(X\) so that

\[
\dim(A \cap B) < \infty, \quad \dim(X/(A + B)) < \infty.
\]

In this case, one defines Index \((A, B) := \dim(A \cap B) - \dim(X/(A + B))\). Also, consider the space of “tangential” boundary traces of Beltrami fields

\[
\mathcal{B}^p_{\lambda, \text{+}}(\partial \Omega) := \left\{ n \times Q|_{\partial \Omega}; \, Q \lambda\text{-Beltrami in } \Omega_+, \, \mathcal{N}(Q) \in L^p(\partial \Omega) \right\}.
\]
The space $\mathcal{B}^p_{\lambda,-}(\partial\Omega)$ is introduced similarly except that, this time, we include the radiation condition (1.14) in the definition.

**Theorem 1.1** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^3$. Then there exists $\epsilon = \epsilon(\partial\Omega) > 0$ such that, for each $2 - \epsilon < p < 2 + \epsilon$ and each $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \cdot \lambda_2 < 0$, the subspaces $\left\{ \mathcal{B}^p_{\lambda_1,+}(\partial\Omega), \mathcal{B}^p_{\lambda_2,+}(\partial\Omega) \right\}$ form a Fredholm pair of index zero for $L^p_{\text{tan}}(\partial\Omega)$.

Our next result is an essentially equivalent reformulation of Theorem 1.1, in which the emphasis is on $(BVP)$. 

**Theorem 1.2** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^3$. Then there exists $\epsilon = \epsilon(\partial\Omega) > 0$ such that, for each $2 - \epsilon < p < 2 + \epsilon$, the problem $(BVP_+)$ is Fredholm solvable, of index zero.

The precise sense in which the above statement should be interpreted is as follows. A solution for $(BVP_+)$ exists if and only if the boundary datum is admissible, i.e. it satisfies finitely many compatibility conditions (which place it in a certain closed space of finite codimension). Also, the space of null solutions of $(BVP_+)$ is finite dimensional of dimension equal to the codimension of the space of admissible data.

Turning attention to exterior traces, we shall establish similar results.

**Theorem 1.3** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^3$. Then there exists $\epsilon = \epsilon(\partial\Omega) > 0$ such that for each $2 - \epsilon < p < 2 + \epsilon$, the subspaces $\left\{ \mathcal{B}^p_{\lambda_1,-}(\partial\Omega), \mathcal{B}^p_{\lambda_2,-}(\partial\Omega) \right\}$ form a Fredholm pair for $L^p_{\text{tan}}(\partial\Omega)$.

Moreover, if in addition $|\lambda_1 + \lambda_2| < \epsilon$, then in fact

$$\mathcal{B}^p_{\lambda_1,-}(\partial\Omega) \oplus \mathcal{B}^p_{\lambda_2,-}(\partial\Omega) = L^p_{\text{tan}}(\partial\Omega).$$

(1.20)

Here the direct sum is topological.

As a corollary of the above result we shall deduce the following.

**Theorem 1.4** For any bounded Lipschitz domain $\Omega$ in $\mathbb{R}^3$ there exists $\epsilon = \epsilon(\partial\Omega) > 0$ such that if $2 - \epsilon < p < 2 + \epsilon$ then the problem $(BVP_-)$ is Fredholm solvable. Furthermore, if also $|\beta| < \epsilon$ (i.e., the chiral admittance is small) then $(BVP_-)$ is actually uniquely solvable for each $p$ near 2.

It should be mentioned that $(BVP_{\pm})$ are in fact well posed for all but a discrete set of values of the electromagnetic parameters of the obstacle; see the remark made at the end of §3 for a more precise statement in this regard.

Lately there has been renewed interest in the study of electromagnetic scattering phenomena in chiral media (the original investigations of Arago, Biot, Cauchy,
Pasteur, Fresnel, etc., on the effect of chirality on the polarization of light date back to the 19th century). The fundamental feature of chirality is a rotation of the polarization of the plane waves propagating through the medium. For physical background, history and other pertinent comments the reader is referred to [17]; cf. also [4], [15] and [16]. The papers [23], [3] deal with transmission problems, [1] studies the behavior of the solution with respect to the chirality parameter, while [9] considers a two-dimensional inverse problem. See also [2], [11] for related work and more references.

The novelty of the present paper resides both in the formulation of the problems as well as in the analytical aspects of the setting we consider. Indeed, here we initiate a systematic study of (time-harmonic) electromagnetic scattering in chiral media from the perspective of theory for elliptic problems with minimal smoothness assumptions. The aim is to extend the theory developed in [21], [20] in the case of a dielectric scatterer to also cover the case of a chiral scatterer. Thus, the emphasis is on domains with irregular boundaries (here, Lipschitz), as well as on discontinuous boundary data (typically, $L^p$ classes). Our method is that of (singular) integral operators. As such, we take advantage of the recent progress made in the case of scattering by rough obstacles in achiral media in [21], [20], [22]. The present work is therefore a natural continuation of the program initiated in these references. In a subsequent paper, we shall consider the issue of (three-dimensional) electromagnetic inverse problems for irregular scatterers in chiral media. For the case of non-smooth, dielectric obstacles see [19] where some of the ideas in [10] are extended in order to prove a uniqueness result.

In the larger context of harmonic analysis and PDE’s with low regularity assumptions, the techniques and results in this paper owe a great deal to earlier work of many people. For a survey of the state of the art in this area up to the early 90’s the reader may consult Kenig’s book [14]. The layout of our paper is as follows. In §2 we introduce the some natural integral operators and review their main properties relevant for the problems under discussion. Here we also detail on the connection between the so-called Beltrami fields and Maxwell’s equations. The proofs of Theorems 1.1-1.4 are discussed in §3. At the end of §3 we also include a remark which complements the statements of these theorems.

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2 Beltrami fields and related operators

In the treatment of $(BPV_{\pm})$ we will use integral operators well suited for the problems under discussion. We debut with an observation pertaining to the relation between Beltrami fields and Maxwell’s equations. For this we need to recall a definition. A vector field $U$, satisfying Helmholtz’s equation $(\Delta + \lambda^2)U = 0$ for some $\lambda \in \mathbb{R} \setminus \{0\}$, is said to radiate if it is defined in a neighborhood of infinity
and satisfies
\[
\text{curl } U \times \frac{x}{|x|} + (\text{div } U) \frac{x}{|x|} - i|\lambda|U = o(|x|^{-1}) \quad \text{as } |x| \to \infty. \tag{2.1}
\]
It is well known (cf., e.g., [6]) that, in addition to being annihilated by \(\Delta + \lambda^2\), \(U\) is also divergence-free, then
\[
U \text{ radiates } \iff \text{curl } U \text{ radiates}. \tag{2.2}
\]

**Proposition 2.1** If \(\lambda \in \mathbb{R} \setminus \{0\}\) and \(Q\) is a \(\lambda\)-Beltrami field, i.e. it satisfies the circulation equation \(\text{curl } Q = \lambda Q\), then
\[
(\Delta + \lambda^2)Q = 0 \quad \text{and} \quad \text{div } Q = 0. \tag{2.3}
\]
Conversely, if a field \(Q\) satisfies (2.3) then
\[
\tilde{Q} := \lambda Q + \text{curl } Q \tag{2.4}
\]
is \(\lambda\)-Beltrami. If, in addition, \(Q\) radiates at infinity then
\[
\text{sign } (\lambda) i\tilde{Q} - \tilde{Q} \times \frac{x}{|x|} = o(|x|^{-1}) \quad \text{as } |x| \to \infty. \tag{2.5}
\]
Finally, a \(\lambda\)-Beltrami field \(Q\) satisfies the radiation condition (2.5) if and only if it satisfies (2.1).

