ON DISSIPATIVE AND NON-UNITARY SOLUTIONS TO OPERATOR COMMUTATION RELATIONS

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Dedicated to the memory of Stanislav Petrovich Merkuriev

ABSTRACT. We study the (generalized) semi-Weyl commutation relations

\[ U_g A U_g^* = g(A) \quad \text{on} \quad \text{Dom}(A), \]

where \( A \) is a densely defined operator and \( G \ni g \mapsto U_g \) is a unitary representation of the subgroup \( G \) of the affine group \( \mathcal{G} \), the group of affine transformations of the real axis preserving the orientation. If \( A \) is a symmetric operator, the group \( G \) induces an action/flow on the operator unit ball of contractive transformations from \( \text{Ker}(A^* - iI) \) to \( \text{Ker}(A^* + iI) \). We establish several fixed point theorems for this flow. In the case of one-parameter continuous subgroups of linear transformations, self-adjoint (maximal dissipative) operators associated with the fixed points of the flow give rise to solutions of the (restricted) generalized Weyl commutation relations. We show that in the dissipative setting, the restricted Weyl relations admit a variety of non-unitarily equivalent representations. In the case of deficiency indices (1,1), our general results can be strengthened to the level of an alternative.

1. Introduction

It is well known (see, e.g., [8] or [24]) that the canonical commutation relations in the Weyl form [42]

\[ U_t V_s = e^{ist} V_s U_t, \quad s, t \in \mathbb{R}, \]

between two strongly continuous unitary groups \( U_t = e^{itB} \) and \( V_s = e^{isA} \) in a (separable) Hilbert space are satisfied if and only if

\[ U_t A U_t^* = A + tI \quad \text{on} \quad \text{Dom}(A), \quad t \in \mathbb{R}. \]

It is not well known, but trivial (being an immediate corollary of the Stone-von Neumann uniqueness result [32]), that if a self-adjoint operator \( A \) satisfies (1.2), then \( A \) always admits a symmetric restriction \( \hat{A} \subset A \) with deficiency indices (1,1) such that

\[ U_t \hat{A} U_t^* = \hat{A} + tI \quad \text{on} \quad \text{Dom}(\hat{A}), \quad t \in \mathbb{R}. \]

In this setting the following natural question arises. Suppose that (1.3) holds for some symmetric operator \( \hat{A} \). More generally, assume that the following commutation relations

\[ U_t \hat{A} U_t^* = g_t(\hat{A}) \quad \text{on} \quad \text{Dom}(\hat{A}), \quad t \in \mathbb{R}, \]

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are satisfied, where $g_t$ is a one-parameter group of affine transformations of the real line and $g_t \mapsto U_t$ is its strongly continuous representation by unitary operators.

**The Extension Problem:** Suppose that (1.4) holds for some symmetric operator $\dot{A}$. Classify all maximal dissipative, in particular, self-adjoint (if any) solutions $A$ of the semi-Weyl relations

$$U_t A U_t^* = g_t(A) \quad \text{on} \quad \text{Dom}(A)$$

that extend $\dot{A}$ such that $\dot{A} \subset A \subset (\dot{A})^*$.

Given the fact that any one-parameter subgroup of the affine group is either of the form

(1.6) \hspace{1cm} g_t(x) = x + vt, \quad t \in \mathbb{R}, \quad \text{for some} \quad v \in \mathbb{R}, \quad (v \neq 0),

or

(1.7) \hspace{1cm} g_t(x) = a^t(x - \gamma) + \gamma, \quad t \in \mathbb{R}, \quad \text{for some} \quad 0 < a \neq 1 \quad \text{and} \quad \gamma \in \mathbb{R},

a formal differentiation of (1.5) yields commutation relations for the generators $B$ and $A$ of the unitary group $U_t = e^{iBt}$ and the semi-group of contractions $V_s = e^{isA}$, $s \geq 0$, respectively. These relations are of the form

(1.8) \hspace{1cm} [A, B] = ivI,

in the case of the subgroup $g_t$ of translations (1.6), and

(1.9) \hspace{1cm} [A, B] = i\lambda A - i\mu I, \quad \lambda = \log a, \quad \mu = \gamma \log a,

for the subgroup $g_t$ given by (1.7). Here $[A, B]$ stands for the commutator $[A, B] = AB - BA$ on $\text{Dom}(A) \cap \text{Dom}(B)$.

If we make the change of variables: take $X = A$, $Y = B$ and $Z = ivI$, in the first case, and set $X = A - \gamma I$ and $Y = \frac{1}{\lambda x}B$, in the second, we arrive at the commutation relations

(1.10) \hspace{1cm} [X, Y] = Z, \quad [X, Z] = 0, \quad [Y, Z] = 0,

and

(1.11) \hspace{1cm} [X, Y] = X,

respectively. Therefore, solving the semi-Weyl relations (1.5) may be considered a variant of non-self-adjoint quantization of the three- and two-dimensional Lie algebras (1.10) and (1.11), respectively. For an alternative approach towards quantization of low-dimensional Lie algebras and their functional models we refer to [43].

In this paper we solve a slightly more general extension problem. Namely, we assume that $G \ni g \mapsto U_g$ is a unitary representation of an arbitrary subgroup of the affine group $G$, the group of affine transformation of the real axis preserving the orientation. We also suppose that $\dot{A}$ is a densely defined symmetric operator satisfying the semi-Weyl relations

(1.12) \hspace{1cm} U_g \dot{A} U_g^* = g(\dot{A}) \quad \text{on} \quad \text{Dom}(\dot{A}), \quad g \in G.

Those $\dot{A}$’s will be called $G$-invariant operators (with respect to the unitary representation $G \ni g \mapsto U_g$ of the group $G$). We remark that the $G$-invariant bounded operator colligations (with respect to the group of linear fractional transformations and the Lorentz group) were studied in [6], [26, Ch. 10], [35].
Our main results are as follows. We show that in the semi-bounded case, that is, if $\dot{A}$ from (1.4) is semi-bounded, the extension problem (1.5) is always solvable and that the solution can be given by $G$-invariant self-adjoint operators. In particular, if a non-negative symmetric operator is $G$-invariant with respect to the group of all scaling transformations of the real axis into itself ($g(x) = ax, a > 0, g \in G$), then its Friedrichs and Krein-von Neumann extensions are also $G$-invariant. In fact, if the symmetric operator has deficiency indices $(1,1)$, then those extensions are the only ones that are $G$-invariant.

To treat the general case, we study a flow of transformations on the set $V$ of contractive mappings from the deficiency subspace $\text{Ker}((\dot{A})^* - iI)$ to the deficiency subspace $\text{Ker}((\dot{A})^* + iI)$ induces by the unitary representation $G \ni g \mapsto U_g$. Based on this study, we show that $\dot{A}$ admits a $G$-invariant maximal dissipative extension if and only if the flow has a fixed point.

If the deficiency indices are finite, applying the Schauder fixed point theorem, we show that the extension problem (1.5) is always solvable in the space of maximal dissipative operators. If, in addition, the indices are equal, the flow restricted to the set $U \subset V$ of all isometries from $\text{Ker}((\dot{A})^* - iI)$ onto $\text{Ker}((\dot{A})^* + iI)$ leaves this set invariant. In this case, one can reduce the search for $G$-invariant self-adjoint solutions to the extension problem to the one of fixed points of the restricted flow. We remark that if $\dot{A}$ is not semi-bounded, then self-adjoint $G$-invariant extensions of $\dot{A}$ may not exist in general, even if the deficiency indices of $\dot{A}$ are equal and finite.

Special attention is paid to the case of deficiency indices $(1,1)$. In particular, we show that if $G$ is a one-parametric continuous subgroup of the affine group, the extension problem always has a self-adjoint affine invariant solution if not with respect to the whole group $G$, but at least with respect to a discrete subgroup of $G$. This phenomenon, see Remark 7.3, can be considered an abstract operator-theoretic counterpart of the fall to the center “catastrophe” in Quantum Mechanics [23]. For discussion of the topological origin for this effect see Remark 6.7. In this connection, we also refer to [7] and [10] for a related discussion of the Efimov Effect in three-body systems and to [31] where the collapse in a three-body system with point interactions has been discovered (also see [29] and references therein).

2. $G$-invariant operators

Throughout this paper $G$ denotes the “$ax+b$”-group which is the non-commutative group of non-degenerate affine transformations (with respect to composition) of the real axis preserving the orientation.

Recall that the group $G$ consists of linear transformations of the real axis followed by translations

\begin{equation}
(2.1) \quad x \mapsto g(x) = ax + b, \quad x \in \mathbb{R},
\end{equation}

with $a > 0$ and $b \in \mathbb{R}$.

Introduce the concept of a unitarily affine invariant operator.

