ONE-DIMENSIONAL MODEL OF THREE-PARTICLE RESONANCES

Yu.A. Kuperin, K.A. Makarov, and B.S. Pavlov

The theory of extensions is used to construct a nontrivial S matrix for a system of three one-dimensional particles in the boundary condition model. The analytic structure of the constructed S matrix on the energy shell is investigated.

One of the important problems of scattering theory in a few-particle system is the study of the resonance singularities of the S matrix and the elucidation of the connection between these singularities and the internal structure of the colliding particles. It is rather difficult to establish the analytic structure of the S matrix in a system of three or more particles with realistic interaction. It is therefore interesting to consider simpler models amenable to exact solution.

In the nonrelativistic problem of three-particle scattering, the method of zero-range potentials is successfully used [1-3]. However, this method is not suited for describing scattering processes accompanied by a change in the internal structure of the particles, since from the very beginning it is assumed that the "internal space" of each of the particles is trivial.

In the present paper, we construct a model of scattering theory for three one-dimensional particles that do have internal structure. In the model, the "internal" and "external" spaces are coupled by means of boundary conditions, and this makes it possible to find the three-particle S matrix explicitly. We demonstrate our approach for the simplest problem of the 3 → 3 scattering process with "inclusion" of internal symmetry only in triple collisions. We trace in detail not only the position of the three-particle resonances and the energies of the three-body bound states but also their motion induced by a change in the "coupling constant" that regulates the exchange of energy between the internal and external spaces.

1. Preliminary Results

The energy operator H of a system of three one-dimensional quantum particles, restricted to the invariant subspace of zero total momentum, acts on the space \( L^2(\Lambda) \) of square-integrable functions of \( \Lambda = \{ x \in \mathbb{R}^3, \sum x_i = 0 \} \) in accordance with

\[
(H\Psi)(x) = -\Delta \Psi + V(x) \Psi,
\]

where \( \Delta \) is the Laplacian on \( \Lambda \), and \( V(x) \), the potential, can be represented in the form

\[
V = \sum_{i<j} v(x_i - x_j).
\]

If the two-body potentials \( v(x), x \in \mathbb{R}^3 \), decrease sufficiently rapidly as \( |x| \to \infty \), then the important interaction region is a neighborhood of the straight lines \( L_i = \{ x \in \mathbb{R}^3: x_k = x_i, (j, k, l) \text{ is a permutation of } (1, 2, 3) \} \). The straight lines \( l_i, i = 1, 2, 3 \), divide \( \Lambda \) into six sectors. The interior of each sector \( \lambda \) consists of vectors \( x \) such that

\[
x_i > x_j > x_k, \quad \sigma = (i, j, k) \text{ is a permutation of } (1, 2, 3).
\]

We shall assume that the permutation \( \sigma \) is the number of the sector and write \( \lambda = \lambda_\sigma \).

The scattering problem in the model of three one-dimensional particles for which the two-body interactions between them are specified by means of the boundary conditions \( \partial \Psi / \partial n_i + h_i \Psi = 0 \) on the straight lines \( l_i \) (\( n_i \) is the normal to \( l_i \)) reduces essentially to the problem of diffraction of plane waves by the system of "screens" \( l_i \) [3, 4]. For the ideal conditions \( h_i = 0 \) or \( h_i = \infty \), it can be shown that the scattering amplitude \( f(p, p', \gamma E) \) depends only on the angular variables \( p, p' \) and not on the energy \( E \) (E is the total energy of the colliding particles in the center-of-mass frame) and, therefore, does not have singularities on the energy shell. The boundary conditions written down above correspond to the assumption that in the scattering processes the particles cannot pass through each other: the flux through screen \( l_i \) is equal to...
The asymptotic conditions on the wave function in the neighborhood of the point $x=0$: $\Psi \sim |x|^n$, $|\dot{\Psi}| \sim |x|^{n-1}$ as $|x| \to 0$, where $2\pi - \gamma$ is the angle between neighboring screens, make it impossible for energy to pass through the point $x=0$ into the other sectors different from the sector in which the initial perturbation is concentrated. Thus, the solution of the scattering problem in the boundary condition model has been reduced to the solution of a diffraction problem within just one of the six possible sectors. In the situation in which the problem of diffraction by a complete system of screens was considered [4], it was not possible to obtain an explicit expression for the amplitude $f(\bar{p}, p', \sqrt{E})$, which would have allowed study of its behavior on the "unphysical" sheet.