**Proof.** The first part of the proposition follows from elementary algebra. The part referring to radiation conditions is a straightforward consequence of (2.1) and (2.2). \(\square\)

In the light of the above proposition it is natural to employ layer potentials associated with the (vector) Helmholtz operator \(\Delta + \lambda^2\). For \(\lambda \in \mathbb{C}\) we set
\[
\Phi_\lambda(x) := -\frac{e^{\pm i|\lambda|x}}{4\pi|x|}, \quad x \in \mathbb{R}^3 \setminus \{0\}, \tag{2.6}
\]
where the sign in the exponent is that of \(\lambda\) if \(\lambda \in \mathbb{R}\) and that of \(\text{Im } \lambda\) otherwise. As is well known, \(\Phi_\lambda\) is the standard radial fundamental solution for the Helmholtz operator \(\Delta + \lambda^2\), \(\lambda \in \mathbb{C}\). The single layer potential operator associated with \(\Phi_\lambda\) is defined for \(f \in L^p(\partial \Omega), 1 < p < \infty\), by
\[
\mathcal{S}_\lambda f(x) := \int_{\partial \Omega} \Phi_\lambda(x - y)f(y)\,d\sigma(y), \quad x \in \mathbb{R}^3 \setminus \partial \Omega. \tag{2.7}
\]
The non-tangential boundary traces of \(\mathcal{S}_\lambda f\) are given by
\[
\mathcal{S}_\lambda f := \mathcal{S}_\lambda f|_{\partial \Omega_+} = \mathcal{S}_\lambda f|_{\partial \Omega_-}. \tag{2.8}
\]
For natural radiations conditions for (derivatives of) $S_\lambda$, the reader may consult [6].

Another basic boundary integral operator is

$$M_\lambda f(x) = n(x) \times \text{p.v.} \int_{\partial \Omega} \text{curl}_x(\Phi_\lambda(x-y)f(y))\, d\sigma(y), \quad x \in \mathbb{R}^3 \setminus \partial \Omega,$$

where p.v. stands for principal value. It turns out that the following jump relations hold

$$\text{curl} S_\lambda f|_{\partial \Omega^\pm} = \mp \frac{1}{2} n \times f + \text{p.v.} \int_{\partial \Omega} (\nabla \Phi_\lambda)(\cdot - y) \times f(y)\, d\sigma(y).$$

(2.10)

It is known that for any $f \in L^p_{\text{tan}}(\partial \Omega)$ one has

$$n \times (\text{curl } S_\lambda f)|_{\partial \Omega^\pm} = (\pm \frac{1}{2} I + M_\lambda) f,$$

(2.11)

where $I$ is the identity operator.

Finally, we will also make use of the operator

$$N_\lambda : L^p_{\text{tan}}(\partial \Omega) \to L^p_{\text{tan}}(\partial \Omega), \quad 1 < p < \infty,$$

(2.12)

defined by

$$N_\lambda f := n \times (\text{curl } S_\lambda f)|_{\partial \Omega^+} = n \times (\text{curl } S_\lambda f)|_{\partial \Omega^-}.\quad (2.13)$$

Indeed, to see that $N_\lambda$ is well defined we combine the formulas

$$\Delta = -\text{curl curl} + \nabla \text{div}$$

(2.14)

$$\text{Div}(n \times u) = -\langle n, \text{curl } u \rangle$$

(2.15)

with the fact that

$$\text{div } S_\lambda f = S_\lambda(\text{Div } f), \quad \text{for } f \in L^p_{\text{tan}}(\partial \Omega).$$

(2.16)

The conclusion is that

$$N_\lambda f = \lambda^2 n \times S_\lambda f + n \times \nabla S_\lambda(\text{Div } f), \quad \text{for } f \in L^p_{\text{tan}}(\partial \Omega).$$

(2.17)

As is well known by now, there are fundamental differences between the case when $\partial \Omega$ is smooth and the case when $\partial \Omega$ is only Lipschitz. As opposed to the former, in the latter situation the operator $M_\lambda$ cease to have a weakly singular kernel, a fact which drastically alters the functional analytic properties of $\pm I + M_\lambda$.

For a more comprehensive discussion, the reader is referred to [21], [20] and the references therein.

Next, we summarize the main properties of the above operators.
Proposition 2.2 Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^3$, $\lambda \in \mathbb{C}$ and $1 < p < \infty$. Then the following are true.

1. $\|N(\nabla S_{\lambda})\|_{L^p(\partial \Omega)} \leq C\|f\|_{L^p(\partial \Omega)}$. Hence, in particular, the operator $S_{\lambda} : L^p(\partial \Omega) \to L^1_p(\partial \Omega)$ is bounded.

2. $M_{\lambda}$ is a bounded operator when acting on $L^p_{\tan}(\partial \Omega)$ and on $L^p_{\tan}(\partial \Omega)$.

3. There exists $\epsilon = \epsilon(\partial \Omega) > 0$ such that $\xi I + M_{\lambda}$ is Fredholm with index zero for each $\xi \in \mathbb{R}$ with $|\xi| \geq 1/2$ both on $L^p_{\tan}(\partial \Omega)$ and on $L^p_{\tan}(\partial \Omega)$, as long as $|p - 2| < \epsilon$.

4. If $2 - \epsilon < p < q < 2 + \epsilon$, $\xi \in \mathbb{R}$ with $|\xi| \geq 1/2$, and $f \in L^p_{\tan}(\partial \Omega)$ is such that $(\xi I + M_{\lambda})f \in L^q(\partial \Omega)$, then $f \in L^q_{\tan}(\partial \Omega)$.

5. If $2 - \epsilon < p \leq q < 2 + \epsilon$, $\xi \in \mathbb{R}$ with $|\xi| \geq 1/2$, and $f \in L^p_{\tan}(\partial \Omega)$ is such that $(\xi I + M_{\lambda})f \in L^q_{\tan}(\partial \Omega)$ then, necessarily, $f \in L^q_{\tan}(\partial \Omega)$.

6. $M_{\lambda_1} - M_{\lambda_2}$ is compact on $L^p_{\tan}(\partial \Omega)$ and on $L^p_{\tan}(\partial \Omega)$.

7. $N_{\lambda_1} - N_{\lambda_2} : L^p_{\tan}(\partial \Omega) \to L^p_{\tan}(\partial \Omega)$ is compact.

8. $N_{\lambda_1} - N_{\lambda_2} : L^p_{\tan}(\partial \Omega) \to L^p_{\tan}(\partial \Omega)$ is bounded.

9. $M_{\lambda}N_{\lambda} = -N_{\lambda}M_{\lambda}$ on $L^p_{\tan}(\partial \Omega)$.

10. $N^2_{\lambda} = -\lambda^2(\frac{1}{2} I + M_{\lambda})(-\frac{1}{2} I + M_{\lambda})$ on $L^p_{\tan}(\partial \Omega)$.

Proof. Relying on the deep results in [5], the points (1) – (5) have been proved in [20]. As for (6) – (8), the key observation is that

$$\nabla_x \nabla_y \left( e^{i|\xi - \eta|} - 1 \right) = O\left( \frac{1}{|x - y|} \right), \quad \text{as } |x - y| \to 0. \quad (2.18)$$

Consider, for example, (7). Since by (2.17)

$$N_{\lambda_1} - N_{\lambda_2} = \lambda^2_1 n \times S_{\lambda_1} - \lambda^2_2 n \times S_{\lambda_2} + n \times \nabla(S_{\lambda_1} - S_{\lambda_2})(\text{Div}), \quad (2.19)$$

and since, by (2.18),

$$S_{\lambda_1} - S_{\lambda_2} : L^p_{\tan}(\partial \Omega) \to L^p_{\tan}(\partial \Omega)$$

the conclusion in (7) readily follows.

To see (9), we write Green’s formula (cf. Corollary 3.3, p.142 in [20]) for $u := \text{curl} S_{\lambda} f$, with $f \in L^p_{\tan}(\partial \Omega)$, to obtain

\[9\]
\[
\text{curl } S_\lambda f = \text{curl } S_\lambda ((\frac{1}{2} I + M_\lambda) f) - \nabla S_\lambda (\langle n, \text{curl } f \rangle) + S_\lambda (N_\lambda f). \tag{2.21}
\]

Applying curl to both sides, going (nontangentially) to the boundary and then taking \( n \times \cdot \) of both sides ultimately yields

\[
N_\lambda f = N_\lambda ((\frac{1}{2} I + M_\lambda) f) + ((\frac{1}{2} I + M_\lambda) N_\lambda f), \tag{2.22}
\]
or \( N_\lambda M_\lambda f + M_\lambda N_\lambda f = 0 \), as desired. Finally, the identity (10) is proved in Lemma 5.10, p. 150 in [20].

