TWO SIDES OF THE COIN: THE EFIMOV EFFECT AND COLLAPSE IN THREE-BODY SYSTEMS WITH POINT-LIKE INTERACTIONS. I

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Three-body systems with point-like interactions and internal structures are considered. A complete classification of these systems is carried out and the conditions for the corresponding energy operators to be semi-bounded from below are studied.

We dedicate this paper to the memory of Stanislav Petrovitch Merkuriev. Three years have passed since he left us. There are no more Thursday seminars in one of the auditoriums of the University's main building. There, Merkuriev, after listening to the speaker, or, sometimes just starting to listen to him, would have already seen the problem from the inside, proposed a possible, efficient way of solving it, or, on the contrary, expressed his critical view on the subject, though always amazingly tolerant. During the preparation of this manuscript, we remembered the paraphrased Hallmosh sentence: “First, choose a potential reader (listener), then write (speak).” Following this advice, we choose Merkuriev as the potential listener in hopes of reproducing the traditional style of his seminar. We invite the reader to become a participant in this seminar where Stanislav Petrovitch is invisibly present.

1. Introduction

In the 1960's, two remarkable events happened in the spectral theory of the multiparticle Schrödinger equation. The first was the discovery of a collapse in the three-particle system with $\delta$-function-type interactions [1, 2], and the second—the Efimov effect—was the discovery of an infinite, discrete spectrum in some three-particle systems with rapidly expanding potentials.

Historically, the first mathematical explanation of the Efimov effect was due to Faddeev.\footnote{L. D. Faddeev, private communication.} He noticed that the existence of an infinite series of bounded states for the energy operator of a three-particle system with rapidly expanding interaction potentials (in the presence of virtual levels in two-particle subsystems) can be proved by studying the solvability conditions at large negative energies for the Skornyakov–Ter-Martirosyan equations [4] realized by Danilov [5]. Investigation of the solvability conditions was carried out by Faddeev and Minlos [1, 2]. Faddeev drew the attention of Yafaev to this problem, which resulted in the appearance of the complete mathematical theory of the Efimov effect [6]. However, this theory has a qualitative character.

In [7], it was clearly stated that there is a close, quantitative connection between the eigenvalue asymptotics at minus infinity in the three-particle system with $\delta$-function-like potentials and the asymptotics of eigenvalue accumulation towards the decay threshold in the Efimov effect. In [7] a universality hypothesis for the Efimov effect was proposed, which claims that these asymptotics are determined by the ratio of the particle masses and do not depend on the details of the two-particle potential behavior (of course, assuming there are virtual levels in subsystems). The proof for a similar (but weaker) statement can be found in [8].

Thus, the collapse in a system of three particles with $\delta$-function interactions and the Efimov effect (at least, if a virtual sublevel exists in each two-particle subsystem) are two sides of the same phenomenon.
However, this analogy is incomplete. The full class of three-particle systems (with rapidly expanding interactions) can be split into four subclasses depending on the number of two-particle subsystems for which the virtual level exists. The Efimov effect takes place only in two of these subclasses. An analogous classification of three-particle systems with $\delta$-function interactions such that the energy operator is nonsemibounded from below in two subclasses and the collapse phenomenon is absent in two other subclasses is, in principle impossible. Every self-adjoint realization of the energy operator in such systems is an operator nonsemibounded from below [9]. (Of course, we do not deal with the trivial case where particles merely do not interact in one (or a few) subsystems.) However, in the wider class of systems whose point-like interactions have an internal structure, which was proposed and partly explored in [10–14], such a classification is possible. The first examples of three-particle systems with point-like interactions where the collapse phenomenon is not observed appeared in papers by Shondin [10] and Thomas [11].

In [15], B. S. Pavlov proposed a model where a $\delta$-function potential endowed with an internal structure appeared from consideration of self-adjoint extensions of the Laplace operator in the case where the basic Hilbert space is enlarged to an arbitrary Hilbert space. After this paper, it became clear that the $S$-matrix class corresponding to usual $\delta$-potentials had been considerably enriched. It was parameterized by an arbitrary $R$-function $\omega(z)$ of parameter $z$,

$$S(k) = \frac{\omega(k^2) - ik}{\omega(k^2) + ik},$$

(1.1)

rather than by the real parameter $\omega$,

$$S(k) = \frac{\omega - ik}{\omega + ik}.$$  

(1.2)

Note that, historically, it was Schrader [16] who first discovered the "internal" structure. He successfully used analogous extensions into the functional one-dimensional space to regularize the Hamiltonian in the nonrelativistic Lie model.

From the representation theorem,

$$\omega(z) = Az + B + \int \frac{zs + 1}{s - z} d\mu(s),$$

(1.3)

where $A \geq 0$, $\text{Im} B \geq 0$, and $\mu$ is the finite Borel measure. The corresponding models can be naturally divided into two classes, depending on whether the $R$-function $\omega(z)$ contains the "linear in $z$ term" in the Herglotz representation (1.3) ($A > 0$) or not ($A = 0$).

We select two cases (from now on, we assume that the measure $\mu$ has a compact support in representation (1.3)).

Case a: the $S$-matrix of the model at high energies has a behavior typical of the usual potential scattering:

$$S(k) \to 1.$$  

(1.4)

Case b: the $S$-matrix has an "anomalous" behavior, which is typical, in particular, of the usual $\delta$-potential:

$$S(k) \to -1.$$  

(1.5)

In [12], Pavlov investigated a three-particle system with an internal structure such that case "a" was realized in each of its two-particle subsystems. He also proved the semiboundedness of the three-particle energy operator. Unfortunately, the interactions proposed in [12], strictly speaking, turned out not to be pairwise. A comprehensive modification to the case of pairwise interactions is contained in the paper by one of the authors [13]. There, the direct proof of the semiboundedness of the corresponding Hamiltonian follows from estimates on its quadratic form. The scattering theory for such systems was developed in [14].
The case of a three-particle system with δ-potentials endowed with internal structures, where at least one of the S-matrices corresponding to some two-particle subsystem has anomalous behavior at high energies, has never been discussed in detail.

In accordance with classifications (1.4), (1.5), it is natural to divide the class of three-particle subsystems with point-like interactions into four subclasses. This classification is in complete analogy with the one for the three-particle systems with rapidly expanding interaction potentials. One should just replace the words “in subsystem α, there exists a virtual level” (the resonance at zero energy) with the words “in subsystem α, the corresponding S-matrix has an anomalous behavior at high energies.” Then the following statement, which is a literal repetition of the Efimov effect formulation, is valid (see, e.g., [6–8]).

**Theorem 1.1.** Each self-adjoint realization of the energy operator for a three-particle system, with the δ-interaction endowed with an internal structure, is semibounded from below iff at least two of the three S-matrices corresponding to the pairwise interactions have the proper behavior at high energies, i.e.,

\[ S_\alpha(k) \to 1 \quad \text{for at least two } \alpha \text{'s from the set } \alpha \in \{\{1,2\}, \{2,3\}, \{1,3\}\}. \]

A particular case of this statement, where all \( S_\alpha(k) \to 1 \) at \( k \to \infty \), was first formulated in [12] (see also [13]). The idea of a classification of interactions in terms of the two-particle subsystem S-matrix’s behavior at high energies also belongs to Pavlov [12].