We construct a model that differs from the one proposed in the quoted papers by including additionally a three-particle zero-range potential. We shall assume that the two-particle zero-range potentials correspond to the ideal condition

$$\frac{\partial \Psi}{\partial n_i}|_{x_i}=0, \quad i=1, 2, 3.$$  

From the physical point of view, the inclusion of the three-particle zero-range potential corresponds to the possibility of energy exchange between different sectors through the triple collision point $x=0$. In this model, one can explicitly calculate the scattering amplitude $f(\bar{p}, p', \sqrt{E})$ and investigate its singularities on the Riemann energy surface. We note immediately that in our case the scattering amplitude is nontrivial, i.e., it actually does depend on the variables $\bar{p}, \hat{p}$; it is piecewise constant.

2. 

**Model of Three-Particle Zero-Range Forces**

We fix the sector $\lambda_{o}$ and consider in $L^2(\lambda_{o})$ the operator $L_0=-\Delta$ with domain of definition $\mathcal{D}(L_0)=\{u \in C_0^\infty(\lambda_{o}), \partial u/\partial n_i|_{x_i=0}=0\}$. The class $C_0^\infty(\lambda_{o})$ consists of functions that vanish in a neighborhood of the apex $x=0$ of the sector $\lambda_{o}$. The closure $\overline{L_0}$ is a symmetric operator having deficiency indices $(1, 1)$. Its deficiency elements $g_{\pm}$ at the complex point $\lambda_0=\frac{k^2}{2}$ can be expressed in terms of the Green's function of the Helmholtz equation

$$(\Delta + k^2)G(x, x', k_0) = -4\pi \delta(x-x'), \quad g_{\pm}=G(x, 0, k_0)=\frac{i\pi}{\sqrt{k_0}}H_0^{(1)}(k_0|x|),$$

so that the adjoint operator is defined on the lineal $\mathcal{D}(L_0^*)=\mathcal{D}(L_0)+\mathcal{L}(g_{\pm}, g_{\mp})$. We introduce in the deficiency subspace the new basis $G_{\pm}(\lambda_{o})=\{g_{+}+g_{-}, g_{-}-g_{+}\}$. Choosing for convenience $k_0=\epsilon^{i\pi/2}$ and using the asymptotic behavior of the Hankel function

$$H_0^{(1)}(k|x|) \sim i\pi - 2\ln(k|x|/2) - 2\gamma, \quad |x| \to 0$$

($\gamma$ is Euler's constant), we obtain for $G_{\pm}$ the representations

$$G_+ = -2\ln\left(\frac{|x|}{2}\right) - 2\gamma + O(|x|^3 \ln |x|), \quad G_- = \frac{3\pi}{2} + O(|x|^3 \ln |x|).$$

Functions from the domain of definition of the adjoint operator, $u \in \mathcal{D}(L_0^*)$, can be uniquely represented in the form $u=\varepsilon_+G_++\varepsilon_-G_-, u \in \mathcal{D}(L_0)$, and their asymptotic behavior at the origin can be readily calculated,

$$u \sim \varepsilon_+ \ln(|x|^2/2) - \frac{3\pi}{2} \varepsilon_- - 2\gamma \varepsilon_+.$$  

Conversely, from known asymptotic behavior at the origin one can recover the quantities $\varepsilon_{\pm}$. In particular, for outgoing wave $G_+$

$$\varepsilon_+ = 1/2nt \quad \varepsilon_- = \frac{i}{3\pi^2} \ln(-1/E), \quad E=k^2.$$  

We have in the $G_{\pm}$ basis

$$e_+ = 1/2nt \quad e_- = \frac{i}{3\pi^2} \ln(-1/E), \quad E=k^2.$$  

Suppose $u, v \in \mathcal{D}(L_0^*)$; then

$$\langle L_0^* u, v \rangle - \langle u, L_0^* v \rangle = \lim_{|x| \to 0} \int_{|x|=\epsilon, \epsilon=0} \left( \frac{\partial \bar{u}}{\partial n} - \frac{\partial u}{\partial n} \right) \, dS = \frac{4}{\pi^2} \{e_+(u)\bar{e}_+(v) - e_+(u)e_-(v)\} = [u, v].$$

We call the form $\langle \cdot, \cdot \rangle_0$ the boundary form in the sector $\lambda_{o}$.

