THE TAN 2Θ-THEOREM IN FLUID DYNAMICS

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Dedicated with great pleasure to Eduard Tsekanovskii at the occasion of his 80th birthday

ABSTRACT. We show that the generalized Reynolds number (in fluid dynamics) introduced by Ladyzhenskaya is closely related to the rotation of the positive spectral subspace of the Stokes block-operator in the underlying Hilbert space. We also explicitly evaluate the bottom of the negative spectrum of the Stokes operator and prove a sharp inequality relating the distance from the bottom of its spectrum to the origin and the length of the first positive gap.

1. INTRODUCTION

It is generally believed that a steady flow of an incompressible fluid is stable whenever the Reynolds number associated with the flow is sufficiently low, while it is experimentally proven that flows become turbulent for high Reynolds numbers (about several hundreds and beyond).

Historically, the first rigorous quantitative stability result for stationary solutions to the 2D-Navier-Stokes equation (in bounded domains) is due to Ladyzhenskaya [21]. Her analysis shows that given a stationary solution $v_{st}$, any other solution $v$ (with smooth initial data and the same forcing) approaches $v_{st}$ exponentially fast

$$v - v_{st} = O(e^{-\alpha t}), \quad t \to \infty,$$

whenever the generalized Reynolds number

$$\text{Re}^* = \frac{2v_s}{\nu\sqrt{\lambda_1(\Omega)}}$$

is less than one. Here $\nu$ is the viscosity of the incompressible fluid, $\lambda_1(\Omega)$ is the principal eigenvalue of the Dirichlet Laplacian in the bounded domain $\Omega$, and $v_s$ stands for the characteristic velocity of the stationary flow $v_{st}$ (see (3.8)).

In fact, the rate of convergence $\alpha$ in (1.1) is given by [21],

$$\alpha = \nu\lambda_1(\Omega)(1 - \text{Re}^*).$$

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To better understand the functional-analytic as well as (Hilbert space) geometric aspects of the Navier-Stokes stability in any dimension, we introduce and study the (model) Stokes block operator, which is the Friedrichs extension of the block operator matrix

\[
S = \begin{pmatrix} -\nu \Delta & v \ast \text{grad} \\ -v \ast \text{div} & 0 \end{pmatrix}
\]

initially defined on the set \(C_0^\infty(\Omega)^n \oplus C^\infty(\Omega)\) of infinitely differentiable vector-valued functions in the Hilbert space \(H = L^2(\Omega)^n \oplus L^2(\Omega), n \geq 2\) (cf. \([3, 10]\)).

One of the principal results of the current paper links the Ladyzhenskaya-Reynolds number \(\text{Re}^*\) to the norm of the operator angle \(\Theta\) between the positive subspace of the Stokes operator and the positive subspace of its diagonal part (see \([7, 19, 27]\) for the concept of an operator angle). That is, the following \(\tan 2 \Theta\) \(\text{Theorem in Fluid Dynamics},\)

\[
\tan 2\|\Theta\| \leq \text{Re}^*,
\]

holds (see Theorem 2.4).

The essence of this estimate is the remarkable fact that \(\text{the magnitude of the Reynolds number limits the rotation of the spectral subspaces of the block Stokes operator.}\)

We also show that the lowest positive eigenvalue \(\lambda_1(S)\) of the Stokes operator \(S\) and the bottom of its negative (essential) spectrum satisfy the inequality

\[
|\inf \text{spec}(S)| \leq \frac{1}{4} \text{[Re}^*\text{]}^2 \lambda_1(S),
\]

which is asymptotically sharp as \(\nu \to \infty\) or \(v \ast \to 0\).

In particular, the Ladyzhenskaya (2D-) stability hypothesis \(\text{Re}^* < 1\) yields the following \(\text{Stability Laws} :\)

- the relative spectral shift \(\delta\) defined as ratio of the shift of the spectrum from the origin to the left to the length of the spectral gap of the Stokes operator is bounded by
  \[
  \delta = \frac{|\inf \text{spec}(S)|}{\lambda_1(S)} < \frac{1}{4};
  \]

- the maximal rotation angle \(\|\Theta\|\) between the positive subspaces of the perturbed and unperturbed Stokes operators is bounded by
  \[
  \|\Theta\| < \frac{\pi}{8};
  \]

- the Friedrichs extension of the block operator matrix
  \[
  T = \begin{pmatrix} -\nu \Delta - \frac{1}{2} \nu \lambda_1(\Omega) & v \ast \text{grad} \\ -v \ast \text{div} & \frac{1}{2} \nu \lambda_1(\Omega) \end{pmatrix}
  \]
  is positive definite.
We also observe that the Ladyzhenskaya decay exponent $\alpha$ provides the lower bound for $\inf \text{spec}(T)$,

$$\alpha = \nu \lambda_1(\Omega)(1 - \Re^*) \leq 2 \cdot \inf \text{spec}(T),$$

which is asymptotically sharp in the sense that

$$\lim_{\Re^* \downarrow 0} \frac{\alpha}{\inf \text{spec}(T)} = 2.$$

All that combined gives the direct operator-theoretic interpretation for the 2D-Ladyzhenskaya result in the framework of the linearization method in hydrodynamical stability theory.

The paper is organized as follows.

In Section 2, the Stokes operator $S$ is defined as a self-adjoint operator in the Hilbert space $\mathcal{H}$. In Theorem 2.1, based on the quadratic numerical range variational principle (see Appendix B), we obtain an estimate for the first positive eigenvalue and explicitly evaluate the lower edge of $S$. Theorem 2.4, the Tan $2\Theta$-Theorem in fluid dynamics, is deduced from a general rotation angle bound obtained in [14] for indefinite forms. In Theorem 2.5, we show that at low Reynolds numbers, the qualitative spectral analysis for the Stokes operator is closely related to the one for its principal symbol.

In Section 3, under the hypothesis that the generalized Reynolds number is less than one, see the delightful paper “Life at low Reynolds number” by E. M. Purcell [23] for an excellent conceptual presentation, we discuss the Stability Laws and provide an operator-theoretic interpretation for the Ladyzhenskaya Stability Theorem [21, Theorem 6.5.12].

Appendix A contains supplementary material beyond the main scope of the exposition and deals with the dimensional analysis of the problem in question.

First, we provide a (heuristic) justification supporting the appearance of the characteristic velocity parameter $v_*$ in the definition of the Stokes operator.

Next, as a result of the dimensional analysis, we naturally arrive at dimensionless variables such as the generalized Reynolds and Strouhal type numbers, see (A.3) and (A.4) for their definition. We also show that at low Reynolds numbers, their product and ratio is proportional to the distance from the bottom of the spectrum of the Stokes operator to the origin and the length of spectral gap of the diagonal part of the Stokes operator, respectively (see (A.5) and (A.6)). This observation is illustrated in the Strouhal-Reynolds-Rotation angle diagram Fig. 2.

In Appendix B, we briefly recall representation theorems for indefinite (saddle-point) forms and provide necessary information on the properties of their quadratic numerical range (cf. [28] for the concept of quadratic numerical ranges for operator matrices).

We adopt the following notation. In the Hilbert space $\mathcal{H}$ we use the scalar product $\langle \cdot, \cdot \rangle$ semi-linear in the first and linear in the second component. $I$.
denotes the identity operator on a Hilbert space \( \mathcal{H} \), where we frequently omit the subscript. Given a self-adjoint operator \( S \) and a Borel set \( M \) on the real axis, the corresponding spectral projection is denoted by \( E_S(M) \).

Given an orthogonal decomposition \( \mathcal{H}_0 \oplus \mathcal{H}_1 \) of the Hilbert space \( \mathcal{H} \) and dense subsets \( \mathcal{K}_i \subset \mathcal{H}_i, \ i = 0, 1 \), by \( \mathcal{H}_0 \oplus \mathcal{K}_1 \) we denote a subset of \( \mathcal{H} \) formed by the vectors \( (x_0, x_1) \) with \( x_i \in \mathcal{K}_i, \ i = 0, 1 \).