In each case, we describe the pre-Hamiltonians of three-particle systems with point-like interactions as symmetrical operators in Hilbert space. We prove the part of this theorem (see Lemma 4.1) that relates to the semiboundedness property of the spectrum: if the S-matrices have proper behaviors at high energies (1.4) for at least two of the two-particle subsystems, then the three-particle energy operator is semibounded.

In Theorem 1.1, the proof of necessity, i.e., the demonstration of nonsemiboundness of the energy operator in the case where more than one two-particle system has an “anomalous” S-matrix (1.5), follows the original proof by Faddeev and Minlos [2]. The technical tools necessary for this proof are provided by [7, 8, 10, 17]. We postpone publication of the “details” to Part II of the present article.

**2. The Hamiltonian of the three-particle system with point-like interactions as a symmetrical operator**

In this section, we describe the energy operator for the three-particle system with an internal structure in momentum representation. The motivation [12, 13] and the discussion of the pairwise character of the interactions under consideration can be found in [13, 14].

In the space \( \mathbb{R}^6 \), we consider three systems of orthogonal coordinates (the Jacobi coordinates),

\[ P = k_\alpha \oplus p_\alpha, \quad \alpha = 1, 2, 3, \]

where the subscript \( \alpha \) enumerates all possible two-particle subsystems connected by the orthogonal transformations

\[ \begin{pmatrix} k_\alpha \\ p_\alpha \end{pmatrix} = U_{\alpha \beta} \begin{pmatrix} k_\beta \\ p_\beta \end{pmatrix}, \quad U_{\alpha \beta} = \begin{pmatrix} c_{\alpha \beta} & s_{\alpha \beta} \\ s_{\alpha \beta} & -c_{\alpha \beta} \end{pmatrix}. \]

For the matrices \( U_{\alpha \beta} \), the condition

\[ U_{\alpha \beta} U_{\gamma \delta} U_{\eta \zeta} = 1 \]

holds. The coefficients \( c_{\alpha \beta} \), \( s_{\alpha \beta} \) \((c_{\alpha \beta}^2 + s_{\alpha \beta}^2 = 1)\) are expressed explicitly via the particle masses [17]. The free Hamiltonian \( H \) in the space \( L^2(\mathbb{R}^3) \) is

\[ H = -\frac{1}{2m_1} \Delta x_1 - \frac{1}{2m_2} \Delta x_2 - \frac{1}{2m_3} \Delta x_3. \]
Here \(x_1, x_2,\) and \(x_3\) are particle coordinates in configuration space after separating the propagation of the center of mass. This Hamiltonian is unitarily equivalent to the operator of multiplication by \(P^2, \ P \in \mathbb{R}^6,\) in the space

\[\mathcal{H}_{\text{ex}} = L_2(\mathbb{R}^6).\]  

To each two-particle subsystem \(\alpha\) (\(\alpha = 1, 2, 3,\)) we put into correspondence a Hilbert space (of internal degrees of freedom) \(\mathcal{H}_\alpha^{\text{lin}},\) a self-adjoint operator \(A_\alpha\) (an internal Hamiltonian acting in the space \(\mathcal{H}_\alpha^{\text{lin}},\)) and some element \(g_\alpha \in \mathcal{H}_\alpha^{\text{lin}}\) (the channel vector). Moreover, we assume that the automorphism of the upper half-plane

\[z \to \frac{a_\alpha + b_\alpha z}{c_\alpha + d_\alpha z},\]  

such that

\[\det \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} = -1\]  

is given. Here the complex numbers \(a_\alpha, b_\alpha, c_\alpha,\) and \(d_\alpha\) lie on one line \(\{z = x + iy : x = \theta_\alpha y\}\) in the complex plane \(\mathbb{C}\) and \(\theta_\alpha \in \mathbb{R}\) is fixed.

These items determine the model with the \(\delta\)-function-like interaction proposed by Pavlov [15]. We follow the notation of [13], where the energy operator \(h_\alpha\) of a two-particle system was described as a self-adjoint operator in the space \(L_2(\mathbb{R}^3) \oplus \mathcal{H}_\alpha^{\text{lin}}.\)

Let us recall the most specific properties of the model (for details, see [12-14]). The resolvent of the operator \(h_\alpha\) has a block structure; its block, corresponding to the space \(L_2(\mathbb{R}^3)\) (the generalized Krein resolvents), is

\[P_{L_2(\mathbb{R}^3)}(h_\alpha - z)^{-1} \big|_{L_2(\mathbb{R}^3)} = \left(\hat{h} - z\right)^{-1} - T(z).\]  

Here \(P_{L_2(\mathbb{R}^3)}\) is the orthogonal projection operator from the space \(L_2(\mathbb{R}^3) \oplus \mathcal{H}_\alpha^{\text{lin}}\) to the subspace \(L_2(\mathbb{R}^3),\) \(\hat{h}\) is the operator of multiplication by \(p^2\) in the space \(L_2(\mathbb{R}^3)\) (the free Hamiltonian), and, eventually, \(T(z)\) is a one-dimensional integral operator with the kernel

\[T(p, k; z) = \frac{1}{p^2 - z} t_\alpha(z) \frac{1}{k^2 - z}.\]  

The function \(t_\alpha(z),\) which is further called the \(t\)-matrix (this is not quite a precise term) of the two-particle subsystem \(\alpha,\) is

\[t_\alpha(z) = \frac{1}{\omega_\alpha(z) + \frac{1}{\sqrt{4\pi}}},\]  

where the function \(\omega_\alpha\) has a positive imaginary part in the upper half-plane (the \(R\)-function). It is expressed, up to a constant term, as

\[\omega_\alpha(z) = \frac{d_\alpha + b_\alpha r(z)}{c_\alpha + a_\alpha r(z)} + \text{const}\]  

via the fractional-linear transformation of the quadratic form of the resolvent (evaluated on the channel vector \(g_\alpha\)) of the internal operator \(A_\alpha:\)

\[r(z) = \left(\left(A_\alpha - z\right)^{-1} g_\alpha, g_\alpha\right).\]  

The behavior of the function \(\omega_\alpha(z)\) at \(z \to \infty\) varies depending on whether the model parameter \(c_\alpha\) in (2.5) is equal to zero or not. For \(c_\alpha = 0,\) the function \(\omega(z)\) tends to infinity as \(z \to \infty;\) for \(c_\alpha \neq 0,\) the function \(\omega_\alpha(z)\) is bounded at infinity. Consequently, the \(t\)-matrix of system (2.9) also has different behaviors:

- **Case a** \((c_\alpha \neq 0):\)

\[t_\alpha(z) = O\left(\frac{1}{|z|}\right), \quad z \to -\infty,\]  

- **Case b** \((c_\alpha = 0,\) in the upper half-plane):
Case b \((c_\alpha = 0)\):

\[
t_\alpha(z) = O\left(\frac{1}{\sqrt{|z|}}\right), \quad z \to -\infty.
\]  

(2.13)

Here, in the limit \(k \to \infty\), the formally defined S-matrix of the model,

\[
S_\alpha(k) = \frac{\omega_\alpha(k^2) - \frac{i \hbar}{2k}}{\omega_\alpha(k^2) + \frac{i \hbar}{2k}},
\]

(2.14)

tends to unity \((1.4)\) in case "a" and manifests "anomalous" behavior \((1.5)\) in case "b".