In the Hilbert space $\bigoplus_{\lambda_{o}} L^2(\lambda_{o})$ we consider the operator $\mathcal{L} = \sum_{\lambda_{o}} \mathcal{L}_{\lambda_{o}}$, where $\mathcal{L}_{\lambda_{o}}$ is the corresponding
operator in each of the sectors \( \lambda_\alpha \). Then for \( \mathcal{U}, \mathcal{V} \in \sum_\alpha L^2(\lambda_\alpha) \), obviously

\[
\langle L^* \mathcal{U}, \mathcal{V} \rangle - \langle \mathcal{U}, L^* \mathcal{V} \rangle = \sum_{\alpha} \left[ \frac{1}{i} \langle e_\alpha^+ (\mathcal{U}), e_\alpha^+ (\mathcal{V}) \rangle - \langle e_\alpha^- (\mathcal{U}), e_\alpha^- (\mathcal{V}) \rangle \right] = \{\mathcal{U}, \mathcal{V}\}.
\]

We shall call the six-component vectors \( e_{\alpha}^{\text{ext}} \) boundary vectors.

The operator \( L \) is a densely defined operator with deficiency indices \((6, 6)\). In accordance with the von Neumann formulas, the domain of definition of its adjoint operator can be represented in the form

\[
\mathcal{D}(L^*) = \mathcal{D}(L) + \mathbb{R} + \mathbb{R}^*,
\]

where \( \mathbb{R} \) and \( \mathbb{R}^* \) are the deficiency subspaces of \( L \) at the points \( i \) and \( -i \), respectively. The components of the boundary vectors \( e_{\alpha}^{\text{ext}} \) of the element \( \mathcal{U} \in \mathcal{D}(L^*) \) are determined uniquely by the representation of \( \mathcal{U} \) in the form \( \mathcal{U} = \mathcal{U}_0 + \sum_{\alpha} e_\alpha^+ G_\alpha^+ + e_\alpha^- G_\alpha^- \), \( \mathcal{U}_0 \in \mathcal{D}(L) \). All the self-adjoint extensions of the operator \( L \) can be obtained from the adjoint operator \( L^* \) by restricting its domain of definition to the lineal \( \mathcal{D} : \mathcal{D}(L) \subset \mathcal{D}(L^*) \), on which the boundary form \( \{\cdot, \cdot\} \) is annihilated. For this, it is sufficient, for example, if the boundary vectors of the elements of \( \mathcal{D} \) satisfy

\[
A e_{\alpha}^{\text{ext}} = e_{\alpha}^{\text{ext}},
\]

where \( A \) is a \( 6 \times 6 \) Hermitian matrix. For a complete description of all self-adjoint extensions, see, for example, [5].

In the complete 36-parameter family of extensions one must choose physically meaningful ones. For example, the requirement that the colliding particles be all on an equal footing leads to a cyclic matrix \( A \). We shall denote the obtained self-adjoint extension of the operator \( L \) by \( L_A \).

3. Spectral Analysis of the Operator \( L_A \)

We make the spectral analysis of the obtained operator \( L_A \) – we describe the solution of the scattering problem and the form of the three-particle bound states.

A plane wave \( \exp i \langle p, x \rangle \), incident on the screen \( l_0 \) is transformed in accordance with the reflection law determined by the boundary condition (1) into the reflected wave \( \exp i \langle \tau p, x \rangle \), where \( \tau \) is an element of the permutation group \( S_3 \) [4]. Geometrically, the action of the group element \( \tau \) on the vector \( p \) corresponds to reflection of \( p \) with respect to the straight line \( l \). Thus, the plane wave \( \exp i \langle p, x \rangle \), arriving from the sector \( \lambda_\mu \) (in this case, we shall write \( p_{\lambda_\mu} = \lambda_0 \)), leaves, having undergone three reflections, in the direction \( \tau_\mu p \). The contribution of the plane wave \( \Psi^d \) in the sector \( \lambda_\mu \) is given by the symmetrization of the incident wave with respect to permutations of the coordinates of the vector \( p \): \( \Psi^d = \sum_{\alpha \beta} \exp i \langle \alpha p, x \rangle \).