Suppose that $G$ is a subgroup of $G$. Assume, in addition, that $G \ni g \mapsto U_g$ is a unitary representation of $G$ on a separable Hilbert space $\mathcal{H}$,

\[ U_f U_g = U_{fg}, \quad f, g \in G. \]

If $G$ is a continuous group, assume that the representation $G \ni g \mapsto U_g$ is strongly continuous.
Definition 2.1. A densely defined closed operator $A$ is said to be unitarily affine invariant, or more specifically, $G$-invariant with respect to a unitary representation $G 
i g \mapsto U_g$, if for all $g \in G$

$$U_g(\text{Dom}(A)) = \text{Dom}(A)$$

and

$$U_g A U^*_g = g(A) \quad \text{on} \quad \text{Dom}(A).$$

We notice that the operator equality (2.2) should be understood in the sense that

$$U_g A U^*_g f = g(A)f = aAf + bf, \quad \text{for all} \quad f \in \text{Dom}(A),$$

whenever the transformation $g \in G$ is of the form $x \mapsto g(x) = ax + b$.

The case of one-parameter continuous subgroups of the affine group deserves a special discussion.

Recall that if $g_t$ is a one-parameter subgroup of the affine group, then either the group $G = \{g_t\}_{t \in \mathbb{R}}$ consists of affine transformations of $\mathbb{R}$ of the form

$$g_t(x) = x + vt, \quad t \in \mathbb{R},$$

for some $v \in \mathbb{R}$, $v \neq 0$, or

$$g_t(x) = a^t(x - \gamma) + \gamma, \quad t \in \mathbb{R},$$

for some $a > 0$, $a \neq 1$, and $\gamma \in \mathbb{R}$.

Remark 2.2. Note that in case (2.5), in contrast to (2.4), all transformations $g_t$, $t \in \mathbb{R}$, have a finite fixed point:

$$g_t(\gamma) = \gamma \quad \text{for all} \quad t \in \mathbb{R}.$$  

The concept of $G$-invariant self-adjoint operators in the case of the group of translations,

$$g_t(x) = x + t,$$

is naturally arises in connection with the canonical commutation relations in the Weyl form.

Theorem 2.3. Suppose that $G = \{g_t\}_{t \in \mathbb{R}}$ is a one-parameter continuous subgroup of the affine transformations

$$g_t(x) = x + t.$$  

Assume that $A$ is a self-adjoint $G$-invariant operator with respect to a strongly continuous unitary representation $t \mapsto U_t$ in a separable Hilbert space. That is,

$$U_t A U^*_t = A + tI \quad \text{on} \quad \text{Dom}(A).$$

Then the unitary group $U_t$ and the unitary group $V_s$ generated by $A$,

$$V_s = e^{iAs}, \quad s \in \mathbb{R},$$

satisfy the Weyl commutation relations

$$U_t V_s = e^{ist} V_s U_t, \quad s, t \in \mathbb{R}.$$  

The converse is also true. That is, if two strongly continuous unitary groups $U_t$ and $V_s$ satisfy the Weyl relations (2.7), then the generator $A$ of the group $V_s$ is $G$-invariant with respect to the shift group (2.6) and its unitary representation $g_t \mapsto U_t$. 

As for the proof of this result we refer to [24, Chapter II, Sec. 7].

**Remark 2.4.** We remark that the Stone-von Neumann uniqueness result [32] states that the shift invariant self-adjoint operator $A$ and the generator $B$ of the unitary group $U_t$ are mutually unitarily equivalent to a finite or infinite direct sum of the momentum and position operators from the Schrödinger representation:

$$ (A, B) \approx \bigoplus_{n=1}^{\ell} (P, Q), \quad \ell = 1, 2, \ldots, \infty. $$

Here

$$ (Pf)(x) = \frac{d}{dx}f(x), \quad \text{Dom}(P) = W_2^1(\mathbb{R}), $$

is the momentum operator, and

$$ (Qf)(x) = xf(x), \quad \text{Dom}(Q) = \{ f \in L^2(\mathbb{R}) \mid xf(\cdot) \in L^2(\mathbb{R}) \}, $$

is the position operator, respectively.

Theorem 2.3 admits a generalization to the case of arbitrary continuous one-parameter affine subgroups.

**Theorem 2.5.** Suppose that $G$ is a one-parameter continuous subgroup of the affine group $G$. Assume that $A$ is a self-adjoint $G$-invariant operator with respect to a strongly continuous unitary representation $G \ni g_t \mapsto U_t$ in a separable Hilbert space.

Then the unitary group $U_t$ and the unitary group $V_s$ generated by $A$,

$$ V_s = e^{iAs}, \quad s \in \mathbb{R}, $$

satisfy the generalized Weyl commutation relations

$$ U_t V_s = e^{i\phi(t)} V_{g_t(0)} U_t, \quad s, t \in \mathbb{R}, $$

where

$$ g_t'(0) = \frac{d}{dx}g_t(x)|_{x=0}. $$

The converse is also true. That is, if two strongly continuous unitary groups $U_t$ and $V_s$ satisfy the generalized Weyl relations (2.8), then the generator $A$ of the group $V_s$ is $G$-invariant with respect to the one-parameter continuous group $G = \{ g_t \}_{t \in \mathbb{R}}$ and its unitary representation $G \ni g_t \mapsto U_t$.

It is easy to see that if $G = \{ g(t) \}_{t \in \mathbb{R}}$ is the group of translations,

$$ g_t(x) = x + t, $$

the commutation relations (2.8) turn into the standard Weyl commutation relations (2.7). Indeed, in this case, $g_t(0) = t$ and $g_t'(0) = 1$, so, (2.8) simplifies to (2.7).

For the further generalizations of the Stone-von Neumann result we refer to [28].

Along with the Weyl commutation relations (2.8), one can also introduce the concept of restricted generalized Weyl commutation relations

$$ U_t V_s = e^{i\phi(t)} V_{g_t(0)} U_t, \quad t \in \mathbb{R}, \quad s \geq 0, $$

involving a strongly continuous unitary group $U_t$ and a strongly continuous semigroup of contractions $V_s$, $s \geq 0$. See [4, 13, 14, 15, 16, 17, 18, 36, 37] where the concept of restricted Weyl commutation relations in the case of the group of affine translations has been discussed.
The following result, which is an immediate generalization of Theorem 2.5 to the case of maximal dissipative $G$-invariant generators, characterizes non-unitary solutions to the restricted generalized Weyl commutation relations (see Section 7, Subsections 7.3 and 7.7, for a number of examples of such solutions).

**Theorem 2.6.** Suppose that $G = \{g_t\}_{t \in \mathbb{R}}$ is a one-parameter continuous group of the affine group $G$. Assume that $A$ is a maximal dissipative $G$-invariant operator with respect to unitary representation $G \ni g_t \mapsto U_t$ in a Hilbert space. Then the strongly continuous unitary group $U_t$ and the strongly continuous semi-group $V_s$ of contractions generated by $A$,

$$V_s = e^{iAs}, \quad s \geq 0,$$

satisfy the restricted generalized Weyl commutation relations

$$U_tV_s = e^{ig_t(0)}V_{g_t(0)s}U_t, \quad t \in \mathbb{R}, \ s \geq 0,$$

where

$$g_t(0) = \frac{d}{dx} g_t(x)|_{x=0}.$$

The converse is also true. That is, if a strongly continuous unitary group $U_t$ and a strongly continuous semi-group of contractions $V_s$, $s \geq 0$, satisfy the generalized restricted Weyl commutation relations (2.9), then the generator $A$ of the semi-group $V_s$ is $G$-invariant with respect to the one-parameter continuous group $G = \{g_t\}_{t \in \mathbb{R}}$ and its unitary representation $g_t \mapsto U_t$.

**Proof.** To prove the assertion, assume that $f \in \text{Dom}(A)$. Then (see, e.g., [21, Theorem 1.3])

$$V_s f = -\frac{1}{2\pi i} \int_{\Gamma} e^{i\lambda s}(A - \lambda I)^{-1} f d\lambda, \quad s \geq 0,$$

where $\Gamma$ is any contour in the lower half-plane parallel to the real axis and the integral is understood in the principal value sense. Since $A$ is a $G$-invariant operator, it is easy to see that

$$U_t(A - \lambda I)^{-1} f = g_{t-}(\lambda)(A - g_{t-}(\lambda)I)^{-1}U_t f$$

and therefore (after a simple change of variable)

$$U_tV_s f = -\frac{1}{2\pi i} \int_{\Gamma} e^{i\lambda s}U_t(A - \lambda I)^{-1} f d\lambda$$

$$= -\frac{1}{2\pi i} \int_{\Gamma'} e^{i\lambda s}g_{t-}(\lambda)(A - g_{t-}(\lambda)I)^{-1}U_t f d\lambda$$

$$= -\frac{1}{2\pi i} \int_{\Gamma'} e^{ig_{t-}(\lambda)s}(A - \lambda I)^{-1} f d\lambda$$

$$= -\frac{1}{2\pi i} \int_{\Gamma'} e^{ig_{t-}'(0)\lambda + g_{t-}(0)s}(A - \lambda I)^{-1} f d\lambda$$

$$= e^{isg_{t-}'(0)}V_{g_{t-}'(0)s}U_t f,$$

with $\Gamma' = g_{t-}(\Gamma)$, a contour in the lower half-plane.