**Remark.** Thanks to (2.18), it can be easily checked that all the compact operators in the statement of Proposition 2.2 are also smoothing. That is, for each \( 1 < p < \infty \), they map \( L^p(\partial \Omega) \) boundedly into \( L^{p+\delta}(\partial \Omega) \) for some \( \delta = \delta(p) > 0 \). Hereafter, we shall let the superscript \( t \) denote transposition of operators.

**Proposition 2.3** For each \( 1 < p < \infty \), \( (\pm \frac{1}{2} I + M_\lambda)^t \), the transposed of \( \pm \frac{1}{2} I + M_\lambda \) acting on \( L^p_{\text{tan}}(\partial \Omega) \) are, respectively, \( n \times (\mp \frac{1}{2} I + M_\lambda) n \times \) acting on \( L^q_{\text{tan}}(\partial \Omega) \), \( 1/p + 1/q = 1 \).

Furthermore, \( (N_{\lambda_1} - N_{\lambda_2})^t \), the transposed of \( N_{\lambda_1} - N_{\lambda_2} \) acting on \( L^p_{\text{tan}}(\partial \Omega) \) is the operator \( n \times (N_{\lambda_1} - N_{\lambda_2}) n \times \) acting on \( L^q_{\text{tan}}(\partial \Omega) \), \( 1/p + 1/q = 1 \).

**Proof.** The first part, dealing with the transposed of \( M_\lambda \) has been proved in [20]; cf. (5.27) there. Hence we are left with identifying the transposed of \( N_{\lambda_1} - N_{\lambda_2} \). In turn, this is an immediate consequence of the fact that \( L^p_{\text{tan}}(\partial \Omega) \) is dense in \( L^p_{\text{tan}}(\partial \Omega) \), the point (7) in Proposition 2.2 and the identity

\[
\int_{\partial \Omega} \langle N_\lambda f, n \times g \rangle \, d\sigma = - \int_{\partial \Omega} \langle f, n \times N_\lambda g \rangle \, d\sigma, \tag{2.23}
\]
valid for each \( f \in L^p_{\text{tan}}(\partial \Omega), g \in L^q_{\text{tan}}(\partial \Omega), 1/p + 1/q = 1 \), which we now tackle. By (2.17), we have

\[
\int_{\partial \Omega} \langle N_\lambda f, n \times g \rangle \, d\sigma = \lambda^2 \int_{\partial \Omega} \langle n \times S_\lambda f, n \times g \rangle \, d\sigma \tag{2.24}
\]

\[
+ \int_{\partial \Omega} \langle n \times \nabla S_\lambda (\text{Div } f), n \times g \rangle \, d\sigma
\]

\[
= \lambda^2 \int_{\partial \Omega} \langle f, S_\lambda g \rangle \, d\sigma + \int_{\partial \Omega} \langle \nabla_{\text{tan}} S_\lambda (\text{Div } f), g \rangle \, d\sigma
\]

\[
=: \, I + II,
\]

using the tangentiality of \( f \) and \( g \) (i.e., \( f = -n \times (n \times f) \), etc.) and the fact that \( S_\lambda^t = S_\lambda \). We continue with \( II \) and make use of the formula (1.5). Thus,
\[ II = \int_{\partial \Omega} \langle S_\lambda(\text{Div } f), \text{Div } g \rangle \, d\sigma = \int_{\partial \Omega} \langle \text{Div } f, S_\lambda(\text{Div } g) \rangle \, d\sigma \quad (2.25) \]

\[ = \int_{\partial \Omega} \langle f, \nabla_{\text{tan}} S_\lambda(\text{Div } g) \rangle \, d\sigma. \]

Combining (2.24), (2.25) and recalling (2.17), the identity (2.23) follows. \(\square\)

In order to state our next result we need to introduce the operators

\[ P^\pm_\lambda := \pm \frac{1}{2} I + M_\lambda + \lambda^{-1} N_\lambda \quad (2.26) \]

for each \( \lambda \neq 0 \).

**Theorem 2.4** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^3 \) and fix an arbitrary \( 1 < p < \infty \). Then, for each \( \lambda \in \mathbb{R} \setminus \{0\} \), the interior Beltrami boundary value problem

\[ (\text{Beltrami}_+) \begin{cases} \text{curl } Q = \lambda Q \text{ in } \Omega_+, \\ \mathcal{N}(Q) \in L^p(\partial \Omega) \\ n \times Q \bigg|_{\partial \Omega} = f \in L^p_{\text{tan}}(\partial \Omega), \end{cases} \]

is solvable if and only if \( f \in \text{Ker } (P^-_\lambda; L^p_{\text{tan}}(\partial \Omega)) \). Also, the exterior Beltrami boundary value problem

\[ (\text{Beltrami}_-) \begin{cases} \text{curl } Q = \lambda Q \text{ in } \Omega_-, \\ \mathcal{N}(Q) \in L^p(\partial \Omega) \\ n \times Q \bigg|_{\partial \Omega} = f \in L^p_{\text{tan}}(\partial \Omega), \end{cases} \]

\[ \frac{x}{|x|} \times Q + \text{sign}(\lambda) iQ = o \left( \frac{1}{|x|} \right) \text{ as } |x| \to \infty, \]

is solvable if and only if \( f \) belongs to \( \text{Ker } (P^+_\lambda; L^p_{\text{tan}}(\partial \Omega)) \). In both cases, whenever it exists the solution is unique.

**Proof.** Green's formula gives

\[ Q = \pm \text{curl } S_\lambda(n \times Q) \mp \nabla S_\lambda(\langle n, Q \rangle) \pm \lambda S_\lambda(n \times Q) \quad \text{in } \Omega_\pm. \quad (2.27) \]

Applying curl to both sides gives

\[ \lambda Q = \pm \lambda \text{curl } S_\lambda(n \times Q) \pm \text{curl } S_\lambda(n \times Q) \quad \text{in } \Omega_\pm, \quad (2.28) \]

and, after going (nontangentially) to the boundary, taking \( n \times \cdot \) of both sides, plus some algebra,
\[
(\mp \frac{1}{2} I + M_\lambda + \lambda^{-1} N_\lambda)(n \times Q|_{\partial \Omega}) = 0. \tag{2.29}
\]
This proves the necessity of the membership of \(f\) to the kernel of \(P^\mp_\lambda\) in order for \((Beltrami_\pm)\) to be solvable.

Conversely, if \(f \in \text{Ker}(P^\mp_\lambda; L^p_{\text{tan}}(\partial\Omega))\), then \(\pm f = (\pm \frac{1}{2} I + M_\lambda + \lambda^{-1} N_\lambda)f\). Granted this, it is easy to check that

\[
Q := \text{curl} S_\lambda f + \lambda^{-1} \text{curl curl} S_\lambda f \quad \text{in} \ \Omega_\pm \tag{2.30}
\]
solves \((Beltrami_\pm)\). Uniqueness for \((Beltrami_\pm)\) is immediate from (2.28). □

For further reference, below we collect the main properties of the operators \(P^\pm_\lambda\).

**Proposition 2.5** Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^3\), \(\lambda \in \mathbb{R}\setminus\{0\}\) and fix \(1 < p < \infty\). Then the following are true.

1. \(P^+_\lambda - P^-_\lambda = I\), the identity of \(L^p_{\text{tan}}(\partial\Omega)\).
2. \(P^+_\lambda P^-_\lambda = P^-_\lambda P^+_\lambda = 0\) and \((P^\pm_\lambda)^2 = \pm P^\pm_\lambda\) on \(L^p_{\text{tan}}(\partial\Omega)\).
3. For each \(f \in L^p_{\text{tan}}(\partial\Omega)\) and \(g \in L^q_{\text{tan}}(\partial\Omega)\), \(1/p + 1/q = 1\), there holds

\[
\int_{\partial\Omega} \langle P^\pm_\lambda f, n \times g \rangle \, d\sigma = -\int_{\partial\Omega} \langle f, n \times P^\pm_\lambda g \rangle \, d\sigma. \tag{2.31}
\]

*Proof.* (1) is clear from definitions, while (2) and (3) follow from Proposition 2.2, Proposition 2.3 and some algebra. □

## 3 The proofs of the main results

First we will prove uniqueness for the exterior problem, stated in §1.