2. The Stokes Operator

Assume that \( \Omega \) is a bounded \( C^2 \)-domain in \( \mathbb{R}^n, \ n \geq 2 \), \( \Delta = \Delta \cdot I_n \) is the vector-valued Dirichlet Laplacian, with \( I_n \) the identity operator in \( \mathbb{C}^n \), and \( \nu > 0 \) and \( v_* \geq 0 \) are parameters. In the direct sum of Hilbert spaces \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \), where \( \mathcal{H}_0 = L^2(\Omega)^n \) and \( \mathcal{H}_1 = L^2(\Omega) \) stand for the “velocity” and “pressure” subspaces, respectively, consider the Stokes block operator matrix (cf. [10]) given by

\[
\begin{pmatrix}
-\nu \Delta & v_* \text{grad} \\
-v_* \text{div} & 0
\end{pmatrix}
\]

We introduce a self-adjoint realization \( S = S(\nu, v_*) \) of (2.1) as a unique self-adjoint operator associated with the symmetric sesquilinear (saddle-point) form \( s[v \oplus p, u \oplus q] \)

\[
= \nu \sum_{j=1}^n \int_{\Omega} \langle D_j v(x), D_j u(x) \rangle \, dx - v_* \int_{\Omega} \text{div} v(x) q(x) \, dx - v_* \int_{\Omega} \text{div} u(x) p(x) \, dx
\]

\[
= \nu \langle \text{grad} v, \text{grad} u \rangle - v_* \langle \text{div} v, q \rangle - v_* \langle p, \text{div} u \rangle
\]

defined on

\( \text{Dom}[s] = \{ v \oplus p \mid v \in H_0^1(\Omega)^n, \ p \in L^2(\Omega) \} \).

Here \( \text{grad} \) denotes the component-wise application of the standard gradient operator initially defined on the Sobolev space \( H_0^1(\Omega) \).

Using the inequality

\[
|\langle \text{div} v, p \rangle| \leq \varepsilon \| (-\Delta)^{1/2} v \|^2 + C(\varepsilon)(\| v \|^2 + \| p \|^2)
\]

valid for any \( \varepsilon > 0 \), with \( C(\varepsilon) \) an appropriately chosen constant, one verifies that \( s \) on \( \text{Dom}[s] \) is a closed semi-bounded form (by the KLMN-Theorem).

We also remark that the closure of the operator matrix (2.1) defined on the Sobolev space \( (H^2(\Omega) \cap H_0^1(\Omega))^n \oplus H^1(\Omega) \) is a self-adjoint operator, see [10], which yields another characterization for the operator \( S = S(\nu, v_*) \).

We now provide more detailed information on the location of the spectrum of the Stokes operator \( S \).
Theorem 2.1. Let $S$ be the Stokes operator. Then

(i) the positive spectrum of $S$ is discrete and

$$\lambda_1(S) \geq \nu \lambda_1(\Omega),$$

where $\lambda_1(S)$ is the smallest positive eigenvalue of $S$ and $\lambda_1(\Omega)$ is the principal eigenvalue of the Dirichlet Laplacian in $\Omega$. Moreover, the asymptotic representation

$$\lambda_1(S) = \nu \lambda_1(\Omega)(1 + o(1)) \quad \text{as} \quad \nu \to \infty \quad \text{or} \quad v_* \to 0,$$

holds;

(ii) the point $\lambda = 0$ is an isolated simple eigenvalue of $S$;

(iii) the bottom of the (essential) spectrum of the Stokes operator is explicitly given by

$$\inf \text{spec}(S) = -\frac{v_*^2}{\nu}.$$

In particular,

$$\inf \text{spec}(S) = -\frac{1}{4} \nu \lambda_1(\Omega)[\text{Re}^*]^2,$$

where

$$\text{Re}^* = \frac{2v_*}{\nu \sqrt{\lambda_1(\Omega)}},$$

is the generalized Reynolds number. Moreover, one has the estimate

$$|\inf \text{spec}(S)| \leq \frac{1}{4} [\text{Re}^*]^2 \lambda_1(S).$$

Proof: (i). It is well known that the essential spectrum of the Stokes operator $S$ is purely negative [3], [10], [12], therefore, the positive spectrum of $S$ is discrete.

The inequality

$$\lambda_1(S) \geq \nu \lambda_1(\Omega),$$

follows from Lemma B.3 (vi) (see Appendix B).

To prove the asymptotics (2.2), we proceed as follows.

Let $\lambda_1(\Omega)$ denote the first positive eigenvalue of the vector-valued Dirichlet problem

$$-\Delta f = \lambda_1(\Omega)f,$$

$$f|_{\partial \Omega} = 0,$$

with $f$ the corresponding eigenfunction. Introducing $v = (f, 0)^T \in \mathcal{H}$, one observes that

$$\|Sv - \nu \lambda_1(\Omega)v\| = v_* \|\text{div} f\|.

Using the standard estimate

$$\text{dist}(\lambda, \text{spec}(T)) \leq \frac{\|(T - \lambda I)x\|}{\|x\|}, \quad x \in \text{Dom}(T),$$
valid for any self-adjoint operator $T$, from (2.7) it follows that
\[
\text{dist}(\nu \lambda_1(\Omega), \text{spec}(S)) \leq v_s \frac{\|\text{div} f\|}{\|f\|}.
\]

This yields the claimed asymptotics (2.2) for $v_s \to 0$ and, by rescaling, for $\nu \to \infty$. Taking into account that the open interval $(0, \nu \lambda_1(\Omega))$ is free of the spectrum of $S$, we even get that
\[
(2.8) \quad \nu \lambda_1(\Omega) \leq \lambda_1(S) \leq \nu \lambda_1(\Omega) + v_s \frac{\|\text{div} f\|}{\|f\|}.
\]

(ii). We claim that
\[
\text{Ker}(S) = \{0 \oplus p \mid p \text{ constant} \} \subset L^2(\Omega)^n \oplus L^2(\Omega).
\]

Indeed, by [25, Theorem 1.3],
\[
(2.9) \quad \text{Ker}(S) = (\text{Ker}(-\Delta) \cap L^2) \oplus L^2 \subset L^2(\Omega)^n \oplus L^2(\Omega),
\]
where
\[
L^+ = \{v \in H^1_0(\Omega)^n \mid \langle \text{div} v, p \rangle = 0 \text{ for all } p \in L^2(\Omega)\}
\]
and
\[
(2.10) \quad L^- = \{p \in L^2(\Omega) \mid \langle \text{div} v, p \rangle = 0 \text{ for all } v \in H^1_0(\Omega)^n\}.\]

Since $\text{Ker}(-\Delta)$ is trivial, we have that $\text{Ker}(S) = L^-$. This means that the pressure $p$ is a constant function since $\langle \text{div} v, p \rangle = 0$ for all $v$.

Since the essential spectrum of the Stokes operator is purely negative, it follows that $\lambda = 0$ is an isolated eigenvalue of $S$ of multiplicity one.

(iii). We prove (2.3) by applying Lemma B.3 (iv) (see Appendix B) that states that
\[
(2.11) \quad \inf \text{spec}(S) = \inf W^2[s],
\]
where
\[
(2.12) \quad W^2[s] = \bigcup_{v \oplus p \in H^1_0(\Omega)^n \oplus L^2(\Omega), \|v\| = \|p\| = 1} \text{spec} \left( \frac{\nu \|\text{grad} v\|^2}{-v_s \langle \text{div} v, p \rangle} \begin{pmatrix} -v_s \langle \text{div} v, p \rangle \\ 0 \end{pmatrix} \right)
\]
is the quadratic numerical range (associated with the decomposition $\mathcal{H} = (L^2(\Omega))^n \oplus L^2(\Omega)$).