Let us turn to the description of a three-particle energy operator with pairwise interactions determined by the two-particle operators \(h_\alpha, \alpha = 1, 2, 3\).

In the space \(\mathbb{R}^6\), each coordinate system \(\{p_\alpha, k_\alpha\}\) determines the three-dimensional plane \(\mathcal{M}_\alpha:\)

\[
\mathcal{M}_\alpha = \{P: k_\alpha = 0\}, \quad \alpha = 1, 2, 3.
\]

(2.15)

The three-particle pre-Hamiltonian \(H\) of the particles with internal structures acts in the Hilbert space of four-component functions

\[
\mathcal{H} = \mathcal{H}^{ex} \oplus \mathcal{H}^{in},
\]

where

\[
\mathcal{H}^{in} = \bigoplus_\alpha \{L^2(\mathcal{M}_\alpha) \otimes \mathcal{H}^{in}_\alpha\}.
\]

(2.17)

Denote by \(L^2_2(\mathbb{R}^n)\) the Hilbert space of functions \(f\) such that

\[
\|f\|_8^2 = \int_{\mathbb{R}^n} (1 + P^2)^4 |f(P)|^2 dP < \infty.
\]

(2.18)

In the space \(\mathcal{H}^{ex} = L^2(\mathbb{R}^6)\), consider the domain

\[
\mathcal{D} = L^2_2(\mathbb{R}^6) + \mathcal{N},
\]

(2.19)

where

\[
\mathcal{N} = \left\{ \Phi \in L^2(\mathbb{R}^6) : \Phi(P) = \sum_\alpha \frac{\varphi_\alpha(p_\alpha)}{P^2 + 1}, \quad \varphi_\alpha \in L^2_2(\mathcal{M}_\alpha) \right\}.
\]

(2.20)

Note that each element \(u^{ex} \in \mathcal{D}\) can be uniquely represented as

\[
u^{ex}(P) = u(P) + \Phi(P),
\]

(2.21)

where

\[
\Phi(P) = \sum_\alpha \frac{\varphi_\alpha(p_\alpha)}{P^2 + 1},
\]

(2.22)

the "densities" \(\varphi_\alpha\) are the elements of the space \(L^2_2(\mathcal{M}_\alpha)\), and

\[
u \in L^2_2(\mathbb{R}^6).
\]

(2.23)

In coordinate representation, the space \(\mathcal{N}\) is the space of functions that can be presented as simple layer potentials with densities \(\varphi_\alpha\) taken from the Sobolev classes \(W^2_2(\mathbb{R}^3)\). These densities are "dispersed" over the planes \(x_i = x_j\), where the particles from the pair \(\alpha = \{ij\}\) interact.

In the domain

\[
\mathcal{D}(H_0) = \mathcal{D} \oplus \{L^2_2(\mathcal{M}_\alpha) \otimes \mathcal{H}^{in}_\alpha\},
\]

(2.24)
consider the operator $H_0$ acting as follows:

$$H_0 : \left(\oplus u_\alpha \right) \rightarrow \left(\oplus \left\{ (A_\alpha + p_\alpha^2)u_\alpha + l_\alpha g_\alpha \right\} \right).$$

Here

$$l_\alpha(p_\alpha) = a_\alpha I_\alpha(u^{\text{ex}}) + b_\alpha F_\alpha(u^{\text{ex}}),$$

where $I_\alpha$ and $F_\alpha$ are unbounded inclusion operators from the space $L_2(\mathbb{R}^3)$ to the space $L_2(\mathcal{M}_\alpha)$:

$$I_\alpha(u^{\text{ex}})(p_\alpha) = \int_{\mathcal{M}_\alpha} dk_\alpha \left( u^{\text{ex}}(p) - \frac{\varphi_\alpha(p_\alpha)}{p_\alpha^2 + 1} \right) + 2\pi \left( \sqrt{1 + p_\alpha^2} - 1 \right) \phi_\alpha(p_\alpha),$$

$$F_\alpha(u^{\text{ex}})(p_\alpha) = \varphi(p_\alpha).$$

The three-particle Hamiltonian $H$ is defined as a restriction of the operator $H_0$ to the set $\mathcal{D}(H)$ of functions from $\mathcal{D}(H_0)$ that satisfy the constraints—the “boundary conditions”—

$$\langle u_\alpha \cdot g_\alpha \rangle_{\mathcal{H}_\alpha}(p_\alpha) = c_\alpha \langle I_\alpha(u^{\text{ex}}) \rangle(p_\alpha) + d_\alpha \langle F_\alpha(u^{\text{ex}}) \rangle(p_\alpha), \quad \alpha = 1, 2, 3.$$  

By means of relations (2.29), one can check that the operator $H$ is symmetrical and has a dense support on the domain $\mathcal{D}(H)$.

Note that if we formally put $g_\alpha = 0$ in the construction under consideration, then the subspace $\mathcal{H}^{\text{ex}} = L_2^2(\mathbb{R}^6)$ is an invariant subspace of the operator $H$ and boundary conditions (2.29) become the so-called Skornyakov-Ter-Martirosyan boundary conditions that connect the values of the functions $I_\alpha(u^{\text{ex}})$ and $F_\alpha(u^{\text{ex}})$ in the $\mathcal{M}_\alpha$ planes. The Hamiltonian resulting from such a formal transition is nothing but the pre-Hamiltonian of the Faddeev-Minlos model [1, 2] (the Mel’nikov-Minlos model [9]) which describes the three-particle system with the usual $\delta$-function-like interactions.