**Proposition 3.1** If a solution for \((BVP_-)\) exists, then it is unique.

*Proof.* We shall treat first the case when \(p = 2\). To this end, let \((Q_1, Q_2)\) be a solution for \((BVP_-)\) for \(f = 0\). For each \(R > 0\) let \(B_R(0)\) be the ball of radius \(R\) centered at the origin. Then, using the radiation conditions in \((BVP_-)\) we have

\[
0 = \lim_{R \to \infty} \int_{\partial B_R(0)} |n \times Q_j|^2 + |Q_j|^2 + 2(-1)^j \text{Im} \langle Q_j, n \times \overline{Q}_j \rangle, \quad j = 1, 2. \tag{3.1}
\]

On the other hand, for \(R\) large enough, integrations by parts for the Beltrami fields \(Q_j, j = 1, 2\), give
\[ \int_{\partial B_R(0)} \langle n \times \overline{Q}_j, Q_j \rangle - \int_{\partial \Omega} \langle n \times \overline{Q}_j, Q_j \rangle = -2i \text{Im}(\lambda_j) \int_{B_R(0) \cap \Omega_-} |Q_j|^2. \quad (3.2) \]

Adding the two versions of (3.1), corresponding to \( j = 1 \) and \( j = 2 \), and then utilizing (3.2), the condition

\[ \langle n \times \overline{Q}_1, Q_1 \rangle = \langle n \times \overline{Q}_2, Q_2 \rangle \text{ on } \partial \Omega, \]

and the fact that \( \text{Im}(\lambda_j) = 0, \ j = 1, 2 \), finally leads to

\[ \lim_{R \to \infty} \int_{\partial B_R(0)} |Q_j|^2 = 0, \ j = 1, 2. \quad (3.4) \]

It should be stressed that here we make essential use of the fact that \( Q_1 \) and \( Q_2 \) have opposite polarizations. Since each \( Q_j \) solves \( (\Delta + \lambda_j^2)Q_j = 0 \), the above implies, thanks to Rellich’s lemma, that \( Q_j \equiv 0 \) near \( \infty, j = 1, 2 \). By the unique continuation principle, the latter entails \( Q_j = 0 \) in \( \Omega_- \), \( j = 1, 2 \), which finishes the case \( p = 2 \).

The general case, \( 2 - \epsilon < p < 2 + \epsilon \), can be reduced to the previous situation via a boot-strap argument which we now sketch. Set \( f := n \times Q_1 = n \times Q_2 \in L^p_{\text{tan}}(\partial \Omega) \) so that \( P_{\lambda_j}^+f = P_{\lambda_j}^+f = 0 \). Now, (2.26) and Proposition 2.2 allow us to write

\[ 0 = \lambda_1 P_{\lambda_1}^+f - \lambda_2 P_{\lambda_2}^+f = (\lambda_1 - \lambda_2)(\frac{1}{2}I + M_{\lambda})f + \text{Smoothing}(f). \quad (3.5) \]

Here, \( \lambda \in \mathbb{R} \) is arbitrary and “Smoothing” stands for a generic soothing operator in the sense of the remark following the proof of Proposition 2.2. Together with the point (4) in Proposition 2.2, this implies that \( f \in L^{p+\delta}_{\text{tan}}(\partial \Omega) \) for some \( \delta > 0 \). Furthermore, in concert with Green’s representation formula (2.28), this yields \( \mathcal{N}(Q_j) \in L^{p+\delta}(\partial \Omega), j = 1, 2 \). This process can be iterated several times and the final conclusion is that \( \mathcal{N}(Q_j) \in L^2(\partial \Omega), j = 1, 2 \), as desired.

Now we are ready to discuss \((BVP_+)\), stated in the introduction.

**The proof of Theorem 1.2.** Inspired by Proposition 2.1, we are looking for a solution of \((BVP_+)\) in the form

\[ Q_j := \lambda_j \text{curl } S_{\lambda_j}g + \text{curl curl } S_{\lambda_j}g, \quad j = 1, 2, \quad (3.6) \]

where \( g \in L^p_{\text{tan}}(\partial \Omega) \) is to be determined later. Then \((Q_1, Q_2)\) verifies the interior conditions in \((BVP_+)\). Also, the boundary condition becomes

\[ f = \lambda_1 \left( \frac{1}{2}I + M_{\lambda_1} \right)g - \lambda_2 \left( \frac{1}{2}I + M_{\lambda_2} \right)g + (N_{\lambda_1} - N_{\lambda_2})g. \quad (3.7) \]

The equation (3.7) suggests the introduction of the operators
\[ T_\pm := \lambda_1 \left( \pm \frac{1}{2} I + M_{\lambda_1} \right) - \lambda_2 \left( \pm \frac{1}{2} I + M_{\lambda_2} \right) + (N_{\lambda_1} - N_{\lambda_2}), \quad (3.8) \]

which are well defined and bounded both on \( L^p_{\text{tan}}(\partial \Omega) \), as well as on \( L^p_{\text{tan}}(\partial \Omega) \), for each \( 1 < p < \infty \).

The next four propositions are devoted to studying finer mapping properties of these operators.

**Proposition 3.2** There exists \( \epsilon = \epsilon(\partial \Omega) > 0 \) such that the operators \( T_\pm \) are Fredholm with index zero both on \( L^p_{\text{tan}}(\partial \Omega) \) and on \( L^p_{\text{tan}}(\partial \Omega) \) for each \( 2 - \epsilon < p < 2 + \epsilon \) as long as \( \lambda_1 \neq \lambda_2 \).

**Proof.** At the level of \( L^p_{\text{tan}}(\partial \Omega) \), this is a direct consequence of (3) in Proposition 2.2, basic functional analysis and the identity

\[ T_\pm = (\lambda_1 - \lambda_2) \left( \pm \frac{1}{2} I + M_z \right) + \text{Compact}(\lambda_1, \lambda_2, z), \quad (3.9) \]

where \( z \in \mathbb{R} \) is fixed, which in turn is derived from (3.8) and (6) – (7) in Proposition 2.2.

Nevertheless, the argument for \( L^p_{\text{tan}}(\partial \Omega) \) requires a new idea since \( N_{\lambda_1} - N_{\lambda_2} \) is no longer compact when acting on the space under discussion. In order to circumvent this difficulty, we shall find it useful to work with the family of operators

\[ T_\xi := \lambda_1 (\xi I + M_{\lambda_1}) - \lambda_2 (\xi I + M_{\lambda_2}) + (N_{\lambda_1} - N_{\lambda_2}), \quad (3.10) \]

defined for each \( \xi \in \mathbb{R} \setminus (-1/2, 1/2) \). Note that, in particular, \( T_{\pm 1/2} = T_\pm \). Much as before, \( T_\xi \) is Fredholm with index zero when acting on \( L^p_{\text{tan}}(\partial \Omega) \), for each \( 2 - \epsilon < p < 2 + \epsilon \). Our aim is to prove a similar statement at the level of \( L^p_{\text{tan}}(\partial \Omega) \). An easy first step in this direction is the observation that, for each \( 2 - \epsilon < p < 2 + \epsilon \),

\[ \dim \ker (T_\xi; L^p_{\text{tan}}(\partial \Omega)) \leq \dim \ker (T_\xi; L^p_{\text{tan}}(\partial \Omega)) < +\infty. \quad (3.11) \]

In order to continue, we shall need a regularity result to the effect that

\[ f \in L^p_{\text{tan}}(\partial \Omega) \text{ and } T_\xi f \in L^p_{\text{tan}}(\partial \Omega) \Rightarrow f \in L^p_{\text{tan}}(\partial \Omega) \]

again, when \( 2 - \epsilon < p < 2 + \epsilon \). Indeed, this is a simple consequence of the points (5) and (8) in Proposition 2.2. Next we claim that, under the current assumptions on \( \xi \) and \( p \), \( T_\xi \) has a closed range on \( L^p_{\text{tan}}(\partial \Omega) \). This is immediate from the fact that \( T_\xi \) has closed image on \( L^p_{\text{tan}}(\partial \Omega) \) and the regularity statement (3.12).
Summarizing, at this stage we have shown that $T_\xi$ has closed range and finite dimensional kernel on $L^p_{\tan}(\partial \Omega)$ for each $2 - \epsilon < p < 2 + \epsilon$ and each $\xi \in \mathbb{R}$ with $|\xi| \geq 1/2$. Thus, $(T_\xi)_\xi$ is a family of semi-Fredholm operators on $L^p_{\tan}(\partial \Omega)$ which depends continuously on the parameter $\xi$. Consequently, in either case $\xi < -1/2$ or $\xi > 1/2$, index $(T_\xi; L^p_{\tan}(\partial \Omega))$ is independent of $\xi$. Since, obviously, $T_\xi$ is invertible for $|\xi|$ large, the desired conclusion follows.