To do so, consider the trial functions
\[
(2.13) \quad u(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)e^{ikx_1(1, 0, \ldots, 0)^T},
\]
\[
p(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)e^{ikx_1},
\]
where \( f \in C_0^\infty(\Omega) \) with \( \|f\| = 1 \) (\( k \) is a large parameter). From (2.11) and (2.12) one gets the estimate

\[
\inf \text{spec}(S) \leq \inf \text{spec} \left( \begin{pmatrix} \nu \| \nabla u \|^2 & -v_* \langle \text{div } u, p \rangle \\ -v_* \langle \text{div } u, p \rangle & 0 \end{pmatrix} \right).
\]

Since for \( u \) and \( p \) given by (2.13) we have that \( \nu \| \nabla u \|^2 = \nu k^2 + \mathcal{O}(k) \) and \( v_* \langle \text{div } u, p \rangle = iv_* k + \mathcal{O}(1) \) as \( k \to \infty \), inequality (2.14) yields

\[
\inf \text{spec}(S) \leq \lim_{k \to \infty} \inf \text{spec} \left( \begin{pmatrix} \nu k^2 + \mathcal{O}(k) & iv_* k + \mathcal{O}(1) \\ -iv_* k + \mathcal{O}(1) & 0 \end{pmatrix} \right) = -\frac{v_*^2}{\nu}.
\]

To prove the opposite inequality, suppose that \( v \in H_0^1(\Omega)^n \) and \( p \in L^2(\Omega) \) are chosen in such a way that \( \|v\| = \|p\| = 1 \). Then it is clearly seen that

\[
-\frac{v_*^2}{\nu} \leq \inf \text{spec} \left( \begin{pmatrix} \nu \| \nabla v \|^2 & v_* \| \nabla v \| \\ v_* \| \nabla v \| & 0 \end{pmatrix} \right)
\]

\[
= \inf \text{spec} \left( \begin{pmatrix} \nu k^2 & iv_* k \\ -iv_* k & 0 \end{pmatrix} \right) = -\frac{v_*^2}{\nu}.
\]

Here, we used the inequality

\[
|\langle \text{div } v, p \rangle| \leq \|\text{div } v\| \cdot \|p\| \leq \| \nabla v \|,
\]

and the observation that the lowest eigenvalue of a symmetric \( 2 \times 2 \) matrix decreases whenever the absolute value of its off-diagonal entries increases. \( \square \)

**Remark 2.2.** Notice that the upper estimate

\[
\inf \text{spec}(S) \leq -\frac{v_*^2}{\nu}
\]

also follows from the known fact that for any bounded \( C^2 \)-domain \( \Omega \subset \mathbb{R}^n \) the essential spectrum of the Stokes operator is a two-point set

\[
\text{spec}_{\text{ess}}(S) = \left\{ -\frac{v_*^2}{\nu}, -\frac{v_*^2}{2\nu} \right\}
\]

(see, e.g., [10, Theorem 3.15] where the corresponding result is proven for \( \nu = 1 \), \( v_* = -1 \), and can be adapted to the case in question by rescaling).

**Remark 2.3.** As it follows from the proof, inequality (2.5) is asymptotically sharp in the sense that

\[
\lim_{\nu \to \infty} \frac{|\inf \text{spec}(S)|}{\lambda_1(S)} = \lim_{v_* \to 0} \frac{|\inf \text{spec}(S)|}{\lambda_1(S)} = \frac{1}{4} |\text{Re}^*|^2.
\]
Our next ultimate goal is to obtain bounds on the maximal rotation angle between the positive subspace of the Stokes operator and the positive subspace of its diagonal part.

Recall that if $P$ and $Q$ are orthogonal projections and $\text{Ran}(Q)$ is a graph subspace with respect to the decomposition $\mathcal{H} = \text{Ran}(P) \oplus \text{Ran}(P^\perp)$, then the operator angle $\Theta$ between the subspaces $\text{Ran}(P)$ and $\text{Ran}(Q)$ is defined to be a unique self-adjoint operator in the Hilbert space $\mathcal{H}$ with the spectrum in $[0, \pi/2]$ such that

$$\sin^2 \Theta = PQ^\perp|_{\text{Ran}(P)}.$$

Without any attempt to give a complete overview of the work done on pairs of subspaces and operator angles, we mention the pioneering works [6, 8, 9, 11, 15, 20]. For more recent results on operator angles and their norm estimates, we refer to [1, 2, 5, 7, 14, 17, 18, 27, 28] and references therein.

We now present our main result.

**Theorem 2.4 (The tan2Θ-Theorem in Fluid Dynamics).** Denote by $\Theta$ the operator angle between the positive subspace $\text{Ran} E_S((0, \infty))$ of the Stokes operator and the positive subspace of its diagonal part $\mathcal{H}_+ = L^2(\Omega)^n \oplus \{0\}$.

Then

$$\tan 2\|\Theta\| \leq \text{Re}^*, \tag{2.17}$$

where $\text{Re}^*$ is the generalized Reynolds number given by (1.2).

**Proof.** Denote by $Q$ be the orthogonal projection from $\mathcal{H}$ onto the positive spectral subspace $\text{Ran} E_S((0, \infty))$ of the Stokes operator $S$ and let $P$ be the orthogonal projection onto $\mathcal{H}_+ = L^2(\Omega)^n \oplus \{0\}$.

From [14, Theorem 3.1] it follows that

$$\sin \|\Theta\| = \|P - Q\| \leq \sin \left(\frac{1}{2} \arctan \gamma \right), \tag{2.18}$$

where

$$\gamma = \inf_{\mu \in (0, \nu \lambda_1(\Omega))} \sup_{v \oplus p \in H^1_0(\Omega)^n \oplus L^2(\Omega)} \frac{2v_\ast |\text{Re} \langle \text{div} v, p \rangle|}{\nu \langle \text{grad} v, \text{grad} v \rangle - \mu \|v\|^2 + \mu \|p\|^2}.$$

Using the Poincaré inequality

$$\|w\| \leq \frac{1}{\sqrt{\lambda_1(\Omega)}} \|\nabla w\|, \quad w \in H_0^1(\Omega),$$

and the bound

$$\|\text{div} v\| \leq \|\text{grad} v\|,$$
one then obtains that
\[
\gamma \leq \inf_{\mu \in (0, \nu \lambda_1(\Omega))} \sup_{v \in H_0^1(\Omega) \cap L^2(\Omega)} \frac{2v_s \|\nabla v\| \cdot \|p\|}{(\nu - (\lambda_1(\Omega))^{-1}\mu) \|\nabla v\|^2 + \mu \|p\|^2}
\]
(2.19)

\[
\leq \inf_{\mu \in (0, \nu \lambda_1(\Omega))} \frac{v_s}{\sqrt{(\nu - (\lambda_1(\Omega))^{-1}\mu)\mu}}.
\]

Since the infimum (2.19) is attained at the midpoint \(\mu_{\text{opt}}\) of the interval \((0, \nu \lambda_1(\Omega))\) with

(2.20) \[\mu_{\text{opt}} = \frac{1}{2} \nu \lambda_1(\Omega),\]

we obtain the estimate

(2.21) \[\gamma \leq \frac{2v_s}{\nu \sqrt{\lambda_1(\Omega)}} = \text{Re}^*\]

The estimate (2.17) now follows from (2.18).

\[\square\]

We remark that performing the spectral analysis of the Stokes operator can essentially be reduced to the one of its principal symbol which is given by the following \(2 \times 2\) numerical matrix (cf. (2.15))

\[s(\nu, v_s; k) = \begin{pmatrix} \nu k^2 & iv_s k \\ -iv_s k & 0 \end{pmatrix}\]

with the right choice for the “wave number” \(k = \sqrt{\lambda_1(\Omega)}\), where \(\lambda_1(\Omega)\) is the principal Dirichlet eigenvalue of the Laplace operator on the domain \(\Omega\).