Thus, (2.10) shows that the representation (2.9) holds in the strong sense on the dense set $\text{Dom}(A)$. Taking into account that the operators $U_t, t \in \mathbb{R}$, and $V_s, s \geq 0$, are bounded, one extends (2.10) from the dense set to the whole Hilbert space which proves the claim.

The converse follows by differentiation of the commutation relations. \qed
3. G-invariant symmetric operators and the extension problem

The search for self-adjoint or, more generally, maximal dissipative G-invariant operators can be accomplished solving the following extension problem: Given a symmetric G-invariant operator, find its all maximal dissipative G-invariant extensions. We remark that the search for G-invariant self-adjoint realizations of the symmetric operator can be undertaken only if the deficiency indices are equal.

We start with the following elementary observation.

**Lemma 3.1.** Assume that $\dot{A}$ is a G-invariant symmetric operator. Suppose that $A$ is a maximal dissipative extension of $\dot{A}$.

Then the restriction $A_g$ of the adjoint operator $(\dot{A})^*$ onto $D_g = U_g(\text{Dom}(A))$,

$$A_g = (\dot{A})^*|_{D_g},$$

(3.1)

is a maximal dissipative extension of $\dot{A}$.

**Proof.** Since $\dot{A}$ is G-invariant, the operator $U_g AU_g^*$, $g \in G$, is a maximal dissipative extension of $g(\dot{A})$, and hence $g^{-1}(U_g AU_g^*)$ is a maximal dissipative extension of $\dot{A}$.

Since

$$\text{Dom}(g^{-1}(U_g AU_g^*)) = \text{Dom}(U_g AU_g^*) = U_g(\text{Dom}(A)),$$

one concludes that the restriction $A_g$ given by (3.1) is a maximal dissipative operator.

\[\square\]

Recall that the set of all maximal dissipative extensions of $\dot{A}$ is in a one-to-one correspondence with the set $V$ of contractive mappings from the deficiency subspace $N_+ = \text{Ker}((\dot{A})^* - iI)$ into the deficiency subspace $N_- = \text{Ker}((\dot{A})^* + iI)$. Note that the set $V$ is an operator unit ball in the Banach space $L(N_+, N_-)$.

**Proposition 3.2.** Let $\dot{A}$ be a closed symmetric operator and $V \in V$ a contractive mapping from $N_+ = \text{Ker}((\dot{A})^* - iI)$ into $N_- = \text{Ker}((\dot{A})^* + iI)$. Then the restriction $A$ of the adjoint operator $(\dot{A})^*$ on

$$\text{Dom}(A) = \text{Dom}(\dot{A}) + (I - V) \text{Ker}((\dot{A})^* - iI)$$

(3.2)

is a maximal dissipative extension of $\dot{A}$.

Moreover, the domain of any maximal dissipative extension $A$ of $\dot{A}$ such that $\dot{A} \subset A \subset (\dot{A})^*$ has a decomposition of the form (3.2), where $V$ is the restriction of the Cayley transform $(A - iI)(A + iI)^{-1}$ onto the deficiency subspace $N_+ = \text{Ker}((\dot{A})^* - iI)$.

**Remark 3.3.** If $\dot{A}$ has equal deficiency indices and $V$ is an isometric mapping from $\text{Ker}((\dot{A})^* - iI)$ onto $\text{Ker}((\dot{A})^* + iI)$, then the representation (3.2), known as von Neumann’s formula, provides a parametric description of all self-adjoint extensions of a symmetric operator $\dot{A}$. The extension of von Neumann’s formulae to the dissipative case can be found in [40] (also see [3, Theorem 4.1.9]).

4. The flow on the set $V$

Given a symmetric $G$-invariant operator $\dot{A}$, the von Neumann’s formula (3.2) from Proposition 3.2 determines a flow of transformations on the set of contractive mappings $V \in V$. More precisely, the mapping

$$G \ni g \mapsto A_g,$$
with $A_g$ given by (3.1), can naturally be “lifted” to an action of the group $G$ on the operator unit ball $\mathcal{V}$.

In order to describe the action of the group $G$ on $\mathcal{V}$ in more detail, suppose that $V \in \mathcal{V}$ and let $A$ be a unique maximal dissipative extension of $\dot{A}$ such that

\begin{equation}
\text{Dom}(A) = \text{Dom}(\dot{A}) + (I - V) \text{Ker}((\dot{A})^* - iI).
\end{equation}

Define the extension $A_g$ as in (3.1) by

\begin{equation}
A_g = (\dot{A})^{|_{\mathcal{U}_g(\text{Dom}(A))}},
\end{equation}

and let, in accordance with Lemma 3.1 and Proposition 3.2, the contractive mapping $V_g \in \mathcal{V}$ be such that

\begin{equation}
\text{Dom}(A_g) = \text{Dom}(\dot{A}) + (I - V_g) \text{Ker}((\dot{A})^* - iI), \quad g \in G.
\end{equation}

Introduce the action of the group $G$ on the set of a contractive mappings $\mathcal{V}$ by

\begin{equation}
\Gamma_g(V) = V_g, \quad g \in G.
\end{equation}

From the definition of $\Gamma_g$ it follows that

\begin{equation}
\Gamma_{f \cdot g} = \Gamma_f \circ \Gamma_g, \quad f, g \in G,
\end{equation}

so that

\begin{equation}
\varphi(f \cdot g, V) = \varphi(f, \Phi(g, V)), \quad f, g \in \mathcal{G}, \quad V \in \mathcal{V},
\end{equation}

with

\begin{equation}
\varphi(g, V) = \Gamma_g(V).
\end{equation}

Clearly, the maximal dissipative extension $A$ given by (4.1) is $G$-invariant if and only if $V \in \mathcal{V}$ is a fixed point of the flow $G \ni g \mapsto \Gamma_g$. That is,

\begin{equation}
\Gamma_g(V) = V \quad \text{for all} \quad g \in G.
\end{equation}

Therefore, the set of maximal dissipative extensions of $\dot{A}$ is in a one-to-one correspondence with the set of all fixed points of the flow $\Gamma_g$, and hence, the search for all $G$-invariant maximal dissipative extensions of $\dot{A}$ can be reduced to the one for the fixed points of the flow (4.3). If, in addition, the symmetric operator $\dot{A}$ has equal deficiency indices, the restriction of the flow $\Gamma_g$ to the set $\mathcal{U} \subset \mathcal{V}$ of all isometric mappings from the deficiency subspace $\text{Ker}((\dot{A})^* - iI)$ onto the deficiency subspace $\text{Ker}((\dot{A})^* + iI)$ leaves the set $\mathcal{U}$ invariant. In turn, the set of all self-adjoint extensions of $\dot{A}$ is in a one-to-one correspondence with the set of fixed points of the restricted flow $\Gamma_g|_{\mathcal{U}}$.

The following technical result provides a formula representation for the transformations $\Gamma_g, g \in \mathcal{G}$, of the operator unit ball $\mathcal{V}$ into itself.

**Lemma 4.1.** Assume that $\dot{A}$ is a $G$-invariant closed symmetric densely defined operator and $\Gamma_g, g \in G$, is the flow defined by (4.1) and (4.3). Then,

\begin{equation}
\Gamma_g(V) = -[P_- U_g(\gamma I - \delta V)] [P_+ U_g(\alpha I - \beta V)]^{-1},
\end{equation}

where $P_\pm$ denotes the orthogonal projections onto $\text{Ker}((\dot{A})^\mp iI)$ and $\alpha = g^{-1}(i) + i$, $\beta = g^{-1}(-i) + i$, $\gamma = g^{-1}(i) - i$, $\delta = g^{-1}(-i) - i$. 

\textbf{Proof.} Given $V \in \mathcal{V}$, denote by $A$ the maximal dissipative operator obtained by the restriction of the adjoint operator $(\hat{A})^*$ onto the domain

$$\text{Dom}(A) = \text{Dom}(\hat{A}) + (I - V) \text{Ker}((\hat{A})^* - iI).$$

Thus, any element $f \in D(A)$ admits the representation

$$f = f_0 + h - Vh,$$

where $f_0 \in D(\hat{A})$, $h \in \text{Ker}((\hat{A})^* - iI)$ and $Vh \in \text{Ker}((\hat{A})^* + iI)$.