**Proposition 3.3** For each $1 < p, q < \infty$ with $1/p + 1/q = 1$, $(T_\pm)^t$, the transposed of $T_\pm$ acting on $L^p_{\tan}(\partial \Omega)$ is $n \times (T_\pm (n \times \cdot))$ acting on $L^q_{\tan}(\partial \Omega)$.

**Proof.** This is a straightforward corollary of definitions and Proposition 2.3. □

**Proposition 3.4** There exists $\epsilon = \epsilon(\partial \Omega) > 0$ such that for each $2 - \epsilon < p < 2 + \epsilon$, there holds

$$\dim(\text{ker}(T_\pm; L^p_{\tan}(\partial \Omega))) = \dim(\text{ker}(T_\pm; L^2_{\tan}(\partial \Omega))) = \dim(\text{ker}(T_\pm; L^{2,\text{Div}}_{\tan}(\partial \Omega))).$$

(3.13)

**Proof.** First, using a boot-strap argument similar in spirit to the one utilized in the proof of Proposition 3.1, it is not difficult to show that $\text{ker}(T_\pm; L^p_{\tan}(\partial \Omega))$ do not actually depend on $p \in (2 - \epsilon, 2 + \epsilon)$.

Next, using Proposition 3.3, and the fact that $T_\pm$ are Fredholm operators on $L^p_{\tan}(\partial \Omega)$ for $p$ near 2, we have

$$\dim[\text{ker}(T_\pm; L^p_{\tan}(\partial \Omega))] = \dim[(\text{im}(T_\pm)^t; L^p_{\tan}(\partial \Omega))^\circ] = \dim[(\text{im}(T_\pm; L^2_{\tan}(\partial \Omega))^\circ] = \dim[\text{ker}(T_\pm; L^2_{\tan}(\partial \Omega))].$$

(3.14)

where $1/p + 1/q = 1$, and $\{\ldots\}^\circ$ stands for the annihilator of $\{\ldots\}$.

Finally, the last equality in (3.13) is a consequence of the regularity result (3.12). □

**Proposition 3.5** $\text{null}(BVP_+)$, the space of null solutions for $(BVP_+)$ is independent of $p \in (2 - \epsilon, 2 + \epsilon)$ and

$$\dim(\text{ker}(T_\pm; L^2_{\tan}(\partial \Omega))) = \dim(\text{null}(BVP_+)).$$

(3.15)

In particular, $\dim(\text{null}(BVP_+)) < \infty$.

**Proof.** The first part of the statement can be seen by reasoning much as we have done in the last part of the proof of Proposition 3.4; we omit the details.
With an eye toward (3.15), let \( g \in \text{Ker}(T_+; L^2_{\text{tan}}(\partial \Omega)) \) so that, by (3.12), \( g \in L^2_{\text{Div}}(\partial \Omega) \), then define \( Q_j, j = 1, 2 \), as in (3.6). It follows that \( T_+ g = 0 \) and we can define the mapping
\[
\text{Ker} T_+ \ni g \mapsto (Q_1, Q_2) \in \text{Null} (BP_+). \tag{3.16}
\]

We now claim that \( \Phi \) is one-to-one. Indeed, let \( g \in \text{Ker} T_+ \) be such that \((Q_1, Q_2) = \Phi(g) = 0 \) in \( \Omega_+ \). In particular, going to the boundary and taking \( n \times \cdot \) of both sides in (3.6) yields \( P^+_j g = 0, j = 1, 2 \). That is, \( g \in \text{Ker} (P^+_j; L^2_{\text{tan}}(\partial \Omega)) \) for \( j = 1, 2 \). Thanks to Theorem 2.4, this further guarantees the existence of \( \hat{Q}_j \), radiating, \( \lambda_j \)-Beltrami field in \( \Omega_- \) such that \( \mathcal{N}(\hat{Q}_j) \in L^2(\partial \Omega) \) and \( n \times \hat{Q}_j|_{\partial \Omega_-} = g, j = 1, 2 \). Consequently, \((\hat{Q}_1, \hat{Q}_2)\) is a null-solution of \((BP_-)\). Proposition 3.1 then forces \( \hat{Q}_1 = \hat{Q}_2 = 0 \) in \( \Omega_- \) so that \( g = n \times \hat{Q}_j|_{\partial \Omega_-} = 0, \) as desired.

Going further, let us observe that if \((Q_1, Q_2) \in \text{Null} (BP_+)\) is arbitrary and we define \( g := n \times Q_1|_{\partial \Omega_+} = n \times Q_2|_{\partial \Omega_+} \), then \( g \in \text{Ker} (P^-_j; L^2_{\text{Div}}(\partial \Omega)) \) for \( j = 1, 2 \). Thus, \( T_- g = \lambda_1 P^-_1 g - \lambda_2 P^-_2 g = 0 \). Therefore, the mapping
\[
\text{Null} (BP_+) \ni (Q_1, Q_2) \mapsto g \in \text{Ker} T_- \tag{3.17}
\]
is well defined and linear. In fact, so we claim, \( \Psi \) is one-to-one. Indeed, if \((Q_1, Q_2) \in \text{Null} (BP_+)\) satisfy \( n \times Q_1 = n \times Q_2 = 0 \), using Green’s formula we get immediately that \( Q_1 = Q_2 = 0 \) in \( \Omega_+ \).

As a corollary of the above reasoning, we have
\[
\dim (\text{Ker} T_+) \leq \dim (\text{Null} (BP_+)) \leq \dim (\text{Ker} T_-). \tag{3.18}
\]
Since all spaces involved are finite dimensional (cf. Proposition 3.2), at this point Proposition 3.5 follows from (3.18) and Proposition 3.4.

**Proposition 3.6** There exists \( \epsilon = \epsilon(\partial \Omega) > 0 \) with the following significance. Fix \( p \in (2 - \epsilon, 2 + \epsilon) \) and let \( f \in L^p_{\text{Div}}(\partial \Omega) \) be arbitrary. Then there exists a solution of \((BP_+)\) for this choice of the boundary datum if and only if \( f \in \text{Im}(T_+; L^p_{\text{tan}}(\partial \Omega)) \).

**Proof.** The right-to-left implication is more or less immediate from (3.6) and jump-relations. In order to see the opposite implication, let \((Q_1, Q_2)\) be a solution of \((BP_+)\) for the boundary datum \( f \). The aim is to prove that \( f \) belongs to \( \text{Im}(T_+; L^p_{\text{tan}}(\partial \Omega)) \).