**Theorem 2.5.** Let \(s = s(\nu, v_s; \lambda_1(\Omega))\) be the principal symbol of the Stokes operator is evaluated at the wave number

(2.22) \[k = \sqrt{\lambda_1(\Omega)}.
\]

Then

(2.23) \[1 = \lim_{v_s \downarrow 0} \inf \text{spec}(s) = \lim_{\nu \to \infty} \inf \text{spec}(s).
\]

Moreover, the operator angle \(\Theta\) referred to in Theorem 2.4 admits the following norm estimate

(2.24) \[\|\Theta\| \leq \theta,
\]

with \(\theta\) the angle between the eigenvectors of the \(2 \times 2\) matrices \(s(\nu, v_s; \sqrt{\lambda_1(\Omega)})\) and \(s(\nu, 0; \sqrt{\lambda_1(\Omega)})\) corresponding to their positive eigenvalues.
Proof. Let $\lambda_- (s)$ and $\lambda_+ (s)$ be the negative and positive eigenvalues of the $2 \times 2$ matrix $s(\nu, v^*; k)$, respectively.

It is easy to see that
$$\lim_{k \to \infty} \lambda_- (s(\nu, v^*; k)) = -\frac{v^*_2}{\nu},$$
and that
$$\lambda_- (s(\nu, v^*; k)) = -\frac{v^*_2}{\nu} (1 + o(1)) \ \text{as} \ \nu \to \infty \ \text{or} \ v_* \to 0.$$

Moreover,
$$\lambda_+ (s(\nu, v^*; \sqrt{\lambda_1 (\Omega)})) = \nu \lambda_1 (\Omega) (1 + o(1)) \ \text{as} \ \nu \to \infty$$
and
$$\lim_{v_* \to 0} \lambda_+ (s(\nu, v^*; \sqrt{\lambda_1 (\Omega)})) = \nu \lambda_1 (\Omega) \ \text{as} \ v_* \to 0.$$

Comparing these asymptotics with the representations (2.2) and (2.3) in Theorem 2.1 proves (2.23).

To prove the estimate (2.24), observe that the rotation angle $\theta$ between the positive eigensubspaces of the $2 \times 2$ matrices
$$s = \begin{pmatrix} \nu \lambda_1 (\Omega) & iv_* \sqrt{\lambda_1 (\Omega)} \\ -iv_* \sqrt{\lambda_1 (\Omega)} & 0 \end{pmatrix} \quad \text{and} \quad s_0 = \begin{pmatrix} \nu \lambda_1 (\Omega) & 0 \\ 0 & 0 \end{pmatrix}$$
is explicitly given by (cf., [14, Example 4.4])
$$\theta = \frac{1}{2} \arctan \frac{2v_*}{\nu \sqrt{\lambda_1 (\Omega)}} = \frac{1}{2} \arctan \text{Re}^*.$$

The estimate (2.24) follows then from Theorem 2.4.

\[\square\]

3. REYNOLDS NUMBER LESS THAN ONE

In this section, we discuss the case of low Reynolds number (in any dimension $n \geq 2$).

First, we observe that by Theorem 2.4, the hypothesis $\text{Re}^* < 1$ implies

(i) the lower edge $\inf \text{spec}(S)$ of the spectrum of the Stokes operator and its first positive eigenvalue $\lambda_1 (S)$ satisfy the inequality

$$| \inf \text{spec}(S) | < \frac{1}{4} \cdot \lambda_1 (S);$$

(ii) the operator angle $\Theta$ between the positive spectral subspaces of the Stokes operator $S = S(\nu, v^*)$ and the unperturbed diagonal operator $S_0 = S(\nu, 0)$ satisfies the inequality

$$\| \Theta \| < \frac{\pi}{8}.$$
Moreover, by Theorem 3.1 below, we also have that
(iii) the Friedrichs extension $T$ of the block operator matrix

$$
\begin{pmatrix}
-\nu \Delta - \frac{1}{2} \nu \lambda_1(\Omega) & v_\ast \text{grad} \\
-v_\ast \text{div} & \frac{1}{2} \nu \lambda_1(\Omega)
\end{pmatrix}
$$

is a positive definite operator.

**Theorem 3.1.** Let $T$ be the Friedrichs extension of the block operator (3.2). Then

$$
\frac{\nu \lambda_1(\Omega)}{2} \left(1 - \text{Re}^\ast \max \left\{ 1, \frac{1}{2} \text{Re}^\ast \right\}\right) \leq \inf \text{spec}(T),
$$

where $\text{Re}^\ast$ is the generalized Reynolds number given by (1.2).

In particular, if $\text{Re}^\ast < 1$, then the operator $T$ is positive definite and the estimate

$$
\frac{1}{2} \nu \lambda_1(\Omega)(1 - \text{Re}^\ast) \leq \inf \text{spec}(T),
$$

holds and is asymptotically sharp as $\text{Re}^\ast \to 0$.

**Proof.** As in the proof of Theorem 2.1 (iii), we apply Lemma B.3 to see that

$$
\inf \text{spec}(T) = \inf W^2[t].
$$

Here

$$
W^2[t] = \bigcup_{v \oplus p \in H_0^1(\Omega)^n \oplus L^2(\Omega), \|v\| = \|p\| = 1} \text{spec}\left( \nu \|\text{grad} v\|^2 - \frac{1}{2} \nu \lambda_1(\Omega) - v_\ast \langle \text{div} v, p \rangle \begin{pmatrix} 1 & \frac{1}{2} \nu \lambda_1(\Omega) \\ \frac{1}{2} \nu \lambda_1(\Omega) & -v_\ast \rangle \begin{pmatrix} \text{div} v, p \end{pmatrix} \right),
$$

is the quadratic numerical range with respect to $H = (L^2(\Omega))^n \oplus L^2(\Omega)$.

We claim that

$$
\frac{\nu \lambda_1(\Omega)}{2} \inf_{1 \leq x} \inf \text{spec} \left( 2x^2 - 1 \begin{pmatrix} \text{Re}^\ast x & 1 \\ 1 & 1 \end{pmatrix} \right) \leq \inf W^2[t].
$$

Indeed, introduce the notation $k = \|\text{grad} v\|$. Then

$$
\|\langle \text{div} v, p \rangle\| \leq \|\text{grad} v\| \|p\| = k
$$
due to the hypothesis that $\|v\| = \|p\| = 1$. Therefore, the lowest eigenvalue of the $2 \times 2$ matrix

$$
\begin{pmatrix}
\nu \|\text{grad} v\|^2 - \frac{1}{2} \nu \lambda_1(\Omega) & -v_\ast \langle \text{div} v, p \rangle \\
-v_\ast \langle \text{div} v, p \rangle & \frac{1}{2} \nu \lambda_1(\Omega)
\end{pmatrix}
$$
does not exceed the one of

$$
\begin{pmatrix}
\nu k^2 - \frac{1}{2} \nu \lambda_1(\Omega) & -v_\ast k \\
-v_\ast k & \frac{1}{2} \nu \lambda_1(\Omega)
\end{pmatrix}.
$$
Due to the Poincaré inequality, one also has the bound
\[ \sqrt{\lambda_1(\Omega)} \leq k. \]
Thus,
\[ \inf_{\lambda_1(\Omega) \leq k} \inf \text{spec} \left( \begin{pmatrix} \nu k^2 - \frac{1}{2} \nu \lambda_1(\Omega) & -v_x k \\ -v_x k & \frac{1}{2} \nu \lambda_1(\Omega) \end{pmatrix} \right) \leq \inf W^2[t] \]
and (3.5) follows.