By the definition of the mapping $\Gamma_g$, $\Gamma_g(V) \in \mathcal{V}$, the domain of the operator $A_g$ given by (cf. (3.1))

$$A_g = (\hat{A})^*|_{U_g(\text{Dom}(A))},$$

is of the form

$$\text{Dom}(A_g) = \text{Dom}(\hat{A}) + (I - \Gamma_g(V)) \text{Ker}((\hat{A})^* - iI).$$

Suppose that $f \in D(A)$ and that (4.8) holds. Taking into account that

$$U_g f \in \text{Dom}(U_g A U_g^*) = \text{Dom}(A_g),$$

from (4.9) it follows that

$$U_g f = U_g f_0 + U_g h - U_g Vh = k_0 + m - \Gamma_g(V)m$$

for some $k_0 \in D(\hat{A})$ and some $m \in \text{Ker}((\hat{A})^* - iI)$. Then

$$m - \Gamma_g(V)m = U_g h - U_g Vh + \ell_0,$$

with $\ell_0 = U_g f_0 - k_0 \in D(\hat{A})$. Applying $(\hat{A})^* + iI$ to both sides of (4.11) one gets

$$2im = ((\hat{A})^* + iI)U_g h - ((\hat{A})^* + iI)U_g Vh + (\hat{A} + iI)\ell_0.$$ 

Here we used that $m \in \text{Ker}((\hat{A})^* - iI)$ and $\Gamma_g(V)m \in \text{Ker}((\hat{A})^* + iI)$. Since by hypothesis $\hat{A}$ is $G$-invariant, the adjoint operator $(\hat{A})^*$ is $G$-invariant as well. One obtains

$$((\hat{A})^* + iI)U_g h = U_g (g^{-1}((\hat{A})^* + iI))h = (g^{-1}(i) + i)U_g h.$$ 

Similarly, one gets that

$$((\hat{A})^* + iI)U_g Vh = (g^{-1}(-i) + i)U_g Vh.$$ 

Combining (4.12), (4.13) and (4.14), one derives the representation

$$2im = (g^{-1}(i) + i)U_g h - (g^{-1}(-i) + i)U_g Vh + (\hat{A} + iI)\ell_0.$$ 

Recall that $P_\pm$ is the orthogonal projection onto $\text{Ker}((\hat{A})^* \mp iI)$. From (4.15) it follows that

$$2im = P_+ U_g ((g^{-1}(i) + i)h - (g^{-1}(-i) + i)Vh).$$

Applying $(\hat{A})^* - iI$ to both sides of (4.11), in a completely analogous way one arrives at the identity

$$-2i\Gamma_g(V)m = P_- U_g ((g^{-1}(i) - i)h - (g^{-1}(-i) - i)Vh).$$

Combining (4.16) and (4.17) proves the equality

$$-\Gamma_g(V)P_+ U_g (\alpha h - \beta Vh) = P_- U_g (\gamma h - \delta Vh),$$

where $\alpha = g^{-1}(i) + i$, $\beta = g^{-1}(-i) + i$, $\gamma = g^{-1}(i) - i$, and $\delta = g^{-1}(-i) - i$.

Since

$$U_g(\text{Dom}(A)) = \text{Dom}(U_g A U_g^*),$$

we obtain

\begin{align*}
-2i\Gamma_g(V)m & = \frac{1}{(\hat{A}^* + iI)(\hat{A} - iI) - \Gamma_g(V)} [P_+ U_g ((\hat{A}^* - iI)h - (\hat{A} - iI)\ell_0)] - P_- U_g ((\hat{A}^* - iI)h - (\hat{A} - iI)\ell_0), \\
\end{align*}

where $\ell_0 = U_g f_0 - k_0 \in D(\hat{A})$. Applying $(\hat{A}^* - iI)$ to both sides of (4.11), in a completely analogous way one arrives at the identity

$$-2i\Gamma_g(V)m = \frac{1}{(\hat{A}^* + iI)(\hat{A} - iI) - \Gamma_g(V)} [P_+ U_g ((\hat{A}^* - iI)h - (\hat{A} - iI)\ell_0)] - P_- U_g ((\hat{A}^* - iI)h - (\hat{A} - iI)\ell_0).$$

Thus, any element $f \in D(A)$ admits the representation

$$f = f_0 + h - Vh,$$
from (4.10) and (4.16) it follows that the operator $P_+U_g(\alpha I - \beta V)$ maps the deficiency subspace $\text{Ker}((\dot{A})^* - iI)$ onto itself. Moreover, the null space of $P_+U_g(\alpha I - \beta V)$ is trivial. Indeed, let 

$$P_+U_g(\alpha I - \beta V)h = 0$$

for some $h \in \text{Ker}((\dot{A})^* - iI)$ and $m$ be as above (cf. (4.10)). From (4.16) it follows that $m = 0$. Therefore, coming back to (4.10), one concludes that 

$$U_g(\alpha I - \beta V)h \in \text{Dom}(\dot{A})^*.$$ 

Since $\dot{A}$ is $G$-invariant, this means that $(\alpha I - \beta V)h \in \text{Dom}(\dot{A})^*$ which is only possible if $h = 0$ ($\alpha$ is never zero). By the closed graph theorem the bounded mapping $P_+U_g(\alpha I - \beta V)$ from the deficiency subspace $\text{Ker}((\dot{A})^* - iI)$ onto itself has a bounded inverse and hence from (4.18) one concludes that 

$$\Gamma_g(V) = -[P_-U_g(\gamma I - \delta V)][P_+U_g(\alpha I - \beta V)]^{-1},$$

completing the proof. □

Corollary 4.2. Assume that $G$ is a continuous subgroup of the affine group $\mathcal{G}$. Assume, in addition, that $\dot{A}$ is a $G$-invariant closed symmetric operator with finite deficiency indices. Then the mapping 

$$(g, V) \mapsto \Gamma_g(V)$$

from $G \times \mathcal{V}$ to $\mathcal{V}$ is continuous. In particular, if $G = \{g_t\}_{t \in \mathbb{R}}$ is a continuous one-parameter subgroup of the affine group $\mathcal{G}$, then for any $V \in \mathcal{V}$ the trajectory $\mathbb{R} \ni t \mapsto \Gamma_{g_t}(V)$ is continuous.

Remark 4.3. To validate the usage of the term “flow” in our general considerations, recall that a flow on a closed, oriented manifold $\mathcal{M}$ is a one-parameter group of homomorphisms of $\mathcal{M}$. That is, a flow is a function 

$$\varphi : \mathbb{R} \times \mathcal{M} \mapsto \mathcal{M}$$

which is continuous and satisfies 

(i) $\varphi(t_1 + t_2, x) = \varphi(t_1, \varphi(t_2, x))$, 

(ii) $\varphi(0, x) = x$

for all $t_1, t_2 \in \mathbb{R}$ and $x \in \mathcal{M}$. If $G = \{g_t\}_{t \in \mathbb{R}}$ is a continuous one-parameter group of affine transformations and the deficiency indices of the symmetric operator $\dot{A}$ are finite, then by Corollary 4.2, relation (4.5) defines a flow, in the standard sense, on the $n$-manifold $\mathcal{V}$, where

$$n = \dim(\text{Ker}((\dot{A})^* - iI)) \times \dim(\text{Ker}((\dot{A})^* + iI)).$$

5. Fixed point theorems

The main goal of this section is to present several fixed point theorems for the flow $\Gamma_g$ given by (4.3) on the operator unit ball $\mathcal{V}$ introduced in the previous section.

We start with recalling the following general fixed point theorem obtained in [30]. This theorem solves the problem of the existence of fixed points for the flow (4.3) in the case where the symmetric operator $\dot{A}$ is semi-bounded.

To formulate the result we need some preliminaries.

Recall that if $\dot{A}$ is a densely defined (closed) positive operator, then the set of all positive self-adjoint extensions of $\dot{A}$ has the minimal $A_{K\!\!\!\!N}$ (Krein–von Neumann extension) and the maximal $A_F$ (Friedrichs extension) elements. That is, for any
positive self-adjoint extension $\tilde{A}$ the following two-sided operator inequality holds \cite{20}:

$$ (A_F + \lambda I)^{-1} \leq (\tilde{A} + \lambda I)^{-1} \leq (A_{K\nu N} + \lambda I)^{-1}, \quad \text{for all } \lambda > 0. $$

For the convenience of the reader, recall the following fundamental result characterizing the Friedrichs and Krein–von Neumann extensions \cite{2}.

\textbf{Proposition 5.1.} \cite{2} Let $\dot{A}$ be a closed densely defined positive symmetric operator. Denote by $a$ the closure of the quadratic form

\begin{equation}
\dot{a}[f] = (\dot{A}f, f), \quad \text{Dom} [\dot{a}] = \text{Dom}(\dot{A}).
\end{equation}

Then the Friedrichs extension $A_F$ coincides with the restriction of the adjoint operator $(\dot{A})^*$ on the domain

$$ \text{Dom}(A_F) = \text{Dom}((\dot{A})^*) \cap \text{Dom}[a]. $$

The Krein–von Neumann extension $A_{K\nu N}$ coincides with the restriction of the adjoint operator $(\dot{A})^*$ on the domain $\text{Dom}(A_{K\nu N})$ which consists of the set of elements $f$ for which there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$, $f_n \in \text{Dom}(\dot{A})$, such that

$$ \lim_{n,m \to \infty} a[f_n - f_m] = 0 $$

and

$$ \lim_{n \to \infty} \dot{A}f_n = (\dot{A})^* f. $$

Now we are ready to present the aforementioned fixed point theorem in the case where the underlying $G$-invariant symmetric operator is semi-bounded.