Due to the a priori regularity of \( f \) and thanks to the regularity result (3.12), it suffices to show that
\[
f \in \text{Im}(T_+; L^p_{\text{tan}}(\partial \Omega)). \tag{3.19}
\]
Since \( T_+ \) has closed range, (3.19) is further equivalent with checking that
where $1/p + 1/q = 1$. To see this, let $g \in L^q_{\tan}(\partial \Omega)$ be such that $n \times T_-(n \times g) = 0$. Clearly, this entails $T_-(n \times g) = 0$. Next, for $j = 1, 2$ set

$$\hat{Q}_j := \lambda_j \text{curl} S_{\lambda_j}(n \times g) + \text{curl curl} S_{\lambda_j}(n \times g) \quad \text{in} \quad \Omega_\pm. \quad (3.21)$$

When considered in $\Omega_-$, $\hat{Q}_j$ is a radiating, $\lambda_j$-Beltrami field, $j = 1, 2$, and

$$n \times \hat{Q}_1|_{\partial \Omega_-} - n \times \hat{Q}_2|_{\partial \Omega_-} = T_-(n \times g) = 0. \quad (3.22)$$

In other words, when restricted to $\Omega_-$, the pair $(\hat{Q}_1, \hat{Q}_2)$ becomes a null-solution of $(BVP_-)$. Thus, $\hat{Q}_1 = \hat{Q}_2 = 0$ in $\Omega_-$ by Proposition 3.1. As a consequence, $n \times \hat{Q}_j|_{\partial \Omega_-} = 0$, $j = 1, 2$. This, (3.21), and the jump-relations for layer potentials next imply that

$$\frac{1}{\lambda_1} n \times \hat{Q}_1|_{\partial \Omega_+} - n \times \hat{Q}_2|_{\partial \Omega_+} = n \times g. \quad (3.23)$$

With this at hand and recalling that $f = n \times Q_1|_{\partial \Omega_+} - n \times Q_2|_{\partial \Omega_+}$, repeated integrations by parts give

$$\int_{\partial \Omega} \langle f, g \rangle d\sigma = \int_{\partial \Omega} \langle f, -n \times (n \times g) \rangle d\sigma \quad (3.24)$$

$$= \frac{1}{\lambda_1} \int_{\partial \Omega} \langle n \times Q_1, -n \times (n \times \hat{Q}_1) \rangle d\sigma$$

$$- \frac{1}{\lambda_2} \int_{\partial \Omega} \langle n \times Q_2, -n \times (n \times \hat{Q}_2) \rangle d\sigma$$

$$= \frac{1}{\lambda_1} \iint_{\Omega} \langle \text{curl} Q_1, \hat{Q}_1 \rangle - \langle Q_1, \text{curl} \hat{Q}_1 \rangle$$

$$- \frac{1}{\lambda_2} \iint_{\Omega} \langle \text{curl} Q_2, \hat{Q}_2 \rangle - \langle Q_2, \text{curl} \hat{Q}_2 \rangle$$

$$= 0,$$

since $Q_j, \hat{Q}_j$ are $\lambda_j$-Beltrami. This proves (3.20) and completes the proof of Proposition 3.6. \qed

Here are the last details in the proof of Theorem 1.2. Choosing the $Q_j$’s as in (3.6), existence, granted finitely many compatibility conditions, follows from Proposition 3.6 and Proposition 3.2. Also, uniqueness modulo a finite dimensional space is contained in Proposition 3.5 and Proposition 3.2. Finally, the fact that the
Next we are ready to present The proof of Theorem 1.1. An inspection of the proof of Theorem 1.2 shows that

\[ B_{\lambda_1, +}^p (\partial \Omega) \cap B_{\lambda_2, +}^p (\partial \Omega) \simeq \text{Null} (BVP_+), \tag{3.25} \]

whereas

\[ B_{\lambda_1, +}^p (\partial \Omega) + B_{\lambda_2, +}^p (\partial \Omega) = \text{Im} (T_+; L_{\tan}^{p, \text{Div}} (\partial \Omega)). \tag{3.26} \]

The codimension of the latter space in \( L_{\tan}^{p, \text{Div}} (\partial \Omega) \) is, by Proposition 3.2 and Proposition 3.5, equal to \( \dim \left[ \text{Ker} \left( T_+; L_{\tan}^{p, \text{Div}} (\partial \Omega) \right) \right] = \dim \left[ \text{Null} (BVP_+) \right] \). The conclusion is that the subspaces under discussion form a Fredholm pair of index zero for \( L_{\tan}^{p, \text{Div}} (\partial \Omega) \). This finishes the proof of Theorem 1.1.

We now turn to our attention to the proof of Theorems 1.3-1.4. First, recall the usual scale of Besov spaces \( B_{s}^{p,q} (\partial \Omega), 1 < p < q < \infty, -1 < s < 1 \), and introduce

\[ H^p(\text{curl}, \Omega) := \{ u \in L^p(\Omega); \text{curl} \ u \in L^p(\Omega) \}, \]
\[ H^p(\text{div}, \Omega) := \{ u \in L^p(\Omega); \text{div} \ u \in L^p(\Omega) \}. \tag{3.27} \]

Appropriate versions of the above spaces can be defined in the unbounded domain \( \Omega_- \), via localization. The corresponding spaces are denoted by \( H^p_{\text{loc}}(\text{curl}, \Omega_-) \), etc.

For any vector field \( u \in H^p(\text{curl}, \Omega) \) we define the (vector-valued) distribution \( n \times u \in \mathbb{R}^3 \) (which is actually supported on \( \partial \Omega \)) by

\[ \langle n \times u, \varphi \rangle := \iint_\Omega \langle \text{curl} \ u, \varphi \rangle - \iint_\Omega \langle u, \text{curl} \varphi \rangle \tag{3.28} \]

for each test vector field \( \varphi \) in \( \mathbb{R}^3 \). As is well known (cf. Lemma 2.3 in [20]), \( n \times u \in B_{-1/p}^{p,p} (\partial \Omega) \) and there exists a positive constant \( C \) depending only on the Lipschitz character of \( \partial \Omega \) so that

\[ \| n \times u \|_{B_{-1/p}^{p,p} (\partial \Omega)} \leq C \left( \| u \|_{L^p(\Omega)} + \| \text{curl} \ u \|_{L^p(\Omega)} \right). \tag{3.29} \]

For a vector field \( u \in H^p(\text{div}, \Omega) \), the distribution \( \langle n, u \rangle \in B_{-1/p}^{p,p} (\partial \Omega) \) is defined similarly.

Next, define

\[ \mathcal{X}^p(\partial \Omega) := \left\{ f \in B_{-1/p}^{p,p} (\partial \Omega); \exists u \in H^p(\text{curl}, \Omega) \text{ so that } n \times u = f \right\}, \tag{3.30} \]
equipped with the natural norm

$$\|f\|_{X^p(\partial\Omega)} := \inf\{\|u\|_{L^p(\Omega)} + \|\text{curl } u\|_{L^p(\Omega)}; \ u \in H^p(\text{curl}, \Omega), \ n \times u = f\}. \quad (3.31)$$

The surface divergence operator naturally extends to the present setting:

$$\text{Div} : X^p(\partial\Omega) \longrightarrow B^{-p-1}_{-1/p}(\partial\Omega), \quad (3.32)$$

$$\text{Div}(n \times u) := -\langle n, \text{curl } u \rangle, \ \forall u \in H^p(\text{curl}, \Omega).$$

Note that this is meaningful since $\text{curl } u \in H^p(\text{div}, \Omega)$ for any $u \in H^p(\text{curl}, \Omega)$.

Our next result, collecting some of the main properties of the spaces $X^p(\partial\Omega)$, has been proved in [18] (even at the level of differential forms of arbitrary degree).

**Proposition 3.7** Let $\Omega$ be Lipschitz and $1 < p < \infty$. Then the following hold.

(i) $X^p(\partial\Omega) \equiv n \times H^p_{\text{loc}}(\text{curl}, \overline{\Omega}_-)$, in the sense that the two spaces coincide as sets and the norms are equivalent.

(ii) $X^p(\partial\Omega)$ is a reflexive Banach space, and $\{X^p(\partial\Omega)\}_{1 < p < \infty}$ is a complex interpolation scale, i.e.

$$[X^{p_0}(\partial\Omega), X^{p_1}(\partial\Omega)]_{\theta} = X^{p^*}(\partial\Omega), \quad (3.33)$$

for any $1 < p_0, p_1 < \infty$, $\theta \in (0, 1)$ and $p^* := ((1 - \theta)/p_0 + \theta/p_1)^{-1}$.

(iii) $L^p_{\tan}(\partial\Omega) \hookrightarrow X^p(\partial\Omega) \hookrightarrow B^{-p-1}_{-1/p}(\partial\Omega)$ continuously and densely.

(iv) Div defined in (1.4)-(1.5) is compatible with Div defined in (3.32).