Next, it is an elementary exercise to see that the lowest eigenvalue of the matrix
\[ M(x) = \begin{pmatrix} 2x^2 - 1 & -\text{Re}^* x \\ -\text{Re}^* x & 1 \end{pmatrix} \]
is a monotone function in \( x \) on \([1, \infty)\) (increasing for \( \text{Re}^* < 2 \) and decreasing for \( \text{Re}^* > 2 \)), and therefore
\[ \frac{\nu \lambda_1(\Omega)}{2} \min \left\{ 1 - \text{Re}^*, 1 - \frac{1}{2} [\text{Re}^*]^2 \right\} \leq \inf W^2[t], \]
which proves the bound (3.3).

To show that the estimate is asymptotically sharp, denote by \( \lambda_1(\Omega) \) the first eigenvalue of the Dirichlet problem
\[ -\Delta f = \lambda_1(\Omega) f, \]
\[ f|_{\partial \Omega} = 0, \]
with \( f \) the corresponding eigenfunction. Introducing
\[ v = (f, 0, \ldots, 0)^T \in \mathcal{H}, \]
one observes that
\[ (3.6) \quad \| Tv - \frac{1}{2} \nu \lambda_1(\Omega) v \| = v_x \| f_x \|, \]
which implies
\[ \text{dist} \left( \frac{1}{2} \nu \lambda_1(\Omega), \text{spec}(T) \right) \leq v_x \frac{\| f_x \|}{\| f \|}. \]
Thus,
\[ \frac{1}{2} \nu \lambda_1(\Omega) \leq \inf \text{spec}(T) \leq \frac{1}{2} \nu \lambda_1(\Omega) + v_x \frac{\| f_x \|}{\| f \|} \]
and hence
\[ \frac{1}{2} \nu \lambda_1(\Omega)(1 - \text{Re}^*) = \inf \text{spec}(T)(1 + \mathcal{O}(\text{Re}^*)) \quad \text{as} \quad \text{Re}^* \to 0, \]
which completes the proof.
Remark 3.2. The positive definiteness of the operator \( T \) is the manifestation of a geometric variant of the Birman-Schwinger principle for off-diagonal perturbations as presented in [14, Corollary 3.4]).

Indeed, the operator \( T_\mu = S - \mu J, \mu \in (0, \nu \lambda_1(\Omega)) \), is positive definite if and only if
\[
\gamma(\mu) = \sup_{v \in p \in H^1_0(\Omega) \oplus L^2(\Omega)} \frac{2v_\ast |\text{Re} \langle \text{div} v, p \rangle|}{\nu (\text{grad} v, \text{grad} v) - \mu \|v\|^2 + \mu \|p\|^2} < 1.
\]

But we have already seen (cf. (2.20) and (2.21)) that
\[
\gamma(\mu_{\text{opt}}) \leq \text{Re}^* < 1, \quad \text{with} \quad \mu_{\text{opt}} = \frac{1}{2} \nu \lambda_1(\Omega),
\]
which shows that \( T = T_{\mu_{\text{opt}}} \) is positive definite.

Remark 3.3. By [3, Corollary 2.1], the essential spectrum of the operator matrix \( T \) can be computed explicitly as
\[
\text{spec}_{\text{ess}}(T) = \left\{ \frac{1}{2} \nu \lambda_1(\Omega) - \frac{v_\ast^2}{\nu}, \frac{1}{2} \nu \lambda_1(\Omega) - \frac{1}{2} \frac{v_\ast^2}{\nu} \right\}
\]
and thus
\[
(3.7) \quad \inf \text{spec}_{\text{ess}}(T) = \frac{1}{2} \nu \lambda_1(\Omega) - \frac{v_\ast^2}{\nu} = \frac{1}{2} \nu \lambda_1(\Omega) \left( 1 - \frac{1}{2} |\text{Re}^*|^2 \right).
\]

In particular, it follows from (3.3) that for large values of the Reynolds number the essential spectrum of \( T \) coincides with the lower edge of its spectrum,
\[
\inf \text{spec}_{\text{ess}}(T) = \inf \text{spec}(T) = \frac{1}{2} \nu \lambda_1(\Omega) \left( 1 - \frac{1}{2} |\text{Re}^*|^2 \right) \quad \text{for} \quad \text{Re}^* \geq 2,
\]
and hence inequality (3.3) turns into an equality.

We also remark that if \( \text{Re}^* < 2 \), then by Theorem 2.1, the spectra of the absolute value \(|S|\) of the Stokes operator restricted to the non-negative subspace of \( S \) and to its orthogonal complement are subordinated (cf. [14, Theorem 4.2]). That is,
\[
\max \text{spec}(|S|_{\text{Ran}(E_S(-\infty,0))}) < \min \text{spec}(|S|_{\text{Ran}(E_S(0,\infty))}).
\]

Our main motivation for the discussion of the low Reynolds number hypothesis (\( \text{Re}^* < 1 \)) and its implications is to better understand the functional-analytic aspects of the Ladyzhenskaya stability result that concerns the asymptotic behavior of solutions to the 2D Navier-Stokes equation
\[
\frac{\partial v}{\partial t} + \langle v, \nabla \rangle v - \nu \Delta v = -\frac{1}{\rho} \text{grad} p + f,
\]
\[
\text{div} v = 0, \quad v|_{\partial \Omega} = 0, \quad v|_{t=0} = v_0.
\]
Here, as usual, we are dealing with a (nonstationary) flow \( v \) of an incompressible fluid that does not move close to the (smooth) boundary \( \partial \Omega \) of a (bounded) domain \( \Omega \) and \( u, p \), and \( f \) stand for the velocity field, pressure, and the acceleration due to
external forcing, respectively. Furthermore, \( \rho \) and \( \nu \) are the constant density and viscosity of the fluid and \( v_0 \) is the initial velocity of the flow.

**Proposition 3.4** ([21, Theorem 6.5.12]). Suppose that \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) with \( C^2 \)-boundary and that \( v_{st} \) is a stationary solution of the two dimensional Navier-Stokes equation

\[
(v_{st} \cdot \nabla)v_{st} - \nu \Delta v_{st} + \frac{1}{\rho} \nabla p = f,
\]

\[
\text{div} v_{st} = 0, \quad v_{st}|_{\partial \Omega} = 0,
\]

such that the generalized Reynolds number

\[
Re^* = \frac{2v_s}{\nu \sqrt{\lambda_1(\Omega)}}
\]

is less than one, where

\[
v_s = \left( \iint_{\Omega} \left( \left| \frac{\partial v_{st}}{\partial x} \right|^2 + \left| \frac{\partial v_{st}}{\partial y} \right|^2 \right) \, dx \, dy \right)^{1/2}.
\]

Let \( v \) be a solution for the non-stationary problem corresponding to the same force \( f \) with the initial data \( v|_{t=0} \in H^2(\Omega)^2 \cap J_{0,1} \), where \( J_{0,1} \) is the closure in \( H^1 \)-norm of the smooth solenoidal vector fields of compact support in \( \Omega \).

Then the difference \( u = v - v_{st} \) between these two solutions satisfies the inequality

\[
\|u(x,t)\| \leq \|u(x,0)\| \exp(-\alpha t), \quad x \in \Omega, \quad t \geq 0,
\]

where

\[
\alpha = \nu \lambda_1(\Omega)(1 - Re^*).
\]

We remark that the Ladyzhenskaya stability hypothesis \( Re^* < 1 \) of Proposition 3.4 implies the Stability Laws (i), (ii), and (iii). Moreover, applying Theorem 3.1 now shows that the decay exponent \( \alpha \) provides a lower bound for \( \inf \text{spec}(T) \),

\[
\alpha = \nu \lambda_1(\Omega)(1 - Re^*) \leq 2 \cdot \inf \text{spec}(T),
\]

which is asymptotically sharp in the sense that

\[
\lim_{Re^* \downarrow 0} \frac{\alpha}{\inf \text{spec}(T)} = 2.
\]

Also notice that the lowest eigenvalue of the principal symbol

\[
t(\nu, v^*; k) = \begin{pmatrix}
\nu k^2 - \frac{1}{2}\nu \lambda_1(\Omega) & iv_s k \\
-iv_s k & \frac{1}{2} \nu \lambda_1(\Omega)
\end{pmatrix}
\]

of the operator \( T \) evaluated at \( k = \sqrt{\lambda_1(\Omega)} \),

\[
t = \frac{1}{2} \nu \lambda_1(\Omega) \begin{pmatrix}
1 & i Re^* \\
-i Re^* & 1
\end{pmatrix}
\]
equals one half of the decay exponent in (3.9), that is,
\[ \alpha = 2 \inf \text{spec}(t). \]

All that combined together now sheds some light on the functional-analytic nature of the 2D stability in fluid dynamics.