\textbf{Theorem 5.2.} Let $G$ be a subgroup of the affine group $G$. Suppose that $\dot{A}$ is a closed, densely defined, semi-bounded from below symmetric $G$-invariant operator with the greatest lower bound $\gamma$. Then both the Friedrichs extension $A_F$ of the operator $\dot{A}$ and the extension $A_K = (\dot{A} - \gamma I)_{K\nu N} + \gamma I$, where $(\dot{A} - \gamma I)_{K\nu N}$ denotes the Krein–von Neumann extension of the positive symmetric operator $\dot{A} - \gamma I$, are $G$-invariant.

We provide a short proof the idea of which is due to Gerald Teschl \cite{12} (see \cite{30} for the original reasoning).

\textbf{Proof.} Suppose that $\dot{A}$ is $G$-invariant with respect to a unitary representation $G \ni g \mapsto U_g$. Denote by $\mathcal{I}$ the operator interval of self-adjoint extensions of $\dot{A}$ with the greatest lower bound greater or equal to $\gamma$. Recall that for any $A \in \mathcal{I}$ one has the operator inequality $A_K \leq A \leq A_F$. Since the symmetric operator $\dot{A}$ is $G$-invariant, the correspondence

$$ A \rightarrow g^{-1}(U_g A U_g^*), \quad g(x) = ax + b, $$

with $A$ a self-adjoint extension of $\dot{A}$, maps the operator interval $\mathcal{I} = [A_K, A_F]$ onto itself. Moreover, this mapping is operator monotone, that is, it preserves the order. Therefore, the end-points $A_K$ and $A_F$ of the operator interval $\mathcal{I}$ has to be fixed points of this map which completes the proof. \hfill $\square$

\textbf{Remark 5.3.} We remark that in the situation of Theorem 5.2, the greatest lower bound $\gamma$ has to be a fixed point for all transformations from the group $G$. Therefore, the hypothesis that $\dot{A}$ is semi-bounded is rather restrictive. In particular, this hypothesis implies that the group $G$ is necessarily a proper subgroup of the whole
affine group $G$. For instance, the group $G$ does not contain the transformations
$g(x) = x + b$, $b \neq 0$, without (finite) fixed points.

Next, we claim (under mild assumptions) that for finite deficiency indices of
$\dot{A}$ the flow (4.3) always has a fixed point. Therefore, any $G$-invariant symmetric
operator with finite deficiency indices possesses either a self-adjoint or a maximal
dissipative $G$-invariant extension of $\dot{A}$ (cf. [15, Theorem 15]).

**Theorem 5.4.** Let $G$ be either a cyclic or a one-parameter continuous subgroup of
affine transformations of the real axis into itself preserving the orientation. Suppose
that $G \ni g \mapsto U_g$ is a unitary representation in a Hilbert space $\mathcal{H}$. Assume that $\dot{A}$
is a closed symmetric $G$-invariant operator with finite deficiency indices.

Then the operator $\dot{A}$ admits a $G$-invariant maximal dissipative extension.

**Proof.** For any $g \in G$, the mapping $\Gamma_g : \mathcal{V} \to \mathcal{V}$ is a continuous mapping from the
unit ball $\mathcal{V}$ into itself. Since the deficiency indices of $\dot{A}$ are finite, $\mathcal{V}$ is a compact
convex set and therefore, by the Schauder fixed point theorem (see, e.g., [27], page
291), the mapping $\Gamma_g$ has a fixed point $V \in \mathcal{V}$. That is,

$$\Gamma_g(V) = V.$$

In particular,

$$\Gamma_g^n(V) = V \quad \text{for all } n \in \mathbb{Z}.$$ 

Therefore, the restriction $A$ of the adjoint operator $(\dot{A})^*$ onto the domain

$$\text{Dom}(A) = \text{Dom}(\dot{A}) + (I - V) \text{Ker}((\dot{A})^* - iI)$$

is a $G[g]$-invariant operator, where $G[g]$ is the cyclic group generated by the element
$g \in G$, proving the claim in the case of cyclic groups $G$.

In order to treat the case of one-parameter continuous subgroups $G = \{g_t\}_{t \in \mathbb{R}}$,
we remark that by the first part of the proof for any $n \in \mathbb{N}$ there exists a $V_n \in \mathcal{V}$
such that

$$(5.2) \quad \Gamma_{g_q}(V_n) = V_n \quad \text{for all } q \in \mathbb{Z}/n.$$

Since by hypothesis $\dot{A}$ has finite deficiency indices and hence the unit ball $\mathcal{V}$ is
compact, one can find a $V \in \mathcal{V}$ and a subsequence $\{n_k\}_{k \in \mathbb{N}}$ such that

$$(5.3) \quad \lim_{k \to \infty} V_{n_k} = V.$$

Given $t \in \mathbb{R}$, choose a sequence of rationals $\{q_k\}_{k \in \mathbb{N}}$ such that

$$(5.4) \quad \lim_{k \to \infty} q_k = t, \quad g_k \in \mathbb{Z}/n_k.$$

Combining (5.2) and (5.3), one obtains that

$$\Gamma_{g_{q_k}}(V_{n_k}) = V_{n_k}.$$

Going to the limit $k \to \infty$ in this equality, from (5.3) and (5.4) one concludes that

$$\Gamma_{g_t}(V) = V \quad \text{for all } t \in \mathbb{R}$$

upon applying Corollary 4.2. The proof is complete. $\square$
Remark 5.5. If $\hat{A}$ has equal deficiency indices and if a fixed point $V$ is an isometry from $\text{Ker}((\hat{A})^* - iI)$ onto $\text{Ker}((\hat{A})^* + iI)$, then the corresponding maximal dissipative extension is self-adjoint. We also remark that Theorem 5.4 does not guarantee the existence of a $G$-invariant self-adjoint extensions of $\hat{A}$ in general. In other words, the restricted flow $\Gamma_g|_U$, with $U$ the set of all isometries from $\text{Ker}((\hat{A})^* - iI)$ onto $\text{Ker}((\hat{A})^* + iI)$, may not have fixed points at all (see Section 7 for the corresponding examples and Remark 7.3 for a detailed discussion of the group-theoretic descent of those examples).

Remark 5.6. It is worth mentioning that Theorem 5.4 is a simple consequence of the following Lefschetz fixed point theorem for flows on manifolds.

Proposition 5.7. (see, eg., [41, Theorem 6.28]) If $M$ is a closed oriented manifold such that the Euler characteristic $\chi(M)$ of $M$ is not zero, then any flow on $M$ has a fixed point.

Indeed, take $M = V$ and notice that $V$ is a closed convex set with $\chi(V) \neq 0$.

6. The case of deficiency indices $(1, 1)$

In this subsection we provide several results towards the solution of the extension problem (1.5) in the dissipative as well as in the self-adjoint settings that are specific to the case of deficiency indices $(1, 1)$ only.

Hypothesis 6.1. Assume that $G$ is a subgroup of the affine group $G$. Suppose, in addition, that $\hat{A}$ is a $G$-invariant closed symmetric operator with deficiency indices $(1, 1)$.

Under Hypothesis 6.1, based on the results of Proposition 3.2 and Lemma 4.1 one can identify the flow $G \ni g \mapsto \Gamma_g$ with the representation of the group $G$ into the group $\text{Aut}(D)$ of automorphisms of the unit disk $D$ (consisting of linear-fraction transformations $z \mapsto e^{i\theta} \frac{z-a}{1-\bar{a}z}$ for some $|a| < 1$ and $\theta \in \mathbb{R}$, mapping the unit circle $T$ into itself).

Recall the following elementary result.

Proposition 6.2. Any automorphism $\Gamma$ of the (closed) unit disk $D$ different from the identical map has at most two different fixed points in $D$. If $\Gamma$ has exactly two fixed points, then both of them lie on the boundary $T$ of $D$.

Theorem 6.3. Assume Hypothesis 6.1. Suppose that $\hat{A}$ has either

(i) at least three different self-adjoint $G$-invariant extensions

or

(ii) at least two $G$-invariant maximal dissipative extensions one of which is not self-adjoint.

Then any maximal dissipative extension of $\hat{A}$ is $G$-invariant. Moreover, in this case, if the group $G$ is nontrivial, then $\hat{A}$ is not semi-bounded.