3. Auxiliary operator $K(z)$

An analysis of the conditions for the pre-Hamiltonian $H$ (2.25) to be self-adjoint in the domain $\mathcal{D}(H)$ (2.24), (2.29), is closely related to studying the auxiliary operator $K(z)$ through which the three-particle $T$-matrix is expressed. Before defining this object, recall that the $t$-matrices of the models of point-like interactions endowed with internal structures have the following form in the case of the two-particle problem:

$$t(z) = \frac{1}{\omega(z) + i\frac{\sqrt{\nu}}{4\pi}}.$$  

Here $\omega(z)$ is the $R$-function (see (2.10)). If the internal operator $A$ in model (2.25) is bounded, then in the Herglotz representation (1.3), the measure $\mu$, corresponding to the $R$-function $\omega(z)$, has a compact support. Therefore, at $z \to -\infty$, the following asymptotics are valid:

in case $a$:

$$t(z) = \frac{1}{Az + B + i\frac{\sqrt{\nu}}{4\pi}} + \mathcal{O}(|z|^{-2}),$$

in case $b$:

$$t(z) = \frac{1}{B + i\frac{\sqrt{\nu}}{4\pi}} + \mathcal{O}(|z|^{-\frac{3}{2}}),$$

where $A$ and $B$ ($A > 0$) are constants. Thus, for sufficiently large negative values of the parameter $z$, the function $t(z)$ is negative and distant from zero. Note that the leading term in the asymptotic of $t(z)$
coincides with the \( t \)-matrix of the Shondin model [10] in case “a” and corresponds to the \( t \)-matrix of the usual \( \delta \)-interaction [18] in case “b.”

For each \( \alpha \) (\( \alpha = 1, 2, 3 \)), let us consider a self-adjoint, strictly positive (unbounded) operator of multiplication \( W_\alpha(z) \) in the space \( L_2(\mathbb{R}^3) \),

\[
(W_\alpha(z)f)(p) = (-t_\alpha(z - p^2))^{-1} f(p),
\]

which is determined on its natural domain of definition where it is self-adjoint:

\[
\mathcal{D}(W_\alpha(z)) = \{ f \in L_2(\mathbb{R}^3) : W_\alpha(z)f \in L_2(\mathbb{R}^3) \}.
\]

For sufficiently large negative values of the parameter \( z \), the operator \( W_\alpha(z) \) is compatible with the operator of multiplication by the function \( p^2 + 1 \) in case “a,” and by the function \( \sqrt{p^2 + 1} \) in case “b.” Therefore, in case a:

\[
\mathcal{D}(W_\alpha(z)) = L_2^2(\mathbb{R}^3),
\]

in case b:

\[
\mathcal{D}(W_\alpha(z)) = L_1^2(\mathbb{R}^3).
\]

An auxiliary operator \( K(z) \) acts in the space \( \mathcal{H} \) of three-component functions:

\[
\mathcal{H} = L_2(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3).
\]

It is a perturbation,

\[
K(z) = W(z) - R(z),
\]

of the diagonal operator \( W(z) \),

\[
W(z) = \text{diag}(W_1(z), W_2(z), W_3(z)).
\]

Here the perturbation \( R(z) \) is a \( 3 \times 3 \)-matrix operator with a null diagonal. Matrix elements of \( R(z) \) are (unbounded) integral operators with the kernels

\[
R_{\alpha\beta}(p, k; z) = \frac{1}{|s_{\alpha\beta}|} \left( p^2 - 2c_{\alpha\beta}(p, k) + k^2 - z s_{\alpha\beta}^2 \right)^{-1},
\]

where \( s_{\alpha\beta} \) and \( c_{\alpha\beta} \) are matrix elements of transformation (2.2) from one Jacobi coordinate system to another.

From time to time, where the dependence in \( z \) of the objects under consideration seems to be unessential, we drop the corresponding parametrization in the notations. The operator \( r \) is considered in the domain

\[
\mathcal{D}(R) = \mathcal{H}^{2,2,2},
\]

where it is obviously correctly defined and symmetric. Here \( \mathcal{H}^{a,b,c} \) denotes the space

\[
\mathcal{H}^{a,b,c} = L_2^a(\mathbb{R}^3) \oplus L_2^b(\mathbb{R}^3) \oplus L_2^c(\mathbb{R}^3).
\]

The operator \( K(z) \) is given on the domain

\[
\mathcal{D}(K(z)) = \mathcal{D}(W(z)) \cap \mathcal{D}(R) = \mathcal{H}^{2,2,2}.
\]

It has been noted already that there are four types of three-particle systems with internal structures:

- Case I: in each subsystem, case “a” is realized;
- Case II: case “b” is realized in one (and only one) subsystem;
- Case III: case “a” is realized in one (and only one) subsystem;
- Case IV: in each subsystem, case “b” is realized.

The following two statements are crucial both for describing the self-adjoint extensions of the pre-Hamiltonian \( H \) (2.25), (2.26)–(2.29), which is initially determined on \( \mathcal{D}(H) \), and for finding the discrete spectrum of such extensions.
Lemma 3.1. The auxiliary operator $K(z)$ for $z \in \mathbb{R}_-$ is essentially self-adjoint in the domain $\mathcal{H}^{2,2,2}$ iff the three-particle pre-Hamiltonian $H$ is essentially self-adjoint in $\mathcal{D}(H)$.

Note that the essential self-adjointness of $K(z)$ does not depend on the choice of the concrete value for parameter $z$ (lying to the left of some fixed $z_0$) since the difference $K(z) - K(z')$ is a bounded operator, or, more precisely, admits a bounded extension.

Each self-adjoint extension of the operator $K(z)$ is in one-to-one correspondence with a unique self-adjoint extension of the pre-Hamiltonian $H$. This correspondence can be established such that the following property is satisfied: let $\tilde{K}(z)$ be a self-adjoint extension of the operator $K(z)$ and $\tilde{H}$ be the corresponding self-adjoint extension of the operator $H$. Let $\tilde{K}(z')$ be a self-adjoint extension of the operator $K(z')$ corresponding to another point $z'$ and $\tilde{H}$ be the corresponding self-adjoint extension of $H$. Then $\tilde{H} = \tilde{H}$ iff $\mathcal{D}(\tilde{K}(z)) = \mathcal{D}(\tilde{K}(z'))$.

Lemma 3.2. Let $\tilde{K}(z)$ be a self-adjoint extension of the operator $K(z)$ and $\tilde{H}$ be the corresponding self-adjoint extension of the pre-Hamiltonian $H$. Then the real point $z$ belongs to the resolvent set of operator $H$ iff the inverse operator to $\tilde{K}(z)$ is bounded.

The proofs of these statements are quite standard and, in fact, are absolutely similar to the reasoning contained in [2.9, 19].

Therefore, the theory of self-adjoint extensions of the pre-Hamiltonian $H$ is reduced to studying the self-adjoint extensions of the operator $K(z)$. The proof of the energy operator semiboundedness is reduced to studying the invertibility of the operator $K(z)$ or, more precisely, its self-adjoint extensions at sufficiently large values of the parameter $z$.

Technically, the key point of the construction is the following lemma.

Lemma 3.3. In the space $L^2(\mathbb{R}^3)$, consider the integral operator $Q$ with the kernel

$$Q(p,k) = \frac{1}{(p^2 + 1)^{\frac{1}{2}}(p^2 + k^2 + 1)(k^2 + 1)^{\frac{1}{2}}}.$$  

(3.13)

This operator is bounded in the space $L^2(\mathbb{R}^3)$. Moreover, $Q$ continuously maps the space $L^2_\delta(\mathbb{R}^3)$, $0 \leq \delta < 1$, onto itself and, thus, the operator $Q$ continuously maps the space $L^2_\delta(\mathbb{R}^3)$, $\delta \geq 1$, onto the space $L^2(\mathbb{R}^3)$ for each $\kappa < 1$.