**APPENDIX A. DIMENSIONAL ANALYSIS**

In this appendix we present (i) a heuristic consideration that motivated the particular choice of the Stokes block operator and (ii) apply the general dimensional theory to perform spectral analysis of the Stokes system.

(i). Assume that the Navier-Stokes equation has a steady-state solution \( v_{st} \) and linearize the equation in a neighborhood of this solution to get
\[
\frac{\partial u}{\partial t} + \langle v_{st}, \nabla \rangle u + \langle u, \nabla \rangle v_{st} - \nu \Delta u = -\frac{1}{\rho} \text{grad} \hat{p} + f,
\]
\[
\text{div} u = 0, \quad u|_{\partial \Omega} = 0, \quad u|_{t=0} = u_0.
\]

We assume that the field of external mass forces \( f \) is time-independent.

Then, one observes that for smooth data, the solution \((u, \hat{p})\) of the corresponding stationary problem satisfies the system of equations
\[
\left( \begin{array}{c}
\langle v_{st}, \nabla \rangle + \langle \cdot, \nabla \rangle v_{st} - \nu \Delta \text{grad} \\
-\text{div}
\end{array} \right) \left( \begin{array}{c}
u \\
0
\end{array} \right) \left( \begin{array}{c} u \\
\hat{p} \rho
\end{array} \right) = \left( \begin{array}{c} f \\
0
\end{array} \right),
\]
which can equivalently be rewritten as
\[
(A.1) \quad \left( \begin{array}{c}
\langle v_{st}, \nabla \rangle + \langle \cdot, \nabla \rangle v_{st} - \nu \Delta \\
-\nu_s \text{div}
\end{array} \right) \left( \begin{array}{c}
u \\
0
\end{array} \right) \left( \begin{array}{c} u \\
\hat{p} \rho
\end{array} \right) = \left( \begin{array}{c} f \\
0
\end{array} \right).
\]

Here we choose the parameter \( v_s \) as a characteristic velocity of the stationary flow. Note that in dimension \( n = 2 \), \( v_s \) given by (3.8) has indeed the dimension of velocity. Introducing a characteristic velocity \( u^* \) of the stationary solution \( u \) of the system (A.1), one observes (on a heuristic level) that the terms \( \langle v_{st}, \nabla \rangle u \) and \( \langle u, \nabla \rangle v_{st} \) are of the order \( v_s u^* \sqrt{\lambda_1(\Omega)} \) each, meanwhile \( \nu \Delta u \) is of the order \( \nu \lambda_1(\Omega) u^* \), and thus
\[
\langle v_{st}, \nabla \rangle u + \langle u, \nabla \rangle v_{st} = \text{Re}_s \mathcal{O}(-\nu \Delta u).
\]

For this reason, the lower order terms \( \langle v_{st}, \nabla \rangle u + \langle u, \nabla \rangle v_{st} \) can be neglected in the limit \( \text{Re}_s \to 0 \) to obtain the system
\[
(A.2) \quad \left( \begin{array}{c}
-\nu \Delta \\
-v_s \text{div}
\end{array} \right) \left( \begin{array}{c}
u_s \text{grad} \\
0
\end{array} \right) \left( \begin{array}{c} v \\
q
\end{array} \right) = \left( \begin{array}{c} f \\
0
\end{array} \right),
\]
where the “renormalized pressure” \( \hat{q} \) is given by
\[
\hat{q} = \frac{\hat{p}}{v_s \rho}.
\]
Finally, it remains to observe that the left-hand side of (A.2) is nothing but the Stokes operator matrix $S = S(\nu, v_*)$ defined by (1.3).

(ii). Clearly, typical physical dimensional variables associated with the steady motion $v_{st}$ of incompressible fluid in a bounded domain are $T$ (time), $V$ (velocity), $\nu$ (viscosity) and $L$ (length). Recall that in the framework of general dimensional analysis (see, e.g., [22, §19], [26]), given the fundamental units which are in our case the ones of length and time, to every monomial power $T^{\alpha} V^{\beta} \nu^{\gamma} L^{\delta}$ of the physical variables, one assigns the vector $(\alpha, \beta, \gamma, \delta)$ in a 4-dimensional space

$$T^{\alpha} V^{\beta} \nu^{\gamma} L^{\delta} \mapsto (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4.$$ 

In this setting, the dimensionless quantities/monomials form a 2-dimensional plane $\mathbb{P}$ in $\mathbb{R}^4$ determined by the equations

$$\mathbb{P} : \begin{cases} 
\alpha - \beta - \gamma = 0 \\
\beta + 2\gamma + \delta = 0.
\end{cases}$$

It is easy to see that the two-dimensional square lattice $\Lambda = \mathbb{Z}^4 \cap \mathbb{P}$ has an orthogonal basis $(r, s)$ associated with the dimension free variables

$$\frac{VL}{\nu} \mapsto r = (0, 1, -1, 1) \quad \text{and} \quad \frac{TV}{L} \mapsto s = (1, 1, 0, -1).$$

That is,

$$\Lambda = \{ ms + nr \mid m, n \in \mathbb{Z} \} = \mathbb{Z}^4 \cap \mathbb{P}.$$ 

The lattice $\Lambda$ has the square sublattice $\Lambda'$ of index 2 (see [4, I.2.2] for the definition of the index),

$$\Lambda' = \{ ms + nr \mid m = n \pmod{2}, m, n \in \mathbb{Z} \} \subset \Lambda.$$ 

In turn, the sublattice $\Lambda'$ has an orthogonal basis $(c, d)$ (of minimal Euclidean length) associated with the new pair of dimension free variables (see Fig. 1)

$$\frac{TV^2}{\nu} \mapsto c = s + r = (1, 2, -1, 0)$$ 

and

$$\frac{TV}{L^2} \mapsto d = s - r = (1, 0, 1, -2).$$

That is,

$$\Lambda' = \{ mc + nd \mid m, n \in \mathbb{Z} \}.$$
Fig. 1. Lattice $\Lambda$ versus sublattice $\Lambda'$ of index 2.

Introduce the characteristic length scale $L \sim k^{-1}$, where the wave number $k$ is given by (2.22) (see Theorem 2.5), the characteristic velocity $v_*$, and finally the (characteristic) time scale $\tau$, which, in the current setting, is at our disposal. Then one observes that the dimensionless quantity $\frac{V L}{\nu}$ transforms into the Reynolds number (cf. [22, §19])

(A.3) \[ \frac{V L}{\nu} \rightarrow Re_* = \frac{v_*}{\nu \sqrt{\lambda_1(\Omega)}} \quad \text{as } V \rightarrow v_* \text{ and } L \rightarrow 1/\sqrt{\lambda_1(\Omega)}. \]

In turn, the dimension-free variable $\frac{TV}{L}$ gives rise to the Strouhal type number (cf. [22, §19])

(A.4) \[ \frac{TV}{L} \rightarrow St_* = \tau \left( v_* \sqrt{\lambda_1(\Omega)} \right) \quad \text{as } T \rightarrow \tau. \]

Note that the factor $v_* \sqrt{\lambda_1(\Omega)}$ in (A.4) has the dimension of a frequency and can be interpreted as the “circulation frequency” associated with the stationary flow $v_{st}$ in the bounded domain $\Omega$.