In addition, in each of the sectors \( \lambda_\nu \) there arises a divergent wave \( \tilde{G}(|p| |x|) \) with amplitude \( \eta_{\nu\mu} \). In our model, the "partial" amplitudes \( \eta_{\nu\mu} \) corresponding to scattering from the sector \( \lambda_\mu \) to the sector \( \lambda_\nu \) do not depend on the angular variables, and in this sense the scattering in each of the sectors takes place in the S-wave channel. Therefore, the continuum eigenfunctions of the Hamiltonian \( L_A \), which have in asymptotia plane waves only in the sector \( \lambda_\nu \), can be represented in the sector \( \lambda_\nu \) in the form

\[
\Psi_{\nu}(x, p) = \delta_{\nu\nu} \Psi^d + \eta_{\nu\lambda_0} G(|p| |x|).
\]

(4)

To calculate the amplitudes \( \eta_{\nu\mu} \), we find the boundary vectors of the eigenfunction \( \Psi_{\nu\mu} \). The asymptotic behavior of \( \Psi_{\nu\mu} \) in the neighborhood of the point \( x = 0 \) is given by (see (2))

\[
\Psi_{\nu}(x, p) \sim 6 \delta_{\nu\nu} + \eta_{\nu\lambda_0} \left( \frac{1}{2\pi} G_{\nu}^+ - \frac{1}{2\pi} \ln(-1/E) G_{\nu}^- \right) \sim e_{\nu} G_{\nu}^+ + e_{\nu} G_{\nu}^-.
\]

We introduce the vectors \( \eta_{\mu} \) and \( e_{\mu} \) with components \( \eta_{\nu\mu} \) and \( -8i \delta_{\nu\mu} \) respectively. Then

\[
e_{\nu}^{\text{ext}}(\Psi_{\nu}) = \frac{\eta_{\nu\lambda_0}}{2\pi i}, \quad e_{\nu}^{\text{ext}}(\Psi_{\nu}) = \frac{1}{2\pi i} (e_{\nu} + \Phi(E) \eta_{\nu}).
\]

(5)
where $\Phi(E) = -\frac{2}{3\pi} \ln(-1/E)$, $E = p^2$. The condition (3) connecting the boundary vectors leads to the following representation for the amplitude $\eta$: $\eta = (1 - \Phi(E)A)^{-1}ae_m$ where $I$ is the identity operator in $\mathbb{R}^4$. The function $\Phi(E)$ is multiplied up on the plane of the energy $E$. To separate a single-valued branch of $\Phi(E)$, we take a cut along the positive half-axis and fix the branch of the logarithm in such a way that on the negative half-axis the function $\Phi(E)$ takes real values. Knowing the eigenvalues $\lambda_s$ and eigenvectors $a_s$ of the matrix $A$: $A a_s = \lambda_s a_s$, we obtain an explicit expression for the vector of the partial amplitudes:

$$\eta_m(\sqrt{E}) = \sum_{s=1}^{\infty} \lambda_s a_s \langle e_s, a_s \rangle (1 - \Phi(E)\lambda_s)^{-1}.$$  

The singularities of the amplitudes $\eta_m$ lie on the first sheet of the energy on the negative half-axis and coincide with the energies of the three-particle bound states:

$$E_s = -\exp \left( -\frac{3\pi i}{2} \lambda_s \right), \quad s = 1, 2, \ldots, 6. \tag{6}$$  

The corresponding eigenfunctions of the discrete spectrum have the form

$$\Psi_s(x) = H_s^0 \left( i e^{-\frac{3\pi}{4} \lambda_s^{-1}} \right) \frac{1}{\sqrt{2}} \sum_{t} \chi_t(x) a_{st}, \tag{7}$$  

where $\chi_t(x)$ is the indicator of the sector $\lambda_t$, $a_{st}$ is the $t$ component of the eigenvector $a_s$.

Knowing the continuum functions $\Psi(x, p)$, we can find the scattering matrix $S_E$ as an operator in $L^2(S^1)$. The kernel $S_k(q, p)$ of this operator is determined by the following asymptotic behavior as $x \to \infty$ [4]:

$$\int \Psi(x, p) g(x) dx = \int S_k(x, p) g(x) dx \mathcal{G}(|p| |x|) + O(|x|^{-\nu}),$$

in which $g$ is a smooth function on $S^1$ with support that does not contain the vector $\hat{p}$, $dx$ is the normalized measure on $S^1$, and $E = p^2$. The scattering operator has the natural block structure $S_\nu S_\nu^*$ associated with decomposition of the plane $\Lambda$ into sectors; here, $\nu$ denotes the number of the sector of the incoming wave, $\mu$ the number of the scattering sector $x \in \lambda_\mu, p \in \lambda_\mu^*$. The scattering matrix $S_\nu^*$ can be expressed in terms of the amplitude $\eta_{\mu}$ as follows: $S_\nu^*(q, p) = \delta(q, \tau p) + \eta_{\mu}(\sqrt{E})$, $q \in \lambda_\nu, p \in \lambda_\nu^*$, where $\delta(q, \tau p)$ is the delta function on $S^1$.