Proof. In case (i), for any $g \in G$ the automorphism $\Gamma_g$ of the unit disk $D$ has at least three different fixed points. Hence, by Proposition 6.2, $\Gamma_g$ is the identical map on $D$ and thus any maximal dissipative extension $A$ of $\hat{A}$ such that $\hat{A} \subset A \subset (\hat{A})^*$ is $G$-invariant.

In case (ii), for any $g \in G$ the automorphism $\Gamma_g$ has at least two fixed points, one of which is not on the boundary of the unit disk and hence, again, by Proposition
6.2, the automorphism $\Gamma_g$, $g \in G$, is the identical map, proving that any maximal dissipative extension of $\dot{A}$ is $G$-invariant.

To prove the last statement of the theorem, assume, to the contrary, that $\dot{A}$ is a semi-bounded symmetric operator. Suppose, for definiteness, that $\dot{A}$ is semi-bounded from below.

Assume that one can find a point $\lambda_0 \in \mathbb{R}$ to the left from the greatest lower bound of $\dot{A}$ such that the set $\Sigma = \bigcup_{g \in G} g(\lambda_0)$ is unbounded from below, that is,

$$\inf \Sigma = -\infty.$$  \hspace{1cm} (6.1)

By Krein’s theorem (see [20], Theorem 23) there exists a self-adjoint extension $\hat{A}$ of the symmetric operators $\dot{A}$ such that $\lambda_0$ is a simple eigenvalue of $\hat{A}$. Since any maximal dissipative extension of $\dot{A}$ is $G$-invariant, the self-adjoint operator $\hat{A}$ is also $G$-invariant. Since the spectrum of a $G$-invariant operator is a $G$-invariant set, one concludes that the set $\Sigma$ belongs to the pure point spectrum of $\hat{A}$ and hence $\hat{A}$ is not semi-bounded from below for (6.1) holds. The latter is impossible: any self-adjoint extension of a semi-bounded symmetric operator with finite deficiency indices is semi-bounded from below.

To justify the choice of $\lambda_0$ with the property (6.1) one can argue as follows.

By hypothesis, the group $G$ is nontrivial and, therefore, $G$ contains a transformations $g$ either of the form $g(x) = x + b$ for some $b \neq 0$ or $g(x) = a(x - \gamma) + \gamma$ for some $a > 0$, $a \neq 1$, and $\gamma \in \mathbb{R}$.

In the first case, the set $\Sigma$ contains the orbit $\bigcup_{n \in \mathbb{Z}} \{\lambda_0 + nb\}$ and hence one can take any $\lambda_0$ to the left from the greatest lower bound of $\dot{A}$ to meet the requirement (6.1). In the second case,

$$\bigcup_{n \in \mathbb{Z}} \{a^n(\lambda_0 - \gamma) + \gamma\} \subset \bigcup_{g \in G} g(\lambda_0) = \Sigma.$$  

Hence, one can choose any $\lambda_0 < \gamma$ to the left from the greatest lower bound of $\dot{A}$ for (6.1) to hold.

\[ \square \]

Our next result shows that a semi-bounded $G$-invariant symmetric operator with deficiency indices $(1,1)$ possesses self-adjoint $G$-invariant extensions only.

**Theorem 6.4.** Assume Hypothesis 6.1. Suppose, in addition, that $\hat{A}$ is semi-bounded from below.

If $A$ is a maximal dissipative $G$-invariant extension of $\dot{A}$, then $A$ is necessarily self-adjoint. Moreover, either $A = A_F$ or $A = A_K$ where $A_F$ and $A_K$ are the operators referred to in Theorem 5.2.

**Proof.** Assume to the contrary that $A$, with $\hat{A} \subset A \subset (\hat{A})^*$, is a $G$-invariant maximal dissipative extension of $\dot{A}$ different from $A_F$ and $A_K$.

If $A_F \neq A_K$, then $\hat{A}$ has at least three different $G$-invariant extensions. Therefore, by Theorem 6.3, the operator $\hat{A}$ is not semi-bounded from below which contradicts the hypothesis. Thus, either $A = A_F$ or $A = A_K$.

If the extensions $A_F$ and $A_K$ coincide, that is,

$$A_F = A_K,$$  \hspace{1cm} (6.2)

one proceeds as follows.
By hypothesis, $A \neq A_F = A_K$, the $G$-invariant maximal dissipative extension $A$ is necessarily self-adjoint. Indeed, otherwise, $\dot{A}$ has at least two $G$-invariant maximal dissipative extensions, $A_F$ and $A$, one of which is not self-adjoint. Applying Theorem 6.3 shows that the symmetric operator $\dot{A}$ is not semi-bounded which contradicts the hypothesis. Thus, $\dot{A} = A_F = A_K$.

Denote by $\tilde{\gamma}$ the greatest lower bound of $A$. Since $A_K = A_F$, the greatest lower bound $\tilde{\gamma}$ of $A$ is strictly less than the greatest lower bound $\gamma$ of the symmetric operator $\dot{A}$. Following the strategy of proof of the second statement of Theorem 6.3, one concludes that the orbit $\Sigma = \bigcup_{g \in G} g(\tilde{\gamma})$ belongs to the spectrum of the $G$-invariant self-adjoint operator $A$. By Remark 5.3, the greatest lower bound $\gamma$ is a fixed point for any transformation $g$ from the group $G$,

$g(\gamma) = \gamma, \quad g \in G,$

that is, $g(x) = a(x - \gamma) + \gamma$ for some $a > 0$ (cf. (2.5)). Since $\tilde{\gamma} < \gamma$ and (6.3) holds, it is also clear that $\inf \Sigma = -\infty$. Therefore, $A$ is not semi-bounded from below, so is $\dot{A}$. A contradiction. \hfill \Box

We recall, see Remark 5.5, that the fixed point result, Theorem 5.4, does not guarantee the existence of a self-adjoint unitarily affine invariant extension of a symmetric invariant operator in general. However, a $G$-invariant symmetric operator with deficiency indices $(1, 1)$ always has an affine invariant self-adjoint extension if not with respect to the whole group $G$ but at least with respect to a cyclic subgroup of $G$, provided that $G$ is a continuous one-parameter group.

**Theorem 6.5.** Assume Hypothesis 6.1. Suppose that $G = \{g_t\}_{t \in \mathbb{R}}$ is a continuous one-parameter group and that $\dot{A}$ has no self-adjoint $G$-invariant extension.

Then there exists a discrete cyclic subgroup $G[g]$ generated by some element $g \in G$ such that any maximal dissipative, in particular, any self-adjoint extension of $\dot{A}$ is $G[g]$-invariant.

**Proof.** Let $V \in \mathbb{T}$ be the von Neumann parameter on the unit circle and $A_V$ the associated self-adjoint extension of $A$. Introduce the continuous flow on the circumference $\mathbb{T}$ by

$\varphi(t, V) = \Gamma_{g_t}(V), \quad t \in \mathbb{R}, \quad g_t \in G.$

To prove the assertion, it is sufficient to show that the map $t \mapsto \varphi(t, V)$ is not injective for some $V \in \mathbb{T}$.

Indeed, if this map is not injective, then there exist two different values $t_1$ and $t_2$ of the parameter $t$ such that

$\varphi(t_1, V) = \varphi(t_2, V).$

Therefore, $V$ is a fixed point of the transformation $\varphi(t_2 - t_1, \cdot)$. Hence, for any $n \in \mathbb{Z}$

$V = \varphi(n(t_2 - t_1), V),$

and, therefore, the extension $A[g] = A_V$ is $G[g]$-invariant with respect to the discrete subgroup $G[g]$ generated by the element

$g = g_{t_1 - t_2}.$

By hypothesis the self-adjoint operator $A[g]$ is not $G$-invariant (with respect to the whole subgroup $G$) and therefore $V$ is not a fixed point for the whole family of
the transformations $\gamma_t$. In particular, the orbit $S_V = \cup_{t \in \mathbb{R}} \varphi(t, V)$ is an infinite set. Next, if $W = \varphi(t, V) \in S_V$ for some $t \in \mathbb{R}$, then
\[ \varphi(T, W) = \varphi(T + t, V) = \varphi(t, V) = W, \]
where
\[ (6.6) \quad T = t_2 - t_1. \]
Therefore $\gamma_n T(W) = W$ for all $W \in S_V$ and all $n \in \mathbb{Z}$ and hence the corresponding self-adjoint operators $A_W$, $W \in S_V$, are $G[g]$-invariant. Thus, since the set $S_V$ is infinite, the operator $\hat{A}$ has infinitely many, and, therefore, at least three different $G[g]$-invariant self-adjoint extensions. Therefore, by Theorem 6.3, every maximal dissipative extension of $\hat{A}$ (obtained as a restriction of $(\hat{A})^*$) is $G[g]$-invariant.