Upon the identifications above, it is striking to observe that the new set of dimensionless variables $\frac{TV^2}{\nu}$ and $\frac{T \nu}{L^2}$ carries important spectral information on the Stokes operator in the following sense. The product $St_* Re_*$ is proportional to the distance from the bottom of the spectrum of the Stokes operator to the origin and the ratio $St_*/Re_*$ is proportional to the length of spectral gap of the diagonal part of the Stokes operator. That is,

\[ \frac{TV^2}{\nu} \rightarrow St_* Re_* = \tau \frac{v_*^2}{\nu} = \tau |\inf \text{spec}(S)| \]

and

\[ \frac{T \nu}{L^2} \rightarrow St_*/Re_* = \tau \nu \lambda_1(\Omega). \]
Moreover, from (2.23) one also derives that

\[
\lim_{\text{Re}^* \to 0^+} \inf \sigma(S) = \lim_{\text{Re}^* \to 0^+} \frac{\inf \sigma(S)}{\text{Re}^*} = -\frac{1}{\tau},
\]

and that

\[
\lim_{\text{Re}^* \to 0^+} \frac{\lambda_1(S)}{\text{St}^*} = \lim_{\text{Re}^* \to 0^+} \frac{\lambda_1(S)}{\text{Re}^*} = \frac{1}{\tau}.
\]

We also notice that the upper bound \(\theta\) for the norm \(\|\Theta\|\) of the operator angle (2.24) in Theorem 2.5 can be read off from the following diagram.

![Fig. 2. Strouhal-Reynolds-Rotation angle diagram.](image)

Here, the Strouhal number \(\text{St}^*\), the height of the right triangle on the diagram, coincides with the geometric mean of the dimensionless quantities \(2\text{St}^*\text{Re}^*\) and \(\frac{1}{2}\frac{\text{St}^*}{\text{Re}^*}\). Moreover,

\[
\tan 2\|\Theta\| \leq \frac{\text{St}^*}{\frac{1}{2}\text{St}^*/\text{Re}^*} = \frac{2v^*}{\nu \sqrt{\lambda_1(\Omega)}} = 2\text{Re}^* = \text{Re}^*.
\]

**APPENDIX B. SADDLE-POINT FORMS AND NUMERICAL RANGES**

In this appendix we recall the concept of a saddle-point form with respect to a decomposition \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\).

Assume that \(A_\pm \geq 0\) are non-negative self-adjoint operators acting in \(\mathcal{H}_\pm\).

On \(\text{Dom}[a] = \text{Dom}(A^{1/2}_+) \oplus \text{Dom}(A^{1/2}_-) \subseteq \mathcal{H}\) introduce the diagonal saddle-point sesquilinear form

\[
a[x, y] = a_+[x_+, y_+] - a_- [x_-, y_-],
\]

where \(x_\pm, y_\pm \in \text{Dom}[a_\pm] = \text{Dom}(A^{1/2}_\pm)\) and

\[
a_\pm [x_\pm, y_\pm] = (A^{1/2}_\pm x_\pm, A^{1/2}_\pm y_\pm), \quad x_\pm, y_\pm \in \text{Dom}[a_\pm] = \text{Dom}(A^{1/2}_\pm),
\]

are the non-negative closed forms associated with the self-adjoint operators \(A_+\) and \(A_-\), respectively.
We say that a form \( b \) is a saddle-point form with respect to the decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) if it admits the representation
\[
b[x, y] = a[x, y] + v[x, y], \quad x, y \in \text{Dom}[b] = \text{Dom}[a],
\]
where \( v \) is a symmetric off-diagonal form with respect to the decomposition, that is
\[
v[x, Jy] = -v[Jx, y],
\]
with \( J = I_{\mathcal{H}_+} \oplus (-I_{\mathcal{H}_-}) \).

We also require that the off-diagonal form \( v \) is a form bounded perturbation of the diagonal form \( a \) in the sense that
\[
|v[x]| \leq \beta(\langle |A|^{1/2}x, |A|^{1/2}x \rangle + \|x\|^2), \quad x \in \text{Dom}[v],
\]
for some \( \beta \geq 0 \).

We start with citing the First Representation Theorem proven in [25, Theorem 2.7], [13], [14] in a more general setting and adapted here to the case of saddle-point forms.

**Theorem B.1.** Let \( b \) be a saddle-point form with respect to the decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \).

Then there exists a unique self-adjoint operator \( B \) such that
\[
\text{Dom}(B) \subseteq \text{Dom}[b]
\]
and
\[
b[x, y] = \langle x, By \rangle \quad \text{for all} \quad x \in \text{Dom}[b] \quad \text{and} \quad y \in \text{Dom}(B).
\]

We say that the operator \( B \) associated with the saddle-point form \( b \) via Theorem B.1 satisfies the domain stability condition if
\[
\text{Dom}[b] = \text{Dom}(|A|^{1/2}) = \text{Dom}(|B|^{1/2}).
\]

For completeness sake, we cite the corresponding Second Representation Theorem (see [25, Theorem 3.1], [13], [14]).

**Theorem B.2.** Let \( b \) be a saddle-point form with respect to the decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) and \( B \) the associated operator referred to in Theorem B.1.

If the domain stability condition (B.1) holds, then the operator \( B \) represents this form in the sense that
\[
b[x, y] = \langle |B|^{1/2}x, \text{sign}(B)|B|^{1/2}y \rangle \quad \text{for all} \quad x, y \in \text{Dom}[b] = \text{Dom}(|B|^{1/2}).
\]

Recall that the numerical range of an operator \( B \) is denoted as
\[
W(B) := \{ \langle x, Bx \rangle \mid x \in \text{Dom}(B), \|x\| = 1 \}.
\]
Accordingly, we define the numerical range of a saddle-point form \( b \) as
\[
W[b] := \{ b[x] \mid x \in \text{Dom}[b], \|x\| = 1 \}.
\]
Next, we generalize the concept of the quadratic numerical range for operator matrices presented in [28] to the case of saddle-point forms.

Given a saddle-point form \( b \) with respect to the decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \), we define its quadratic numerical range

\[
W^2[b] := \bigcup_{x_+ \oplus x_- \in \text{Dom}(a_+ \oplus a_-), \|x_+\| \|x_-\| = 1} \text{spec} \left( \frac{a_+[x_+]}{v[x_+, x_-]} - \frac{a_-[x_-]}{v[x_+, x_-]} \right).
\]

Here we use the standard shorthand notation \( a_\pm[x_\pm] = a_\pm[x_\pm, x_\pm] \).

**Lemma B.3** (cf. [28, 29]). Let \( b \) be a saddle-point form with respect to the decomposition \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) associated with the self-adjoint operator \( B \) and let \( a = a_+ \oplus (-a_-) \) be the diagonal part of \( b \).