We note finally that the Laplacians acting on the sectors are combined as a result of our constructions "in the S channel" into a single operator, which, without danger of confusion, we have denoted by $L_A$. On the orthogonal complement to the S channel, the Laplacians are decoupled, since for us only the S-wave scattering in each fixed sector is of interest.

We summarize the above results in the following proposition.

THEOREM 1. The operator $L_A$ has a finite number of negative eigenvalues $E_s$ determined by Eq. (6). The corresponding eigenfunctions of the discrete spectrum are given by Eq. (7). The continuous spectrum of $L_A$ is a six-fold spectrum and fills the interval $[0, \infty)$. The complete system of continuum eigenfunctions is determined by the relation (4).

4. Three-Particle Interaction with Internal Structure

The $S$ matrix obtained in Sec. 3 has poles at the points of the discrete spectrum of the operator $L_A$ and does not have singularities on the "unphysical" sheet. An $S$ matrix richer in singularities corresponds to the model in which it is assumed that triple collision results in the formation of a new complex of particles, described by the Hamiltonian $A^{in}$, this complex decaying in the passage of time by virtue of "external" factors. To construct the model, we need a variant of the theory of extensions of nondensely defined symmetric operators. Such a theory was used for the first time in [5] to describe the model of a point atom. We develop this theory in its application to three-particle scattering. We show how, proceeding from a given self-adjoint operator $A^{in}$ acting on the "internal" Hilbert space $\mathcal{K}$, one can construct an operator that differs from the original one by a finite-dimensional operator, and we parametrize all such extensions.
Let \( \mathcal{R} \) be the finite-dimensional generating subspace of the operator \( A^* \), \( U = (A^* + \iota)^{-1}(A^* - \iota) \) be its Cayley transform, \( \mathcal{R}^* = \mathcal{R}^\perp \mathcal{R} \). We construct a nondensely defined symmetric operator, the restriction of \( \mathcal{R} \), \( \mathcal{R}^* \). For this, we define on the lineal \( \mathcal{D}(A_R) = (I-U)\mathcal{R}^* \) the operator \( A_R \) by \( A_R h = A^* h, h \in \mathcal{D}(A_R) \).

**Theorem 2.** The operator \( A_R \) is a symmetric operator with deficiency subspaces

\[
\mathcal{R} = \mathcal{H} \ominus (A_R - iI) \mathcal{D}(A_R), \quad \mathcal{R}^* = \mathcal{H} \ominus (A_R + iI) \mathcal{D}(A_R).
\]

The extension of the symmetric operator \( L \) in Sec. 2 reduced to the restriction of its adjoint to a smaller domain, on which it became self-adjoint. In the present case, we cannot proceed in this manner, since the adjoint does not exist for a nondensely defined operator. Nevertheless, by analogy with the "dense" theory, one would want to assume that the deficiency subspaces are eigenspaces of a formally adjoint operator,

\[
A_R^* \mathcal{R} = i, \quad A_R^* \mathcal{R}^* = i
\]

the relation \( A_R^* |\mathcal{D}(A_R) = A_R |\mathcal{D}(A_R) \) holding. In general, it is not possible to construct a single operator satisfying these conditions, since although the sum \( \mathcal{R} + \mathcal{R}^* \) is not a direct sum the subspaces \( \mathcal{R} + \mathcal{R}^* \) and \( \mathcal{D}(A_R) \) may nevertheless intersect. Nevertheless, separating in \( \mathcal{R} + \mathcal{R}^* \) in a special manner the subspace \( \mathcal{D} \) and defining on \( \mathcal{D} + \mathcal{D}(A_R) \) the operator \( A_R \) on the basis of the conditions (8), we construct all the extensions of the nondensely defined operator \( A_R \).

It is clear that every extension of the operator \( A_R \) is determined uniquely by an operator \( \mathcal{B} \) that maps \( \mathcal{R} \) onto \( \mathcal{R} \) in such a manner that the operator \( U_{\mathcal{B}} = U P_{\mathcal{R}^*} + \mathcal{B} P_{\mathcal{R}^*} \) does not have as eigenvalue \( \lambda = 1 \). Here, \( P_{\mathcal{R}^*} (P_{\mathcal{R}^*}) \) is the projector onto \( \mathcal{R}^* (\mathcal{R}^*) \). Indeed, in this case there is defined the, possibly unbounded, operator \( A_{\mathcal{B}} \) related to \( U_{\mathcal{B}} \) by the Cayley transformation

\[
U_{\mathcal{B}} = (A_{\mathcal{B}} + \iota I)^{-1}(A_{\mathcal{B}} - \iota I), \quad A_{\mathcal{B}} = i(1 - U_{\mathcal{B}})^{-1}(1 + U_{\mathcal{B}}).
\]