To complete the proof it remains to justify the claim that the map $t \mapsto \varphi(t, V)$ is not injective for some $V \in \mathbb{T}$. To do that assume the opposite. That is, suppose that the map $t \mapsto \varphi(t, V)$ is injective for all $V \in \mathbb{T}$. Then, by Brouwer’s invariance of domain theorem [5], the set $S_V = \bigcup_{t \in \mathbb{R}} \varphi(t, V)$ is an open arc.\(^1\) Thus,
\[ \mathbb{T} = \bigcup_{V \in \mathbb{T}} S_V \]
is an open cover. Since $\mathbb{T}$ is a compact set, choosing a finite sub cover,
\[ \mathbb{T} = \bigcup_{k=1}^N S_{V_k}, \quad \text{for some} \quad N \in \mathbb{N}, \]
we get that $\mathbb{T}$ is covered by finitely many open arcs and therefore some of them must intersect. If so, the circumference $\mathbb{T}$ would be the trajectory of some point $V_\ast \in \mathbb{T}$. That is, $\mathbb{T} = S_{V_\ast}$, which means that the map $t \mapsto \varphi(t, V_\ast)$ is not injective. A contradiction.

The proof is complete. \(\square\)

**Remark 6.6.** It follows from Theorems 5.4 and 6.3 that under hypothesis of Theorem 6.5, that is, if $\hat{A}$ has no $G$-invariant self-adjoint extensions, the symmetric operator $\hat{A}$ admits a unique non-self-adjoint maximal dissipative $G$-invariant extension $A$ with the property $\hat{A} \subset A \subset (\hat{A})^*$.

**Remark 6.7.** Concrete examples show (see Examples 7.1 and 7.2 and Subsections 7.1 and 7.5, respectively) that our hypothesis that $\hat{A}$ has no self-adjoint $G$-invariant extension is not empty. Accordingly, this hypothesis implies that the flow $\varphi(t, V) = \gamma_t(V)$ on the circumference $M = \mathbb{T}$ given by (6.4) has no fixed point. It is no surprise: the Euler characteristics of $\mathbb{T}$ is zero, $\chi(\mathbb{T}) = 0$, and hence Proposition 5.7 cannot be applied. Instead, the flow is periodic with period $T$ given by (6.6) and thus
\[ \varphi(T, \cdot) = \text{Id}, \]
So, any point $V \in \mathbb{T}$ is a fixed point,
\[ \varphi(nT, V) = V, \quad n \in \mathbb{Z}. \]

\(^{1}\)We are indebted to N. Yu. Netsvetaev who attracted our attention to this fact.
7. EXAMPLES

In this section we provide several examples that illustrate our main results. In particular, we show that, even in the simplest cases, the restricted standard or generalized Weyl commutation relations possess a variety of non-unitarily equivalent solutions.

Example 7.1. Suppose that $G = \{ g_t \}_{t \in \mathbb{R}}$ is the one-parameter group of translations

$$ g_t(x) = x + t. $$

Let $\hat{A}$ be the differentiation operator in $L^2(0, \ell)$,

$$ (\hat{A} f)(x) = i \frac{d}{dx} f(x) \quad \text{on} \quad \text{Dom}(\hat{A}) = \{ f \in W^1_2(0, \ell) \mid f(0) = f(\ell) = 0 \}. $$

It is well known [1] that $\hat{A}$ has deficiency indices $(1, 1)$. Moreover, $\hat{A}$ is obviously $G$-invariant with respect to the group of unitary transformations given by

$$ (U_t f)(x) = e^{i \Theta} f(x), \quad f \in L^2(0, \ell). $$

7.1. First, we notice that the symmetric $G$-invariant operator $\hat{A}$ from Example 7.1 does not admit self-adjoint $G$-invariant extensions.

Indeed, if $A$ is a self-adjoint extension of $\hat{A}$, then there exists a $\Theta \in [0, 2\pi)$ such that $A$ coincides with the differentiation operator on

$$ \text{Dom}(A) = \{ f \in W^1_2(0, \ell) \mid f(\ell) = e^{i \Theta} f(0) \}. $$

Therefore, as a simple computation shows, the spectrum of the operator $A$ is discrete and it consists of simple eigenvalues forming the arithmetic progression

$$ \text{spec}(A) = \bigcup_{n \in \mathbb{Z}} \left\{ \frac{\Theta + 2\pi n}{\ell} \right\}. $$

However, the spectrum of a $G$-invariant operator is a $G$-invariant set: if $\lambda$ is an eigenvalue of $A$, then for all $t \in \mathbb{R}$ the point $g_t(\lambda) = \lambda + t$ is also an eigenvalue of $A$ which is impossible, for the underlying Hilbert space is separable.

Thus, $\hat{A}$ has no $G$-invariant self-adjoint extension.

This observation justifies the claim from Remark 5.5 that the restricted flow $\Gamma_g|_{\mathcal{U}}$ may not have fixed points at all, even in the case of equal deficiency indices.

7.2. Next, since $\hat{A}$ from Example 7.1 has no $G$-invariant self-adjoint extension, our general fixed point Theorem 5.4 states that there exists a maximal dissipative $G$-invariant extension. In turn, Theorem 6.3 shows then that this extension is unique.

Clearly, that $G$-invariant extension is given by the differentiation operator with the Dirichlet boundary condition at the left end-point of the interval,

$$ (Af)(x) = i \frac{d}{dx} f(x) \quad \text{on} \quad \text{Dom}(A) = \{ f \in W^1_2(0, \ell) \mid f(0) = 0 \}. $$

7.3. It is worth mentioning that the dissipative extension $A$ defined by (7.3) generates a nilpotent semi-group of shifts $V_s$ with index $\ell$. Indeed,

$$ (V_s f)(x) = \begin{cases} 
  f(x - s), & x > s \\
  0, & x \leq s
\end{cases}, \quad f \in L^2(0, \ell), $$

and therefore $V_s = 0$ for $s \geq \ell$. 

An immediate computation shows that the restricted Weyl commutation relations
\begin{equation}
U_t V_s = e^{ist} V_s U_t, \quad t \in \mathbb{R}, \quad s \geq 0,
\end{equation}
for the unitary group $U_t$ and the semi-group of contractions $V_s$ are satisfied.

We remark that the generators $A$, the differentiation operators on a finite interval defined by (7.3), are not unitarily equivalent for different values of the parameter $\ell$, the length of the interval $[0, \ell]$ (see [1]). Therefore, the restricted Weyl commutation relations (7.4) admit a variety of non-unitarily equivalent solutions.

7.4. Finally, we illustrate Theorem 6.5.

One observes that any maximal dissipative, in particular, self-adjoint extension of $A$ has the property that
\begin{equation}
U_t (\text{Dom}(A)) = \text{Dom}(A) \quad \text{for all} \quad t = \frac{2\pi n}{\ell}, \quad n \in \mathbb{Z}.
\end{equation}
Therefore, $A$ is $G[g]$-invariant, where $G[g]$ is the discrete subgroup of the one-parameter group $G$ (7.1) generated by the translations
\[ g(x) = x + \frac{2\pi}{\ell}. \]

Indeed, as it is shown in [1], the domain of any maximal dissipative extensions admits the representation
\[ \text{Dom}(A) = \{ f \in W^1_2(0, \ell) | f(0) = \rho f(\ell) \} \]
for some $|\rho| \leq 1$. Therefore, the domain invariance property (7.5) automatically holds. In particular, the spectrum of $A$ is a lattice in the upper half-plane with period $\frac{2\pi}{\ell}$, whenever $A$ is maximal dissipative non-self-adjoint. If $|\rho| = 1$, and hence the operator $A$ is self-adjoint, the spectrum coincides with the set of points forming the arithmetic progression (7.2) (for some $\Theta$).

Our second example relates to the homogeneous Schrödinger operator with a singular potential. In this example, the underlying one-parameter group coincides with the group of scaling transformations of the real axis.

**Example 7.2.** Suppose that $G = \{ g_t \}_{t \in \mathbb{R}}$ is the group of scaling transformations
\[ g_t(x) = e^{t} x. \]
Let $\hat{A}$ be the closure in $L^2(0, \infty)$ of the symmetric operator given by the differential expression
\begin{equation}
\tau = -\frac{d^2}{dx^2} + \frac{\gamma}{x^2}
\end{equation}
initially defined on $C^\infty_0(0, \infty)$.

It is known, see, e.g., [38, Ch. X], that
\begin{enumerate}
  \item if $\gamma < \frac{1}{4}$, the operator $\hat{A}$ has deficiency indices $(1,1)$,
  \item it is non-negative for $-\frac{1}{4} \leq \gamma$,
  \item $\hat{A}$ is not semi-bounded from below for $\gamma < -\frac{1}{4}$.
\end{enumerate}
Regardless of the magnitude of the coupling constant $\gamma$, the operator $\dot{A}$ is $G$-invariant (homogeneous) with respect to the unitary group of scaling transformations

\[(U_t f)(x) = e^{\frac{t}{2}} f(e^{\frac{t}{2}} x), \quad f \in L^2(\mathbb{R}_+).\]

7.5. First, we notice that as in Example 7.1, the symmetric $G$-invariant operator $\dot{A}$ from Example 7.2 with $\gamma < -\frac{1}{4}$ does not admit self-adjoint $G$-invariant extensions at all.