Proof. The fact that operator $Q$ is bounded in the space $L^2(\mathbb{R}^3)$ is proved in [6]. In order to prove that it is bounded in the space $L^2_\delta(\mathbb{R}^3)$, $0 < \delta < 1$, we need a minor improvement to the estimate proposed in [6].

Let $\delta < 1$. Choose $\epsilon > 0$ such that $\delta + \epsilon < 1$. Let $f \in L^2_\delta(\mathbb{R}^3)$ and $q = Qf$. From the Cauchy–Bunyakovskii inequality, it follows that

$$|q(p)|^2 \leq \frac{1}{\sqrt{p^2 + 1}} \int_{\mathbb{R}^3} \frac{dk}{(p^2 + k^2 + 1)((k^2 + 1)^{\frac{1}{2}} + \delta + \epsilon)} \int_{\mathbb{R}^3} \frac{dk |f(k)|^2(k^2 + 1)^{\delta + \epsilon}}{(p^2 + k^2 + 1)}.  

(3.14)$$

For the integral

$$I(p) = \int_{\mathbb{R}^3} \frac{dk}{(p^2 + k^2 + 1)((k^2 + 1)^{\frac{1}{2}} + \delta + \epsilon)},$$

(3.15)

the following estimate holds:

$$I(p) \leq \int_{\mathbb{R}^3} \frac{dk}{(p^2 + k^2 + 1)|k|^{1+2\delta+2\epsilon}} \leq \text{const.}(p^2 + 1)^{-2(\delta + \epsilon)}.$$  

(3.16)

Thus,

$$\|q\|_{L^2_\delta} \leq \int_{\mathbb{R}^3} \frac{dp}{(p^2 + 1)^{\epsilon + \frac{1}{2}}} \int_{\mathbb{R}^3} \frac{dk |f(k)|^2(k^2 + 1)^{\delta + \epsilon}}{(p^2 + k^2 + 1)} = \int_{\mathbb{R}^3} dk |f(k)|^2(k^2 + 1)^{\delta + \epsilon} \int_{\mathbb{R}^3} \frac{dp}{(p^2 + 1)^{\epsilon + \frac{1}{2}}(p^2 + k^2 + 1)} \leq \|f\|_{L^2_\delta}^2.$$  

(3.17)
Note that by including the parameter $\epsilon$ "in the play," we guarantee the convergence of the last integral in estimate (3.17). This inequality proves the first statement of the lemma.

The proof of the remaining statement is based on the fact that each space of "rapidly expanding functions" $L^2_\delta(\mathbb{R}^3)$ for $\delta \geq 1$ is contained in the space $L^2_\kappa(\mathbb{R}^3)$ for $\kappa < 1$; the latter space is mapped onto itself by the operator $Q$.

Two important observations follow from Lemma 3.3. First, it turns out that the operator $R$ determined on $\mathcal{D}(R) = \mathcal{H}^{2,2,2}$ admits the symmetrical extension $\hat{R}$ if it is considered as an integral operator with kernel (3.10) on the extended domain:

$$\mathcal{D}(\hat{R}) = \mathcal{H}^{1,1,1}.$$

To prove this, let us note that for the kernels of integral operators $R_{\alpha\beta}(z)$, the following estimates are valid:

$$|R_{\alpha\beta}(k, p, z)| \leq \frac{\text{const}}{p^2 + k^2 + 1},$$

(3.18)

and that $R_{\alpha\beta}(z)f \in L^2(\mathbb{R}^3)$ for $f \in L^2(\mathbb{R}^3)$, due to the continuity of operator $Q$ from Lemma 3.3 in the space $L^2_\delta$. Therefore, the operator $\hat{R}$ is correctly determined in the domain $\mathcal{D}(\hat{R})$ of the space $\mathcal{H}$. One can directly check that it is symmetrical in this domain.

From (3.6), (3.7), the domain of definition of operator $W$ coincides with $\mathcal{H}^{2,2,2}$ only for case I and is wider for other cases. say, for case IV,

$$\mathcal{D}(W) = \mathcal{H}^{1,1,1}.$$

Nevertheless, in each of cases I–IV,

$$\mathcal{D}(W) \subset \mathcal{D}(\hat{R}) = \mathcal{H}^{1,1,1}.$$

Second, operator $K$, treated as a perturbation to operator $W$,

$$\hat{K} = W - \hat{R},$$

(3.19)

is defined not only on the domain $\mathcal{H}^{2,2,2}$ but on the wider (in cases II–IV) domain $\mathcal{D}(W)$:

$$\mathcal{D}(\hat{K}) = \mathcal{D}(W) \cap \mathcal{D}(\hat{R}) = \mathcal{D}(W).$$

(3.20)

We write the operator $\hat{K}(z)$ in a slightly different symmetrical form. For this, let us introduce the operators

$$B_{\alpha\beta}(z) = W^{-\frac{1}{2}}(z)R_{\alpha\beta}(z)W^{-\frac{1}{2}}(z)$$

(3.21)

initially determined on the class of rapidly expanding functions. The operators $B_{\alpha\beta}$ are integral operators with the kernels

$$B_{\alpha\beta}(p, k; z) = \left(-t_{\alpha}(z - p^2)^{\frac{1}{2}}R_{\alpha\beta}(p, k; z)(-t_{\beta}(z - k^2))^{\frac{1}{2}},$$

(3.22)

which admit the estimate

$$|B_{\alpha\beta}(p, k; z)| \leq \frac{\text{const}}{(p^2 + 1)^{\delta_{\alpha}}(p^2 + k^2 + 1)(k^2 + 1)^{\delta_{\beta}}}. $$

(3.23)

Here $\delta_{\alpha} = \frac{1}{2}$ if the $t$-matrix $t_{\alpha}(z)$ of the two-particle subsystem $\alpha$ satisfies condition "a," and $\delta_{\alpha} = \frac{1}{4}$ otherwise.