In applications, it is important to describe the domain of definition of the extended operator \( A_{\mathcal{B}} \). Although in the case considered we cannot directly use the von Neumann formulas, the description they give of the domain of definition remains valid. Namely, let \( \mathcal{D}_{\mathcal{R}} = (I - \mathcal{B}) \mathcal{R}^* \) be the subspace in \( \mathcal{R} + \mathcal{R}^* \) uniquely determined by the operator \( \mathcal{B} \). Then we have

**Theorem 3.** The domain of definition of the operator \( A_{\mathcal{R}} \) can be represented in the form \( \mathcal{D}(A_{\mathcal{R}}) = (1 - U) \mathcal{R}^* + (1 - \mathcal{B}) \mathcal{R}^* = \mathcal{D}(A_R) + \mathcal{D}_{\mathcal{R}} \). The operator \( A_{\mathcal{R}} \) is self-adjoint if and only if \( \mathcal{B} : \mathcal{R}^* \to \mathcal{R} \) is an isometry of \( \mathcal{R}^* \) onto \( \mathcal{R} \).

Let \( \{ \psi_i \}_{i=1}^n \) be an orthonormal basis in \( \mathcal{R} \); then \( \{ U^i \psi_i \}_{i=1}^n \) is an orthonormal basis in \( \mathcal{R}^* \). In \( \mathcal{R} + \mathcal{R}^* \) we introduce the new basis \( g^*_k = \frac{1}{\sqrt{2}} (U^i \psi_i), g^*_k = (1/2) (U^i \psi_i - \psi_i) \). Every element \( \xi \in \mathcal{D}_{\mathcal{R}} \) can be uniquely represented in the form

\[
1 = \sum_k (1/2i) \alpha_k (1 - U) U^* \psi_i = \sum_k e_+^k g_+^k + e_-^k g_-^k.
\]

Let \( \mathcal{B} \) be the matrix of the operator \( \mathcal{B} : \mathcal{B} U^* \psi_i = \sum_k B \alpha_k \psi_k, \) and \( e_\alpha, \alpha \) be vectors with the components \( e_\alpha^k, \alpha \). We shall call the vectors \( e_\pm \) boundary vectors of the element \( 1 \). From (9), we obtain the equations for the boundary vectors:

\[
e_+ + e_- = \alpha, \quad i e_+ - e_- = B^* \alpha.
\]

We define the boundary vectors of elements in the domain of definition of the operator \( A_{\mathcal{R}} \) as the boundary vectors of their \( \mathcal{D}_{\mathcal{R}} \) components. It follows from the conditions (10) that the boundary vectors of the element \( u \in \mathcal{D}(A_{\mathcal{R}}) \) satisfy the relations

\[
e_- = i (1 - B^*) (1 + B^*)^{-1} e_+.
\]

In the direct sum of the "internal" and "external" Hilbert spaces, \( \mathcal{H} \otimes L^2 (\Lambda) \), we consider the symmetric operator \( \mathcal{A} = A_{\mathcal{R}} \otimes L \). It has deficiency subspaces \( \mathcal{R} \otimes \mathcal{H} \) and \( \mathcal{R}^* \otimes \mathcal{H}^* \) at the points \( i \) and \( -i \), respectively. We denote the boundary vectors of the elements of the lineal \( \mathcal{R} \) \( \mathcal{R}^* \) (\( \mathcal{R} \) \( \mathcal{R}^* \)) by \( e_{\alpha \pm} (e_{\alpha \mp} \)) of \( (e_{\alpha \uparrow} \) \( e_{\alpha \downarrow} \)). We define as the boundary vector of the element \( u \in (\mathcal{R} \otimes \mathcal{H}) \otimes (\mathcal{R}^* \otimes \mathcal{H}^*) = \) the two-component vector \( (e_{\alpha \uparrow}, e_{\alpha \downarrow}) \). We consider in \( \mathcal{D} \) a lineal \( \mathcal{D} \subset \mathcal{D} \), for which the boundary vectors of the elements satisfy the relation
Theorem 4. The operator $\mathcal{A}$,

$$\mathcal{A} = \mathcal{H} + (iI-[\mathfrak{a}] + iI-[\mathfrak{a}^*])D,$$

with domain of definition

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{H}) \oplus \mathcal{D}(L) + \mathcal{D},$$

is self-adjoint.