Indeed, it is well known (see, e.g., [34]) that if $A$ is a self-adjoint extension of $\dot{A}$, then the domain of $\dot{A}$ can be characterized by the following asymptotic boundary condition: there exists a $\Theta \in [0, 2\pi)$ such that for any $f \in \text{Dom}(A) \subset W^2_{2,\text{loc}}(0, \infty)$

the asymptotic representation

\[(7.8) \quad f(x) \sim C \sqrt{x} \sin (\nu \log x + \Theta) \quad \text{as} \quad x \downarrow 0,
\]

holds for some $C \in \mathbb{C}$, where

\[\nu = \sqrt{|\gamma + \frac{1}{4}|}.\]

Clearly,

\[U_t (\text{Dom}(A)) = \text{Dom}(A) \quad \text{for all} \quad t = \frac{4\pi n}{\nu}, \quad n \in \mathbb{Z},\]

where the scaling group $U_t$ is given by (7.7). Therefore, the extension $A$ is $G[g]$-invariant where $G[g]$ denotes the discrete cyclic subgroup generated by the linear transformation $g \in G$,

\[(7.9) \quad g(x) = \kappa x \quad \text{for} \quad \kappa = e^{4\pi/\nu}.\]

In particular, if $\lambda_0$ is a negative eigenvalue of the self-adjoint extension $A$, then the two-sided geometric progression

\[(7.10) \quad \lambda_n = \kappa^n \lambda_0, \quad n \in \mathbb{Z},\]

fills in the (negative) discrete spectrum of $A$.

It remains to argue exactly as in the case of the differentiation operator on a finite interval from Example 7.1 to show that $\dot{A}$ has no $G$-invariant self-adjoint extension. Also, applying Theorem 6.5 shows that any maximal extension of $\dot{A}$ is invariant with respect to the cyclic group $G[g]$, where $g$ is given by (7.9).

Remark 7.3. We remark that the self-adjoint realizations of the Schrödinger operator

\[H = -\frac{d^2}{dx^2} + \frac{\gamma}{x^2}, \quad \gamma < -\frac{1}{4},\]

defined by the asymptotic boundary conditions (7.8), are well-suited for modeling the fall to the center phenomenon in quantum mechanics [23, Sec. 35], see also [33] and [34].

From the group-theoretic standpoint this phenomenon can be expressed as follows. The initial symmetric homogeneous operator (with respect to the scaling
transformations) is neither semi-bounded from below nor admits self-adjoint homogeneous realizations, while any self-adjoint realization of the corresponding Hamiltonian is homogeneous with respect to a cyclic subgroup of the transformations only. As a consequence, its negative discrete spectrum as a set is invariant with respect to the natural action of this subgroup of scaling transformations of the real axis. That is,

\[ \text{spec}(A) \cap (-\infty, 0) = \bigcup_{n \in \mathbb{Z}} \{ \kappa^n \lambda_0 \}. \]

7.6. Theorem 5.4 combined with Theorem 6.3 shows that \( \dot{A} \) has a unique maximal dissipative \( G \)-invariant extension which coincides with the following homogeneous operator \( A \), the restriction of the adjoint operator \( (\dot{A})^* \) on

\[ \text{Dom}(A) = \left\{ f \in \text{Dom}((\dot{A})^*) \left| \lim_{x \downarrow 0} x^{1/2+\sqrt{|\gamma+\frac{1}{4}}}} f(x) \text{ exists} \right. \right\}. \]

It is clear that \( A \) is \( G \)-invariant. We claim without proof that \( A \) is a maximal dissipative operator.

7.7. The dissipative Schrödinger operator \( A \) with a singular potential from subsection 7.6 generates the contractive semi-group \( V_s = e^{is\dot{A}} \), \( s \geq 0 \). In accordance with Theorem 2.6, the restricted generalized Weyl commutation relations

\[ U_t V_s = \exp(is\epsilon t)V_s U_t, \quad t \in \mathbb{R}, \quad s \geq 0, \]

hold. One can also show that the homogeneous extensions \( A \) are not unitarily equivalent for different values of the coupling constant \( \gamma \) satisfying the inequality \( \gamma < -\frac{1}{4} \). Therefore, the restricted generalized Weyl commutation relations (7.12) admit a continuous family of non-unitarily equivalent solutions (cf. Subsection 7.3).

7.8. Finally, we illustrate Theorem 6.4 which deals with the case of semi-bounded \( G \)-invariant operators.

Assuming that

\[ -\frac{1}{4} \leq \gamma < \frac{3}{4}, \]

one observes that the symmetric operator \( \dot{A} \) is non-negative. Moreover, it is clear that the following two homogenous extensions, the restrictions of the adjoint operator \( (\dot{A})^* \) on

\[ \text{Dom}(A_F) = \left\{ f \in \text{Dom}((\dot{A})^*) \left| \lim_{x \downarrow 0} x^{1/2+\sqrt{|\gamma+\frac{1}{4}}}} f(x) \text{ exists} \right. \right\} \]

and on

\[ \text{Dom}(A_K) = \left\{ f \in \text{Dom}((\dot{A})^*) \left| \lim_{x \downarrow 0} x^{1/2-\sqrt{|\gamma+\frac{1}{4}}}} f(x) \text{ exists} \right. \right\} , \]

respectively, are \( G \)-invariant with respect to the unitary representation (7.7). We remark that the self-adjoint realization \( A_F \) and \( A_K \) (cf. [9, 11, 19]) can be recognized as the Friedrichs and Krein-von Neumann extensions of \( \dot{A} \), respectively, which agrees with the statement of Theorem 6.4.

We also remark that under hypothesis (7.13), the corresponding flow has two different fixed point on the unit circle. In the critical case of

\[ \gamma = -\frac{1}{4}, \]
the extensions $A_K$ and $A_F$ coincide. Hence, the flow (4.3) has only one fixed point on the boundary $T$ of the unit disk $\mathbb{D}$ (cf. Proposition 6.2).

We conclude this section by an example of a $G$-invariant generator that has deficiency indices $(0, 1)$. We will see below that this example is universal in the following sense. The semi-group of unilateral shifts $V_s = e^{is\hat{A}}$, $s \geq 0$, generated by $\hat{A}$ solves the restricted standard as well as generalized Weyl commutation relations (2.8) and (2.9).

**Example 7.4.** Let $\hat{A}$ be the Dirichlet differentiation operator on the positive semi-axis (in $L^2(0, \infty)$),

$$(\hat{A}) f(x) = i \frac{d}{dx} f(x) \quad \text{on} \quad \text{Dom}(\hat{A}) = \{ f \in W^1_2(0, \infty) \mid f(0) = 0 \}.$$ 

It is well known, see, e.g., [1], that the symmetric operator $\hat{A}$ has deficiency indices $(0, 1)$ and it is simultaneously a maximal dissipative operator.

Suppose that $G^{(1)} = \{ g_t \}_{t \in \mathbb{R}}$ is the group of affine transformations

$$g_t(x) = x + t.$$ 

Clearly, $\hat{A}$ is $G^{(1)}$-invariant with respect to the group of unitary transformations given by

$$(U_t^{(1)}) f(x) = e^{ixt} f(x), \quad f \in L^2(0, \infty).$$

Denoting by $G^{(2)} = \{ g_t \}_{t \in \mathbb{R}}$ the group of scaling transformations

$$g_t(x) = e^t x,$$

one also concludes that $\hat{A}$ is $G^{(2)}$-invariant with respect to the group of unitary transformations given by

$$(U_t^{(2)}) f(x) = e^\frac{x}{2} f(e^t x), \quad f \in L^2(0, \infty).$$

In particular, for the semi-group of unilateral right shifts $V_s = e^{is\hat{A}}$ generated by the differentiation operator $A = \hat{A}$, one gets the restricted Weyl commutation relations

$$U_t^{(1)} V_s = e^{ist} V_s U_t^{(1)}, \quad t \in \mathbb{R}, \quad s \geq 0,$$

as well as the restricted generalized Weyl commutation relations

$$U_t^{(2)} V_s = \exp(ise^t) V_s U_t^{(2)}, \quad t \in \mathbb{R}, \quad s \geq 0.$$

We also remark that the operator $A = \hat{A}$ is $G$-invariant with respect to the unitary representation $\mathcal{G} \ni g \mapsto U_g$ of the whole affine group $\mathcal{G}$ given by

$$(U_g) f(x) = a^{1/2} e^{ibx} f(ax), \quad f \in L^2(0, \infty), \quad g \in \mathcal{G}, \quad g(x) = ax + b.$$ 

**REFERENCES**


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