Then

(i) \( \text{spec}(B) \subseteq W^2[b] \);  
(ii) \( W(B) \subseteq W[b] \subseteq W(B) \subseteq (\inf \text{spec}(B), \sup \text{spec}(B)) \);  
(iii) \( W^2[b] \subseteq W[b] \);  
(iv) \( \inf \text{spec}(B) = \inf W^2[b] \), \( \sup \text{spec}(B) = \sup W^2[b] \);  
(v) \( W[a_\pm] \subseteq W^2[b] \) if \( \dim \mathcal{H}_\pm > 1 \);  
(vi) if \( a_\pm \geq \alpha_\pm I \) for some \( \alpha_\pm \geq 0 \), then

\[
\text{spec}(B) \subseteq (-\infty, -\alpha_-] \cup [\alpha_+, \infty).
\]

**Proof.** (i). First, let \( \lambda \in \mathbb{R} \) be an eigenvalue of \( B \) with corresponding eigenfunction \( u \in \text{Dom}(B) \). Since

\[
\text{Dom}(B) \subseteq \text{Dom}(|A|^{1/2}) = \text{Dom}(A_+^{1/2}) \oplus \text{Dom}(A_-^{1/2}),
\]

we have the unique decomposition \( u = u_+ \oplus u_- \) with \( u_\pm \in \text{Dom}(A_\pm^{1/2}) \). We set \( \hat{u}_\pm := \|u_\pm\|^{-1} u_\pm \) if \( u_\pm \neq 0 \) and choose \( \hat{u}_\pm \in \text{Dom}(A_\pm^{1/2}) \) arbitrary with \( \|\hat{u}_\pm\| = 1 \) if \( u_\pm = 0 \). From the eigenvalue equation, we obtain that

\[
\langle \hat{u}_+, Bu \rangle = \lambda \langle \hat{u}_+, u_+ \rangle, \quad \langle \hat{u}_-, Bu \rangle = \lambda \langle \hat{u}_-, u_- \rangle.
\]

By the First Representation Theorem for saddle-point forms [25, Theorem 2.7] (see also [13]) we can rewrite these equations in a 2 \( \times \) 2 matrix form

\[
\begin{pmatrix}
\frac{a_+[\hat{u}_+]}{v[\hat{u}_+, \hat{u}_-]} & \frac{v[\hat{u}_+, \hat{u}_-]}{-a_-[\hat{u}_-]}
\end{pmatrix}
\begin{pmatrix}
||u_+|| \\
||u_-||
\end{pmatrix}
= \lambda
\begin{pmatrix}
||u_+|| \\
||u_-||
\end{pmatrix}.
\]

As a consequence, \( \lambda \in W^2[b] \).

If \( \lambda \in \sigma(B) \) is not an eigenvalue, the Weyl criterion [24, Theorem VII.12] implies that there exists a sequence \((u^{(n)})_{n \in \mathbb{N}} \subseteq \text{Dom}(B) \) with \( ||u^{(n)}|| = 1 \) and \((B - \lambda)u^{(n)} \to 0, \ n \to \infty \).
In the same way as above, we write \( u^{(n)} = u_+^{(n)} \oplus u_-^{(n)} \in \text{Dom}(|A|^{1/2}) \) and introduce \( u_{\pm}^{(n)} \) for the normalized components. Then, we have that
\[
\langle (B - \lambda)u^{(n)}, \hat{u}_+^{(n)} \oplus 0 \rangle =: v_+^{(n)}, \quad \langle (B - \lambda)u^{(n)}, 0 \oplus \hat{u}_-^{(n)} \rangle =: v_-^{(n)},
\]
both converge to zero. By the First Representation Theorem again, these equations can be rewritten as
\[
(B.2) \quad \begin{pmatrix} (a_+ - \lambda)\hat{u}_+^{(n)} \\ b[\hat{u}_+^{(n)}, \hat{u}_-^{(n)}] \\ - (a_- + \lambda)\hat{u}_-^{(n)} \end{pmatrix} \begin{pmatrix} \|u_+^{(n)}\| \\ \|u_-^{(n)}\| \end{pmatrix} = \begin{pmatrix} v_+^{(n)} \\ v_-^{(n)} \end{pmatrix}.
\]
Let \( B_n - \lambda \) denote the matrix in (B.2). Then
\[
1 = \sqrt{\|u_+^{(n)}\|^2 + \|u_-^{(n)}\|^2} \\
\leq \|\lambda(B_n - \lambda)^{-1}\| \cdot \sqrt{(v_+^{(n)})^2 + (v_-^{(n)})^2} = \sqrt{\frac{(v_+^{(n)})^2 + (v_-^{(n)})^2}{\text{dist}(\lambda, \text{spec}(B_n))}}.
\]
Hence
\[
\text{dist}(\lambda, \text{spec}(B_n)) \leq \sqrt{\frac{(v_+^{(n)})^2 + (v_-^{(n)})^2}{n \to \infty}}
\]
and consequently \( \lambda \in W^2[b] \).

(ii). The first inclusion \( W(B) \subseteq W[b] \) follows directly from the First Representation Theorem for saddle-point forms (see [13, 25]) noting that \( \text{Dom}(B) \subseteq \text{Dom}[b] \).

For the second inclusion, \( W[b] \subseteq W(B) \), one can use [16, Lemma VI.3.1] on the form \( b \) to get that
\[
b[x, x] = \langle (|A| + I)^{1/2}x, (J + R)(|A| + I)^{1/2}x \rangle - \langle x, Jx \rangle
\]
holds for \( x \in \text{Dom}[b] = \text{Dom}((|A| + I)^{1/2}) \). Since
\[
B = (|A| + I)^{1/2}(J + R)(|A| + I)^{1/2} - J,
\]
it follows from [13, Theorem 2.3] that \( \text{Dom}(B) \) is a core for the operator \( (|A| + I)^{1/2} \). The claim then is a consequence of the core property.

The last inclusion, \( W(B) \subseteq \left( \inf \text{spec}(B), \sup \text{spec}(B) \right) \), follows directly from the well known convexity of the numerical range and statement [30, Aufgabe VII.5.24(c)] on the extremal points.

(iii). Let \( \lambda \in W^2[b] \). Then, there exist \( x_+ \in \text{Dom}(A_+^{1/2}) \) with \( \|x_+\| = 1 \) and \( c = (c_1, c_2) \in \mathbb{R}^2 \) with \( \|c\| = 1 \) such that
\[
\begin{pmatrix} a_+[x_+] \\ b[x_+, x_-] \\ -a_-[x_-] \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},
\]
Taking the scalar product with \( c \) yields
\[
b[c_1 x_+ \oplus c_2 x_-] = \lambda.
\]
Since \( \|c_1 x_+ \oplus c_2 x_-\| = 1 \), the claim follows.
(iv). Note that by parts (ii) and (iii), we have
\[ W^2[b] \subseteq \left( \inf \text{spec}(B), \sup \text{spec}(B) \right). \]
The claim now follows from part (i) since \( \inf \text{spec}(B), \sup \text{spec}(B) \in W^2[b] \).

(v). Assume that \( \dim \mathcal{H}_- > 1 \). Then, for each \( x_+ \in \text{Dom}[a_+] \), \( \|x_+\| = 1 \), there is an element \( x_- \in \text{Dom}[a_-] \), \( \|x_-\| = 1 \) with \( v[x_+, x_-] = 0 \). To see this, note that by [16, Lemma VI.3.1]
\[ v[x_+, x_-] = \langle R^*(A_+ + I)^{1/2}x_+, (A_- + I)^{1/2}x_- \rangle. \]
Let \( f \in \mathcal{H}_+ \). Then, by \( \dim \mathcal{H}_- > 1 \), there exists an element \( g \in \mathcal{H}_- \) such that \( \langle R^*f, g \rangle_{\mathcal{H}_-} = 0 \).

By the bijectivity of \( (|A| + I)^{1/2} : \text{Dom}((|A| + I)^{1/2}) \rightarrow \mathcal{H}_- \), there exists a suitable \( x_- \) with \( v[x_+, x_-] = 0 \). In this case, we have that \( a_+[x_+] \in \text{spec} \begin{pmatrix} a_+[x_+] & 0 \\ 0 & a_-[x_-] \end{pmatrix} \subseteq W^2[b] \).

(vi). The claim follows directly, noting that the spectrum of the \( 2 \times 2 \) matrix
\[ \begin{pmatrix} a_+ & \overline{v} \\ v & -a_- \end{pmatrix}, \quad 0 \leq a_+ < \infty, \quad v \in \mathbb{C}, \]
is located outside of the interval \( (-a_-, a_+) \). As a consequence, one has that \( W^2[b] \cap (-a_-, a_+) = \emptyset \) and the claim follows.

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