We shall assume that this self-adjoint operator is the Hamiltonian of the three-particle interaction with internal structure. We describe the most interesting spectral characteristics of the operator $\mathcal{A}$.

The scattered waves, corresponding to positive values of the spectral parameter $k > 0$, are determined as the solutions of the homogeneous equation

$$\mathcal{A}u = k^2 u, \quad u = (u^{in}, u^{ext}),$$

the exterior part of which in the sector $\lambda_\nu$ has the form

$$u^{ext}_\nu(x, p) = \delta_{\nu} \Psi^\nu + \eta^{ext}_\nu(|p|) G(|p| |x|), \quad p = k \cdot \hat{p},$$

while the interior part $u^{in} = u^{in}_+ + \sum \varepsilon_+ g^+_n + \varepsilon_- g^+_n$ satisfies the equation

$$\mathcal{A}^{in} u^{in}_+ + \sum \varepsilon_+ g^+_n = k^2 \left( u^{in}_+ + \sum \varepsilon_+ g^+_n \right)$$

with the additional condition (12). From (15),

$$u^{in}_+ = \sum \left( (k^2 \varepsilon^+_n - \varepsilon_- n) (A^{in} - k^2) - i \varepsilon^+_n + (k^2 \varepsilon^+_n + \varepsilon_- n) (A^{in} - k^2) - i \varepsilon_- n \right).$$

The orthogonality condition $(\mathcal{A}^{in} - I) u^{in}_+ \perp \mathfrak{A}$, $\mathfrak{A} \in \mathcal{D}(\mathcal{H})$ imposes an additional constraint on the boundary vectors of the internal problem:

$$\varepsilon_- n = \Delta(k) \varepsilon_+ n,$$

where $\Delta(k)$ is the matrix of the operator $P_{\mathfrak{a}}(I + k^2 A^{in}) (A^{in} - k^2 I)^{-1} P_{\mathfrak{a}}$ in the basis $\{ \phi_i \}_{i=1}^\infty$.

We now calculate the vector of amplitudes in the relation (14). For this, we use the connection (5) between $\varepsilon^{ext}_\nu$ and, solving simultaneously Eqs. (12) and (17), find that

$$\eta^{ext}_\nu = \left[ \mathcal{A} + \frac{Q A^*}{1 - \mathcal{A} \Delta} \Phi(k^2) - A \Phi(k^2) \right]^{-1} \left[ I - \frac{Q A^*}{1 - \mathcal{A} \Delta} \Phi(k^2) \right] e^{ext}_\nu.$$

The vectors $\varepsilon^{in}_n$ can now be found in terms of the known vectors $\varepsilon^{ext}_\nu$, which are connected to the amplitudes $\eta^{ext}_\nu$ by the relation (5),

$$\varepsilon_- n = \Delta(1 - A \Delta)^{-1} Q^* \varepsilon^{ext}_\nu, \quad \varepsilon_+ n = (1 - A \Delta)^{-1} Q^* \varepsilon^{ext}_\nu.$$

Finding $u^{in}_+$, from (16), we obtain the complete solution of the scattering problem in terms of the generating elements.

Note that in the framework of our model nothing hinders us from taking the internal Hamiltonian $A^{in}$ to be an infinite-dimensional operator. For example, to describe inelastic processes, we can use an operator $A^{in}$ with continuous spectrum. Then the scattering matrix corresponding to such an operator will be nonunitary (contractive). For all that, it is interesting to consider first the case when $A^{in}$ is finite dimensional; this is a good approximation for describing the particle interactions at low energies. In this case, the operator function $\Delta(k)$, which contains the basic information about the structure of the internal Hamiltonian, is, having positive imaginary part in the upper half-plane, rational. We trace in this case the connection between the internal Hamiltonian and the singularities of the $S$ matrix of the problem. It can be shown that when a weak coupling takes place between the internal and external spaces the positive eigenvalues...
of the operator $A^{in}$ are transformed into resonances, while the negative eigenvalues generate three-particle bound states.