By Lemma 3.3, operators $B_{\alpha\beta}(z)$ are extended to bounded operators in the space $L^2(\mathbb{R}^3)$ and, hence, determine the bounded matrix operator $B(z)$ with the matrix elements $B_{\alpha\beta}(z)$ in the space $\mathcal{H}$. Thus, we have the following representation on the domain $\mathcal{D}(\hat{K}(z))$:

$$\hat{K} = W - W^{\frac{1}{2}}BW^{\frac{1}{2}}.$$

(3.24)

Its correctness is ensured by the following lemma.
Lemma 3.4. The domain of definition of operator $W^{\frac{1}{2}}$ is invariant w.r.t. operator $B$:

$$B \mathcal{D}(W^{\frac{1}{2}}) \subset \mathcal{D}(W^{\frac{1}{2}}).$$

Proof. As we have mentioned previously, the set of two-particle subsystems with point-like interactions splits into two classes, which we denote by $\mathcal{A}$ and $\mathcal{B}$. We say that the subsystem $\alpha \in \mathcal{A}$ if the corresponding $t$-matrix $t_\alpha(z)$ satisfies condition “a,” and $\beta \in \mathcal{B}$ if $t_\beta(z)$ satisfies condition “b.” Consider the integral operators $Q_{\alpha \beta}$ with the kernels

$$Q_{\alpha \beta}(p, k; z) = (-t_\alpha(z - p^2))^{\theta_\alpha} R_{\alpha \beta}(p, k; z)(-t_\beta(z - k^2))^{\theta_\beta},$$

where $\theta_\alpha = \frac{1}{2}$ if $\alpha \in \mathcal{A}$ and $\theta_\beta = 1$ if $\beta \in \mathcal{B}$. The kernels of operators $Q_{\alpha \beta}$ obviously admit the following estimates:

$$|Q_{\alpha \beta}(p, k; z)| \leq \frac{\text{const}}{(p^2 + 1)^{\frac{1}{2}}}(k^2 + p^2 + 1)(k^2 + 1)^{\frac{1}{2}}.$$

Thus, for operators $Q_{\alpha \beta}$, all of the statements of Lemma 3.3 are valid.

Note that operators $B_{\alpha \beta}$ are related to operators $Q_{\alpha \beta}$ as follows: $B_{\alpha \beta} = W_{\alpha}^{-\frac{1}{2}} Q_{\alpha \beta} W_{\beta}^{-\frac{1}{2}}$ if $\alpha, \beta \in \mathcal{A}$, $B_{\alpha \beta} = W_{\alpha}^{-\frac{1}{2}} Q_{\alpha \beta}$ if $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, $B_{\alpha \beta} = Q_{\alpha \beta} W_{\beta}^{-\frac{1}{2}}$ if $\alpha \in \mathcal{B}$, $\beta \in \mathcal{A}$, and, eventually, $B_{\alpha \beta} = Q_{\alpha \beta}$ for $\alpha, \beta \in \mathcal{B}$.

In each of the cases I–IV, the domain of definition of operator $W^{\frac{1}{2}}$ has the form

$$\mathcal{D}(W^{\frac{1}{2}}) = L^{\delta_\alpha}_{2\alpha}(\mathbb{R}^3) \oplus L^{\delta_\beta}_{2\beta}(\mathbb{R}^3) \oplus L^{\delta_\gamma}_{2\gamma}(\mathbb{R}^3),$$

where $\delta_\alpha = 1$ if $\alpha \in \mathcal{A}$ and $\delta_\beta = \frac{1}{2}$ if $\beta \in \mathcal{B}$. It is clear that the proof of the lemma implies the proof of the inclusion series

$$B_{\alpha \beta} L^{\delta_\alpha}_{2\alpha}(\mathbb{R}^3) \subset L^{\delta_\alpha}_{2\alpha}(\mathbb{R}^3).$$

In each of the cases I–IV, such a proof can be performed by using Lemma 3.3 systematically, accounting for the formulas connecting the operators $B_{\alpha \beta}$ and $Q_{\alpha \beta}$.

For example, let us present such a proof for case I. The assertion of the lemma follows from the invariance of the space $L^{\frac{1}{2}}(\mathbb{R}^3)$ w.r.t. the action of operators $B_{\alpha \beta}$.

Let $f \in L^{\frac{1}{2}}(\mathbb{R}^3)$. Then

$$W_{\beta}^{-\frac{1}{2}} f \in L^{\frac{3}{2}},$$

and, hence, from Lemma 3.3,

$$Q_{\alpha \beta} W_{\beta}^{-\frac{1}{2}} f \in L^{1-\epsilon}(\mathbb{R}^3)$$

for arbitrarily small $\epsilon$. Therefore,

$$B_{\alpha \beta} f \in L^{\frac{3}{2}} - \epsilon \subset L^{1}(\mathbb{R}^3)$$

for $\epsilon \leq \frac{1}{2}$.

Without a proof, let us present a result that is close in spirit and can be obtained by means of the "$\alpha \beta$"-combinatorics, together with a systematic usage of Lemma 3.3.

Lemma 3.5. In case I, if $f \in \mathcal{H}$, then

$$B f \in \mathcal{D}(W^{\frac{1}{2}}).$$

In case II, the same property holds for the square of operator $B$: if $f \in \mathcal{H}$, then

$$B^2 f \in \mathcal{D}(W^{\frac{1}{2}}).$$

Corollary 3.1. In cases I and II, each eigenfunction $\varphi$ of operator $B$, corresponding to a nonzero eigenvalue $\lambda$,

$$B \varphi = \lambda \varphi,$$

belongs to the domain of definition of the quadratic form of operator $W$:

$$\varphi \in \mathcal{D}(W^{\frac{1}{2}}).$$
4. Essential self-adjointness of the $K(z)$ operator in cases I–II. KLMN-theorem

In the previous section, we described the symmetrical extensions $\hat{K}$ of operator $K$ in the domain $\mathcal{D}(W)$, which is wider than the initial domain of definition $\mathcal{D}(K) = \mathcal{H}^{2,2,2}$ given by the formula

$$\hat{K} = W - W^\frac{1}{2}BW^\frac{1}{2}. \quad (4.1)$$

A qualitative notion on the strength of the perturbation $R = W^\frac{1}{2}BW^\frac{1}{2}$ is given by the following theorem.

**Theorem 4.1.** In cases I and II, operator $B$ is compact and belongs to the Hilbert–Schmidt ideal. In cases III and IV, the essential spectrum of the operator $B$ contains the unity

$$1 \in \sigma_{\text{ess}}(B) \quad (4.2)$$

together with a certain neighborhood.

Roughly, the result of Theorem 4.1 means that in cases I and II, the perturbation $R = W^\frac{1}{2}B(z)W^\frac{1}{2}$ is $W$-compact w.r.t. the space of forms, and in cases III and IV, it is only $W$-bounded from below having a $W$-edge not less than one.

Note that the last statement of the theorem concerning the location of the essential spectrum of the operator $B$ in cases III and IV is the central point for proving the existence of the Efimov effect [6, 8], as well as proving that the energy operator of a three-particle system with $\delta$-function-like interactions is unbounded [1, 2, 9].

We give only a sketch of the proof of Theorem 4.1.

**Sketch of the proof.** For the kernel of the integral operator $B_{\alpha\beta}(z)$, estimate (3.23) is valid. Moreover, in cases I and II, at least one of the exponents $\delta_\alpha$ is equal to $\frac{1}{2}$, and, hence, the kernel $B_{\alpha\beta}(p, k; z))$ is a square-summable function of the argument $P = k \oplus p$. Thus, operator $B$ belongs to the Hilbert–Schmidt ideal.