(vi) If $1 < q < \infty$ is the conjugate exponent of $p$, then the mapping

$$n \times \cdot : X^q(\partial\Omega) \longrightarrow (X^p(\partial\Omega))^* \quad (3.34)$$

defined by

$$\langle n \times f, g \rangle := \iint_{\Omega} \langle u, \text{curl } v \rangle - \iint_{\Omega} \langle \text{curl } u, v \rangle \quad (3.35)$$

for $u \in H^q(\text{curl}, \Omega)$ with $f = n \times u$ and $v \in H^p(\text{curl}, \Omega)$ with $g = n \times v$, is an isomorphism.
Remark. The inverse of the operator (3.34) is \( n \times \cdot : (X^p(\partial \Omega))^* \to X^q(\partial \Omega) \), 
\( 1/p + 1/q = 1 \), defined by \( n \times \Phi := -f \) if \( \Phi \in (X^p(\partial \Omega))^* \) is of the form \( \Phi = n \times f \), \( f \in \mathcal{X}^q(\partial \Omega) \). Note that, in this notation, \( (n \times \cdot)^{-1} = -n \times \cdot \) and \( (n \times \cdot)^* = -n \times \cdot \) both on the scale \((X^p(\partial \Omega))^*\) as well as on the scale \( X^p(\partial \Omega) \).

We now consider the action of layer potentials on the scale \( X^p(\partial \Omega) \).

**Proposition 3.8** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^3 \), \( \lambda \in \mathbb{C} \), \( 1 < p < \infty \).

Then the following are true.

1. \( S_\lambda : X^p(\partial \Omega) \to H^{1,p}(\Omega) \) is bounded (here, \( H^{1,p}(\Omega) \) stands for the usual \( L^p \)-based Sobolev space in \( \Omega \)).

2. \( M_\lambda, N_\lambda \) are bounded operators when acting from \( X^p(\partial \Omega) \) into itself.

3. There exists \( \epsilon = \epsilon(\partial \Omega) > 0 \) such that \( \pm \frac{1}{2} I + M_\lambda \) are Fredholm with index zero for \( |p - 2| < \epsilon \) on \( X^p(\partial \Omega) \).

4. \( M_\lambda - M_\lambda \) is compact on \( X^p(\partial \Omega) \).

5. \( M_\lambda N_\lambda = -N_\lambda M_\lambda \) on \( X^p(\partial \Omega) \).

6. \( N_\lambda^2 = -\lambda^2 (\frac{1}{2} I + M_\lambda)(-\frac{1}{2} I + M_\lambda) \) on \( X^p(\partial \Omega) \).

7. The transposed of \( M_\lambda \) acting on \( X^p(\partial \Omega) \) is \(-M_\lambda \) acting on \((X^p(\partial \Omega))^* \equiv X^q(\partial \Omega), 1/p + 1/q = 1 \), where the identification is understood in the sense of (3.34).

8. The transposed of \( N_\lambda \) acting on \( X^p(\partial \Omega) \) is \(-N_\lambda \) acting on \((X^p(\partial \Omega))^* \equiv X^q(\partial \Omega), 1/p + 1/q = 1 \), where the identification is understood in the sense of (3.34).

9. If \( \lambda \in \mathbb{C} \setminus \{0\} \) then \( P^{\pm}_\lambda \), introduced in (2.26), are complementary projections on \( X^p(\partial \Omega) \), in the sense that

\[
P^+ - P^- = I, \quad P^+_\lambda P^- = P^- P^+_\lambda = 0, \quad (P^\pm_\lambda)^2 = \pm P^\pm_\lambda \text{ on } X^p(\partial \Omega),
\]  
(3.36)

and the transposed of \( P^\pm_\lambda \) acting on \( X^p(\partial \Omega) \) is \(-P^\mp_\lambda \) acting on \((X^p(\partial \Omega))^* \equiv X^q(\partial \Omega), 1/p + 1/q = 1 \), where the identification is understood in the sense of (3.34).

10. The operators

\[
T_\pm = \lambda_1 P^\pm_\lambda - \lambda_2 P^\pm_\lambda : X^p(\partial \Omega) \to X^p(\partial \Omega),
\]  
(3.37)
are well-defined and bounded for each $1 < p < \infty$. Moreover, the transposed of $T_\pm$ acting on $\mathcal{X}^p(\partial\Omega)$ is $-T_\mp$ acting on $(\mathcal{X}^p(\partial\Omega))^* \equiv \mathcal{X}^q(\partial\Omega)$, $1/p + 1/q = 1$, where the identification is understood in the sense of (3.34).

Proof. For (1) – (3) the reader is referred to [18] where a more general version has been proved. Next, (4) can justified with the aid of (2.18). As for the remaining points, they all follow from Proposition 2.2, Proposition 2.5, and a density argument.

Next we prove a result which is the counterpart of Proposition 3.2 in the setting of $\mathcal{X}^p(\partial\Omega)$ spaces. It is only here that the smallness condition on the constitutive parameters of the medium is used.

**Proposition 3.9** There exists $\epsilon = \epsilon(\partial\Omega) > 0$ such that if $2 - \epsilon < p < 2 + \epsilon$, $\lambda_1 \neq \lambda_2$, and $|\lambda_1 + \lambda_2| < \epsilon$ then the operators (3.37) are Fredholm with index zero.

Proof. Let us fix some $z \in \mathbb{C}$ and denote by Comp generic compact operators on $\mathcal{X}^p(\partial\Omega)$. Then

$$
T_\pm = (\lambda_1 - \lambda_2) \left[ (\pm \frac{1}{2}I + M_z) + \frac{N_{\lambda_1} - N_{\lambda_2}}{\lambda_1 - \lambda_2} \right] + \text{Comp, on } \mathcal{X}^p(\partial\Omega),
$$

and

$$
N_{\lambda_1}f - N_{\lambda_2}f = \lambda_1^2 n \times S_{\lambda_1}f - \lambda_2^2 n \times S_{\lambda_2}f + n \times \nabla(S_{\lambda_1} - S_{\lambda_2})(\text{Div } f) = (\lambda_1^2 - \lambda_2^2)(n \times S_z f) + \text{Comp } f.
$$

Thus, $(N_{\lambda_1} - N_{\lambda_2})/(\lambda_1 - \lambda_2)$ can be written as a compact operator on $\mathcal{X}^p(\partial\Omega)$ plus an operator with norm $O(|\lambda_1 + \lambda_2|)$ on $\mathcal{X}^p(\partial\Omega)$. The bottom line is that $T_\pm$ are Fredholm with index zero on $\mathcal{X}^p(\partial\Omega)$, as long as $\lambda_1 \neq \lambda_2$, provided $|\lambda_1 + \lambda_2|$ is small and $p$ is near 2.

After these preliminaries we are finally ready to present

**The proof of Theorem 1.4.** The first part of the theorem follows by observing that, much as in the case of the interior problem, $\mathcal{B}^{p}_{\lambda_1, -}(\partial\Omega) \cap \mathcal{B}^{p}_{\lambda_2, -}(\partial\Omega) = 0$, and $\text{Im}(T_\mp; L^p_{\text{Div}}(\partial\Omega)) \subseteq \mathcal{B}^{p}_{\lambda_1, -}(\partial\Omega) + \mathcal{B}^{p}_{\lambda_2, -}(\partial\Omega)$. In particular, the latter space is closed and has finite codimension in $L^p_{\text{Div}}(\partial\Omega)$, as long as $|p - 2|$ is small.

However, proving (1.20) requires more work. As a byproduct of our method we also obtain decompositions similar in spirit to (1.20) valid for other types of spaces; cf. (3.42) below.

To get started, once again as in the case of the interior problem, relying on Theorem 2.4, it suffices to show that
Ker \((P^+_{\lambda_1}; L^p_{\text{tan}}(\partial\Omega)) \oplus \text{Ker}(P^+_{\lambda_2}; L^p_{\text{tan}}(\partial\Omega)) = L^p_{\text{tan}}(\partial\Omega)\). \hfill (3.40)

Note that, by Proposition 2.5, this is equivalent to

\[
\text{Im} (P^+_{\lambda_1}; L^p_{\text{tan}}(\partial\Omega)) \oplus \text{Im}(P^+_{\lambda_2}; L^p_{\text{tan}}(\partial\Omega)) = L^p_{\text{tan}}(\partial\Omega). \hfill (3.41)
\]

The strategy for proving (3.41) is to produce a similar decomposition at a weaker level, i.e.

\[
\text{Im} (P^+_{\lambda_1}; X^p(\partial\Omega)) \oplus \text{Im}(P^+_{\lambda_2}; X^p(\partial\Omega)) = X^p(\partial\Omega) \hfill (3.42)
\]

and then to derive (3.41) from (3.42) by means of a regularity result from [20] to the effect that if \(|p - 2|\) is small then for each \(u \in H^p_{\text{loc}}(\text{curl}, \Omega^\pm) \cap H^p_{\text{loc}}(\text{div}, \Omega^\pm)\),

\[
\langle n \times u \rangle \in L^p(\partial\Omega). \hfill (3.43)
\]

Indeed, granted (3.42), the decomposition (3.41) can be obtained as follows.