Indeed, consider the operator $\mathcal{A}_\alpha$ corresponding to the matrix relation (12) with structure $\tilde{A} = 0$, $Q = Q^* = a\alpha$, where $\alpha$ is a small real parameter. We shall assume that the generating elements are $A^{in}$ orthogonal, $\langle A^{in} \psi_i, \psi_j \rangle = 0$ $i \neq j$. In this case, the matrix $\Delta(k)$ can be diagonalized, $\Delta(k) = \text{diag}\{\Delta, \Delta_3, \ldots, \Delta_t\}$, where

$$\Delta_t(k) = \left\langle \frac{A^{in} \psi_{\Delta_t}^{i+1}}{\lambda_{\Delta_t}^{i+1} - k^2} \psi_{\Delta_t}, \psi_{\Delta_t} \right\rangle = \sum \frac{\lambda_{\Delta_t}^{i+1}}{\lambda_{\Delta_t}^{i+1} - k^2} \langle P_i \psi_i, \psi_i \rangle.$$

Here, $\lambda_{\Delta_t}$ is the $s$-th eigenvalue of the operator $A^{in}$ in the $t$-th reducing subspace generated by the element $\psi_{\Delta_t}$, and $P_{\Delta_t}$ is the corresponding eigenprojector. In accordance with the general theorems on spectrum perturbation, the continuum of the operator $\mathcal{A}_\alpha$ consists of an absolutely continuous branch filling the interval $[0, \infty)$. At the same time, the external part $u^{\text{ext}}$ of the corresponding eigenfunctions is determined by Eqs. (14) and (18), and their internal part $u^{\text{in}}$ can be expressed by means of (16) in terms of the components of the vectors $\psi_{\Delta_t}$, which are uniquely determined by Eqs. (5) and (12).

**Theorem 5.** The discrete spectrum of the operator $\mathcal{A}_\alpha$ consists of a finite number of points $k = i\alpha$, which lie on the positive imaginary axis and are roots of the equation

$$\prod_{i=1}^{s} (\alpha^2 \lambda_i(k) - 1) = 0. \tag{19}$$

The functions of the discrete spectrum corresponding to these eigenvalues can be expressed in terms of the components of the vectors $\psi_{\Delta_t}$, which can be found from the homogeneous system of equations

$$\psi_{\Delta_t} = -A\psi_{\Delta_t} + Q^{*}\psi_{\Delta_t}, \quad \psi_{\Delta_t} = -A\psi_{\Delta_t} + Q\psi_{\Delta_t}, \quad \psi_{\Delta_t} = Q\psi_{\Delta_t} + A\psi_{\Delta_t}, \quad \psi_{\Delta_t} = 0 (-\alpha^2)\psi_{\Delta_t}.$$

Note that each root $k = i\alpha$ of Eq. (19) is a simple pole of the $S$ matrix.

It is interesting to follow the motion of the eigenvalues of the operator $\mathcal{A}_\alpha$ as the channel coupling the "external" and "internal" systems is closed, i.e., as $\alpha \to 0$. In this case, the eigenvalues $-\kappa_{st}$ of the operator $\mathcal{A}_\alpha$ tend as $\alpha \to 0$ to negative eigenvalues $-\kappa_{st}$ of the operator $A^{in}$:

$$\kappa_{st} - a_{st} \sim -\frac{\alpha^2}{3} \frac{\langle P_i \psi_i, \psi_i \rangle}{\lambda_{\Delta_t}} \ln(1/|\lambda_{\Delta_t}|).$$

It is not the same with the positive eigenvalues $\lambda_{\Delta_t} = a_{st}$ of the operator $A^{in}$. At small $\alpha$ they also correspond to the poles of the $S$ matrix, but these poles $k_{st}(\alpha)$ lie in the lower half-plane:

$$k_{st}(\alpha) - a_{st} \sim \frac{\alpha^2}{3} \frac{\langle P_i \psi_i, \psi_i \rangle}{\lambda_{\Delta_t}} \left( \ln \frac{1}{\lambda_{\Delta_t}} - i\alpha \right),$$

and the "external" part of the corresponding solutions of the homogeneous equation (13), which formally have the previous form, is an increasing function as $|x| \to \infty$. Such solutions are usually called resonance states, and the complex numbers $k_{st}(\alpha)$ resonances. Our numbers $k_{st}(\alpha)$ can be interpreted as three-particle resonances generated by the nontrivial internal structure of the colliding particles, while the corresponding growing solutions are to be interpreted as unstable (metastable for small $\alpha$) states.

We emphasize in conclusion that in the framework of our method it would be an easy matter to simulate the internal structure of particles that is manifested already in two-particle collisions. It was the desire to trace separately the contribution of the three- and two-particle forces to the $S$ operator that prompted us not to include the corresponding two-particle interactions in the total Hamiltonian. The analysis corresponding to this case will be published elsewhere.

**Literature Cited**