Let us briefly sketch the proof of the second part of the theorem. First, consider the simplest case, IV. There, for all $\alpha, \alpha = 1, 2, 3$, the $t$-matrices $t_\alpha(z)$ satisfy condition “b”. Thus, we have the asymptotic expansions

$$t_\alpha(z) = \frac{1}{B_\alpha + i\frac{z}{4\pi}} + O(|z|^{-\frac{3}{2}}), \quad \alpha = 1, 2, 3. \quad (4.3)$$

Let $\tilde{t}(z)$ be the $t$-matrix corresponding to a special case of $\delta$-interaction such that in the two-particle system we have a virtual level at zero energy:

$$\tilde{t}(z) = \frac{1}{i\sqrt{z}4\pi}. \quad (4.4)$$

Then, from (4.3),

$$t_\alpha(z) = \tilde{t}(z) + O\left(\frac{1}{|z|}\right). \quad (4.5)$$

Let us introduce the operators $W_\alpha(z)$ (which are the same for all $\alpha$),

$$\tilde{W}_\alpha(z) = (-\tilde{t}(z - p^2))^{-1} \quad (4.6)$$

and the operators $\tilde{B}_{\alpha\beta}(z)$,

$$\tilde{B}_{\alpha\beta}(z) = W_\alpha^\frac{1}{2} R_{\alpha\beta}(z) W_\beta^\frac{1}{2}. \quad (4.7)$$
A straightforward estimate, accounting for asymptotics (4.5), demonstrates that the kernel of difference of the integral operators $B_{\alpha\beta}(z) - B_{\alpha\beta}(z)$ is a Hilbert-Schmidt kernel. Thus, operator $\tilde{B}_{\alpha\beta}(z)$ is a compact perturbation of operator $B_{\alpha\beta}(z)$.

Let $B$ be a unit ball in the space $\mathbb{R}^3$ and $\mathcal{P}$ be the orthogonal projection operator from the space $L_2(\mathbb{R}^3) = L_2(B) \oplus L_2(\mathbb{R}^3 \setminus B)$ to the subspace $L_2(\mathbb{R}^3 \setminus B)$. An analogous estimate for the kernel

$$
\frac{1}{(-z + p^2)^{\frac{1}{4}}} R_{\alpha\beta}(p, k; z) \frac{1 - \theta(p)}{(-z + k^2)^{\frac{1}{4}}} - \frac{1 - \theta(k)}{\sqrt{p}} R_{\alpha\beta} \frac{1}{\sqrt{k}}, \quad (4.8)
$$

where $\theta(p)$ is the indicator of the unit ball $B$ in $\mathbb{R}^3$, demonstrates that the difference $\tilde{B}(z) - \mathcal{P}\tilde{B}(0)\mathcal{P}$ is again a Hilbert–Schmidt operator. Thus, from the Weyl theorem, the essential spectra of operators $B(z)$ and $\mathcal{P}\tilde{B}(0)\mathcal{P}$ coincide.

The spectral analysis of operator $\tilde{B}(0)$, as well as of operator $\mathcal{P}\tilde{B}(0)\mathcal{P}$, in a sense, can be explicitly carried out. We observe first that subspaces $L_2(\mathbb{R}^3)$, corresponding to a fixed angular momentum, are reduced for operators $\tilde{B}_{\alpha\beta}(0)$ and, second, that the kernels of the integral operators $\tilde{B}_{\alpha\beta}$ are homogeneous functions (of degree $-\frac{3}{2}$) of their arguments. In particular, the part of operator $\tilde{B}_{\alpha\beta}(0)$ that corresponds to the invariant subspace of spherically symmetrical functions is unitarily equivalent to the operator of multiplication by functions (admitting meromorphic extensions to the entire complex plane)

$$
M_{\alpha\beta}(s) = \frac{\pi}{2\sqrt{s_{\alpha\beta}}} \frac{\sinh(\arcsin |s_{\alpha\beta}| s)}{s \cosh \frac{\pi s}{2}} \quad (4.9)
$$

in the space $L_2(\mathbb{R})$. Here $s_{\alpha\beta}$ are the matrix elements of transition (2.2) from the Jacobi coordinate system $(k_{\alpha}, p_{\alpha})$ to the system $(k_{\beta}, p_{\beta})$. The proof of Eq. (4.9) which is closest to our way of reasoning is based on the Mellin transform; it can be found in [9] (see also [6]).

Therefore, operator $\tilde{B}(0)$ has a part that is unitarily equivalent in the space $L_2(\mathbb{R}) \oplus L_2(\mathbb{R}) \oplus L_2(\mathbb{R})$ to the operator of multiplication by the $3 \times 3$-matrix-function $M(s)$ with a null diagonal. This function is determined, in accordance with (4.9), by the matrix elements $M_{\alpha\beta}(s) \quad \alpha \neq \beta$. In turn, operator $\mathcal{P}\tilde{B}(0)\mathcal{P}$ contains a part, which is a Wiener-Hopf-type operator, whose spectrum coincides with the spectrum of the operator of multiplication by the matrix-function $M(s)$ “on the half-axis” [22], or, more precisely, in the space $L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_+)$. Thus, the essential spectrum of the operator $\mathcal{P}\tilde{B}(0)\mathcal{P}$ and, hence, due to the Weyl theorem, of the operator $B(z)$, includes the set of eigenvalues of the matrix $M(s)$ for all possible values of the argument $s \in \mathbb{R}_+$. The fact that this set includes an interval containing unity can be proved using the following method (see [9]). Consider a (continuous) function

$$
f(s) = \det(I - M(s)). \quad (4.10)
$$

It is obvious that

$$
\lim_{s \to \infty} f(s) = 1. \quad (4.11)
$$

Evaluating the determinant of $f(s)$ at $s = 0$, we obtain

$$
\frac{\pi}{2} \leq \frac{\sin \varphi}{\varphi},
$$

where the summation runs over the arrows of the cyclical graph $g$: $1 \to 2 \to 3 \to 1$. Further, using the elementary inequality

$$
\frac{2}{\pi} < \frac{\sin \varphi}{\varphi},
$$

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which is valid for \(0 \leq \varphi < \frac{\pi}{2}\), we obtain from (4.9) that
\[
M_{\alpha\beta}(0) > 1, \quad \alpha \neq \beta. \quad (4.13)
\]
Thus, \(f(0) < 0\), and due to the continuity of \(f(s)\) there exists \(s_0\) such that \(f(s_0) = 1\). Since the inequality \(f(0) < 0\) strictly holds, this means that the essential spectrum of operator \(B\) contains some neighborhood of unity.

In case III, in one and only one subsystem, say, with number \(\gamma\), case "a" is realized (i.e., the corresponding scattering matrix has the "proper" behavior at high energies). Therefore, from the first statement of the theorem, the matrix elements \(B_{\alpha\gamma}(z)\) and \(B_{\beta\gamma}(z)\) are Hilbert–Schmidt operators. As a result, similar to case IV, \(B(z)\) is a compact perturbation of some operator, whose part has a spectrum coinciding with the spectrum of the operator of multiplication in the space \(L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_+)\) by a matrix-function \(\tilde{N}(s) = \{N_{\alpha\beta}\}\), which has only two nonzero matrix elements:
\[
N_{\alpha^*,\beta^*}(s) = M_{\alpha^*,\beta^*}(s). \quad (4.14)
\]
Here \(\alpha^*, \beta^*\) are numbers of the two-particle subsystems \((\alpha^* \neq \beta^*)\) that differ from the subsystems with the number \(\gamma\).