First, the fact that the sum in (3.41) is direct is a direct consequence of (3.42) and the first inclusion in \((iii)\) of Proposition 3.7. Second, if \(f \in L^p_{\text{tan}}(\partial\Omega) \hookrightarrow X^p(\partial\Omega)\) is written as

\[
f = P^-_{\lambda_1}g_1 + P^-_{\lambda_2}g_2, \quad g_1, g_2 \in X^p(\partial\Omega), \hfill (3.44)
\]

introduce

\[
Q_j := \text{curl} S_{\lambda_j}g_j + \lambda_j^{-1} \text{curl} \text{curl} S_{\lambda_j}g_j \text{ in } \Omega^\pm, j = 1, 2. \hfill (3.45)
\]

Thus, \(n \times (Q_j|_{\Omega^\pm}) = P^\pm_{\lambda_j}g_j, j = 1, 2\). Our aim is to show that \(P^-_{\lambda_j}g_j\) belongs to \(\text{Im}(P^-_{\lambda_j}; L^p_{\text{tan}}(\partial\Omega))\), for \(j = 1, 2\). Since, modulo a sign, \(P^-_{\lambda_j}\) are idempotent this is, in turn, implied by the membership of \(n \times (Q_j|_{\Omega^-})\) to \(L^p_{\text{tan}}(\partial\Omega)\), which is the issue we tackle next.

The first observation is that, since \(\text{Div}(n \times (Q_j|_{\Omega^-})) = -\lambda_j \langle n, (Q_j|_{\Omega^-}) \rangle\), it suffices to show that

\[
n \times (Q_j|_{\Omega^-}), \langle n, (Q_j|_{\Omega^-}) \rangle \in L^p(\partial\Omega). \hfill (3.46)
\]

To this end, we introduce

\[
u := Q_1 - Q_2 \in H^p_{\text{loc}}(\text{curl}, \Omega^\pm) \cap H^p_{\text{loc}}(\text{div}, \Omega^\pm) \hfill (3.47)
\]

and observe that \(n \times (u|_{\Omega^-}) = f \in L^p_{\text{tan}}(\partial\Omega)\). The regularity statement (3.43) then yields

\[
\langle n, (Q_1|_{\Omega^-}) \rangle - \langle n, (Q_2|_{\Omega^-}) \rangle = \langle n, u \rangle \in L^p(\partial\Omega). \hfill (3.48)
\]

On the other hand,
\[ \lambda_1 \langle n, Q_1 \rangle_{\Omega_\cdot} - \lambda_2 \langle n, Q_2 \rangle_{\Omega_\cdot} = -\text{Div}(n \times u|_{\Omega_\cdot}) = -\text{Div} f \in L^p(\partial \Omega). \quad (3.49) \]

Thus, from (3.48)-(3.49) and the fact that \( \lambda_1 \neq \lambda_2 \), we conclude that \( \langle n, Q_j \rangle_{\Omega_\cdot} \in L^p(\partial \Omega), \) \( j = 1, 2 \). Once again relying on (3.43) we infer that \( n \times Q_j \in L^p(\partial \Omega), \) \( j = 1, 2 \), also. Thus, (3.46) follows.

At this point we are left with proving (3.42), a task to which we now turn. Our first observation is that the uniqueness result in Proposition 3.1 extends without difficulty to the situation when the Beltrami fields \( Q_j \) belong only to \( H^p_{\text{loc}}(\text{curl}, \Omega_\cdot) \), at least when \( p = 2 \). In turn, this translates into

\[ \text{Ker} (P_{\lambda_1}; X^2(\partial \Omega)) \cap \text{Ker} (P_{\lambda_2}; X^2(\partial \Omega)) = 0. \quad (3.50) \]

Going further, this and (9) in Proposition 3.8 yield

\[ \left[ \text{Im} (P_{\lambda_1}; X^2(\partial \Omega)) + \text{Im} (P_{\lambda_2}; X^2(\partial \Omega)) \right]^\circ = 0 \quad (3.51) \]

where \( \{ \ldots \}^\circ \) stands for the annihilator of \( \{ \ldots \} \). Thus, in particular,

\[ \text{Im} (P_{\lambda_1}; X^2(\partial \Omega)) + \text{Im} (P_{\lambda_2}; X^2(\partial \Omega)) \text{ is dense in } X^2(\partial \Omega). \quad (3.52) \]

Next, observe that

\[ \text{Im} (T_-; X^2(\partial \Omega)) \subseteq \text{Im} (P_{\lambda_1}^-; X^2(\partial \Omega)) + \text{Im} (P_{\lambda_2}^-; X^2(\partial \Omega)). \quad (3.53) \]

Recall from Proposition 3.9 (whose hypotheses are satisfied since \( \lambda_1 + \lambda_2 = O(|\beta|) \)) that the first space in (3.53) is closed, of finite codimension in \( X^2(\partial \Omega) \). Consequently, the same conclusion applies to the sum in the right side of (3.53). All in all, (3.52) and (3.53) give that

\[ \text{Im} (P_{\lambda_1}^-; X^2(\partial \Omega)) + \text{Im} (P_{\lambda_2}^-; X^2(\partial \Omega)) = X^2(\partial \Omega). \quad (3.54) \]

Next, we observe that

\[ \text{Im} (P_{\lambda_1}^-; X^2(\partial \Omega)) \cap \text{Im} (P_{\lambda_2}^-; X^2(\partial \Omega)) = 0. \quad (3.55) \]

Indeed, this follows from (3.50) and the point (9) in Proposition 3.8. Note that (3.54) and (3.55) prove the \( p = 2 \)-version of the decomposition we seek. The extension to \( p \neq 2 \) is then obtained by means of a perturbation argument which we now describe.

The first observation is that, due to the existence of common projections (cf. (9) in Proposition 3.8), the subspaces \( \left( \text{Im} (P_{\lambda_j}^p; X^p(\partial \Omega)) \right)_{1 < p < \infty} \) form a complex interpolation scale of Banach spaces. Second, the mapping
\[ \text{Im} (P_{\lambda_1}; X^p(\partial \Omega)) \times \text{Im} (P_{\lambda_2}; X^p(\partial \Omega)) \ni (f, g) \mapsto f + g \in X^p(\partial \Omega) \quad (3.56) \]

is bounded for \(1 < p < \infty\) and is an isomorphism when \(p = 2\). Granted these two ingredients, a general stability result (cf., e.g., the discussion in [12]) guarantees that the mapping (3.56) remains an isomorphism as long as \(|p - 2|\) is small enough. This justifies (3.42) and finishes the proof of Theorem 1.4.

Finally, Theorem 1.3 follows more or less directly from Theorem 1.4; we omit the details.

**Remark.** For any \(\lambda_2 \in \mathbb{R}\) there exists a discrete set \(\Lambda(\lambda_2) \subset \mathbb{R}\) such that for every \(\lambda_1 \in \mathbb{R} \setminus \Lambda(\lambda_2)\), the problems (\(BVP\)) are well-posed, i.e. existence, uniqueness and estimates hold, for each \(2 - \epsilon < p < 2 + \epsilon\).

This is a consequence of the genuine invertibility of \(T_\pm\) on \(L^{p, \text{Div}}_\text{tan}(\partial \Omega)\), for \(2 - \epsilon < p < 2 + \epsilon\), as long as \(\lambda_1 \in \mathbb{R} \setminus \Lambda(\lambda_2)\). The latter statement can be justified with the help of (3.9) and the analytic Fredholm alternative. In the process, it is important to observe that for \(\lambda_1 \in \mathbb{R}\) and \(\lambda_2 \in \mathbb{C} \setminus \mathbb{R}\), \(T_\pm\) is injective since, this time, uniqueness holds for both (\(BVP_\pm\)) (as an inspection of the proof of Proposition 3.1 shows). We leave the straightforward details to the interested reader.

**References**


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