As in case IV,
\[
\lim_{s \to \infty} \det(I - N(s)) = 1
\]
and
\[
\det(I - N(0)) = 1 - N_{\alpha^*,\beta^*}(0) < 0, \quad (4.15)
\]
which again (as in case IV) proves that \(1 \in \sigma_{\text{ess}}(B(z))\) together with some of its neighborhood.

Using the relative compactness (in the sense of forms) of the perturbation \(R = W^\frac{1}{2}BW^\frac{1}{2}\) and Corollary 3.1, we prove that in cases I and II, the relative \(W\)-edge of the perturbation is equal to zero. Applying the KLMN-theorem (Theorem X.17 of [21]), we prove that the operator \(\tilde{K}(z)\) is self-adjoint in the domain \(\mathcal{D}(\tilde{K}(z))\).

One can demonstrate that in cases III and IV, the \(W\)-edge of the perturbation (in the sense of forms) is not less than one, which prohibits the use of the KLMN-theorem. As we shall demonstrate in the second part of this paper (to be published separately), the operator \(\tilde{K}(z)\) is only symmetrical on \(\mathcal{D}(\tilde{K}(z))\), but not essentially self-adjoint there.

**Theorem 4.2.** In cases I and II, the operator \(\tilde{K}\) is self-adjoint on the domain \(\mathcal{D}(\tilde{K})\).

**Proof.** Let us consider the quadratic form
\[
k[\varphi] = w[\varphi] - r[\varphi] \quad (4.16)
\]
in the domain \(\mathcal{D}[k] = \mathcal{D}(W^\frac{1}{2})\). Here \(w\) is the quadratic form of the operator \(W\),
\[
w[\varphi] = \langle W^\frac{1}{2}\varphi, W^\frac{1}{2}\varphi \rangle, \quad \varphi \in \mathcal{D}(W^\frac{1}{2}), \quad (4.17)
\]
and the form \(r\) determined in \(\mathcal{D}[r] = \mathcal{D}(W^\frac{1}{2})\) corresponds to the perturbation \(R\),
\[
r[\varphi] = \langle BW^\frac{1}{2}\varphi, W^\frac{1}{2}\varphi \rangle. \quad (4.18)
\]
We show that the form \(r\) is a \(w\)-restricted form with a null \(w\)-edge, i.e., for each \(\epsilon > 0\), there exists \(b > 0\) such that the inequality
\[
|r[\varphi]| \leq \epsilon w[\varphi] + b\|\varphi\| \quad (4.19)
\]
holds for all \(\varphi \in \mathcal{D}(W^\frac{1}{2})\).
For the first step in the proof of (4.19), we use the compactness of operator $B$ in cases I and II.

Let $P$ be the spectral projection operator of $B$ that corresponds to the spectrum component lying in the interval $[-\varepsilon, \varepsilon]$. Then

$$B = PBP + Q,$$

where $Q$ is a finite-dimensional operator:

$$Q = \sum_{|\lambda_i| \geq \varepsilon} \lambda_i \langle \cdot, \varphi_i \rangle \varphi_i.$$  \hfill (4.21)

Here we denote the eigenvalues and the corresponding eigenfunctions of the compact operator $B$ by $\lambda_i$ and $\varphi_i$. The norm of the operator $PBP$ does not exceed $\varepsilon$, hence, the inequality

$$|r[\varphi]| \leq \varepsilon w[\varphi] + \sum_{|\lambda_i| \geq \varepsilon} |\lambda_i| |(W^{\frac{1}{2}} \varphi, \varphi_i)|^2$$ \hfill (4.22)

holds. From Corollary 3.1, the second term on the r.h.s. of (4.22) is a bounded form. This proves inequality (4.19) at

$$b = \sum_{|\lambda_i| > \varepsilon} |\lambda_i| \cdot \|W^{\frac{1}{2}} \varphi_i\|^2.$$ \hfill (4.23)

From the KLMN-theorem, we conclude that the form $k$ is closed and semibounded in cases I and II. Thus, there exists a self-adjoint operator that corresponds to this form. It is easy to see, however, that it is the operator $\tilde{K}$. This completes the proof of the theorem.

**Lemma 4.1.** In cases I and II, the three-particle Hamiltonian $H$ is semibounded from below and is essentially self-adjoint in $D(H)$.

**Proof.** The KLMN-theorem also states that the operator $\tilde{K}$ is essentially self-adjoint in each domain where the nonperturbed operator (operator $W$ in our case) is also essentially self-adjoint. In particular, the operator $K$ is also essentially self-adjoint in the domain $D(K) = \mathcal{H}_2.2$ for cases I and II (in case I, the operator $K$ is self-adjoint in $D(K) = \mathcal{H}_2.2$). Using the result of Lemma 3.1, we conclude that in cases I and II, the three-particle Hamiltonian $H$ is essentially self-adjoint in $D(H)$. This proves the first part of the assertion of the lemma.

The proof of the invertibility of operator $\tilde{K}(z)$ at sufficiently large negative values of parameter $z$ is based on the following reasoning, which is considerably rougher than in the proof of Theorem 4.2. The existence of the summable majorant for the square of the kernels of operators $B_{\alpha\beta}(z)$, and the fact that

$$|B_{\alpha\beta}(p, k; z)| \rightarrow 0 \quad \text{as } z \rightarrow -\infty$$ \hfill (4.24)

at fixed values of arguments $p$ and $k$, demonstrates that $B(z) \rightarrow 0$ as $z \rightarrow -\infty$ in the Hilbert–Schmidt norm in accordance with the theorem on majorized convergency. Thus, for $z$ lying to the left of some $z_0 \in \mathbb{R}_0$, operator $B$ determines a contraction mapping; then the inequality

$$|r| \leq (1 - \|B(z)\|)w$$ \hfill (4.25)

holds. On the other hand, from representation (3.2)–(3.4), it follows that the lower bound of the form $w(z)$ tends to infinity as $|z|$ in case I and as $\sqrt{|z|}$ in case II,

$$\inf_{|\varphi| = 1} \langle W^{\frac{1}{2}}(z)\varphi, W^{\frac{1}{2}}\varphi \rangle = O(|z|^\delta),$$ \hfill (4.26)

where $\delta = 1$ in case I, and $\delta = \frac{1}{2}$ in case II. Hence, for $z$ lying to the left of $z_0$, $\tilde{K}(z)$ has a bounded inverse operator. From Lemma 3.2, this proves the semiboundedness of the three-particle energy operators in cases I and II.
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