ON A SUBSPACE PERTURBATION PROBLEM

VADIM KOSTRYKIN, Konstantin A. MAKAROV, AND Alexander K. MotoVILOV

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Abstract. We discuss the problem of perturbation of spectral subspaces for linear self-adjoint operators on a separable Hilbert space. Let $A$ and $V$ be bounded self-adjoint operators. Assume that the spectrum of $A$ consists of two disjoint parts $\sigma$ and $\Sigma$ such that $d = \text{dist}(\sigma, \Sigma) > 0$. We show that the norm of the difference of the spectral projections $E_A(\sigma)$ and $E_{A+V}(\{\lambda \mid \text{dist}(\lambda, \sigma) < d/2\})$ for $A$ and $A + V$ is less than one whenever either (i) $\|V\| < \frac{d^2}{2 + d}$ or (ii) $\|V\| < \frac{1}{2}d$ and certain assumptions on the mutual disposition of the sets $\sigma$ and $\Sigma$ are satisfied.

1. Introduction

It is well known (see, e.g., [10]) that if $A$ and $V$ are bounded self-adjoint operators on a separable Hilbert space $H$, then (the perturbation) $V$ does not close gaps of length greater than $2\|V\|$ in the spectrum of $A$. More precisely, if $(a, b)$ is a finite interval and $(a, b) \subset \rho(A)$, the resolvent set of $A$, then

$$(a + \|V\|, b - \|V\|) \subset \rho(A + sV) \quad \text{for all } s \in [-1, 1]$$

whenever $2\|V\| < b - a$. Hence, under the assumption that $A$ has an isolated part $\sigma$ of the spectrum separated from its remainder by gaps of length greater than or equal to $d > 0$, the spectrum of the operators $A + sV$, $s \in [-1, 1]$, will also have separated components, provided that the condition

$$\|V\| < \frac{d}{2} \quad (1.1)$$

holds.

Our main concern is to study the variation of the corresponding spectral subspace associated with the isolated part $\sigma$ of the spectrum of $A$ under perturbations satisfying (1.1).

For notational setup we assume the following hypothesis.

Hypothesis 1. Assume that $A$ and $V$ are bounded self-adjoint operators on a separable Hilbert space $H$. Suppose that the spectrum of $A$ has a part $\sigma$ separated from the remainder of the spectrum $\Sigma$ in the sense that

$$\text{spec}(A) = \sigma \cup \Sigma \quad \text{and} \quad \text{dist}(\sigma, \Sigma) = d > 0.$$
Introduce the orthogonal projections $P = E_A(\sigma)$ and $Q = E_{A+V}(U_{d/2}(\sigma))$, where $U_\varepsilon(\sigma)$, $\varepsilon > 0$, is the open $\varepsilon$-neighborhood of the set $\sigma$. Here $E_A(\Delta)$ and $E_{A+V}(\Delta)$ denote the spectral projections for operators $A$ and $A+V$, respectively, corresponding to a Borel set $\Delta \subset \mathbb{R}$.

In this note we address the following question: Assuming Hypothesis 1, does condition (1.1) imply $\|P - Q\| < 1$?

We give a partially affirmative answer to this question. The precise statement reads as follows.

**Theorem 1.** Assume Hypothesis 1 and suppose that either

(i) $\|V\| < \frac{2}{2 + \pi} d$

or

(ii) $\|V\| < \frac{1}{2} d$

and

(1.2) \quad \text{conv} \text{. hull}(\sigma) \cap \Sigma = \emptyset \quad \text{or} \quad \text{conv} \text{. hull}(\Sigma) \cap \sigma = \emptyset.$

Then

$\|P - Q\| < 1.$

Our strategy of the proof of Theorem 1 does not allow us to relax the condition

(1.3) \quad \|V\| < \frac{2}{2 + \pi} d$

and just assume the natural condition (1.1) with no additional hypotheses. It is an open problem whether Hypothesis 1 alone and the bounds

(1.4) \quad \frac{2}{2 + \pi} \leq \frac{\|V\|}{d} < \frac{1}{2}$

on the perturbation $V$ imply $\|P - Q\| < 1$.

For compact perturbations $V$ satisfying inequality (1.1), we can however state that the pair $(P, Q)$ of the orthogonal projections is a Fredholm pair with zero index. Recall that the pair $(P, Q)$ of orthogonal projections is called Fredholm if the operator $QP$ viewed as a map from $\text{Ran} P$ to $\text{Ran} Q$ is a Fredholm operator [3]. The index of this operator is called the index of the pair $(P, Q)$.

**Theorem 2.** Assume Hypothesis 1 and suppose that $V$ is a compact operator satisfying (1.1). Then the pair $(P, Q)$ is Fredholm with zero index. In particular, the subspaces $\text{Ker}(PQ^\perp - I)$ and $\text{Ker}(P^\perp Q - I)$ are finite-dimensional and

\[ \dim \text{Ker}(PQ^\perp - I) = \dim \text{Ker}(P^\perp Q - I). \]

In the “overcritical” case $\|V\| > d/2$, the perturbed operator $A + V$ may not have separated parts of the spectrum at all. In this case we give an example where the spectral measure of the perturbed operator $A + V$ is “concentrated” on the unit sphere in the space of bounded operators $B(\mathcal{H})$ centered at the point $P = E_A(\sigma)$, with the norm of the perturbation being arbitrarily close to $d/2$. That is, given $d > 0$, for any $\varepsilon > 0$ one can find a self-adjoint operator $A$ satisfying Hypothesis 1 and a self-adjoint perturbation $V$ with $\|V\| = d/2 + \varepsilon$ such that

\[ \|E_A(\sigma) - E_{A+V}(\Delta)\| = 1 \]

for any Borel set $\Delta \subset \mathbb{R}$.
2. Proof of Theorem 1

Our proof of Theorem 1 is based on the following sharp result (see [9] and references cited therein) taken from geometric perturbation theory initiated by C. Davis [6] and developed further in [4], [5], [7], [8], [10].

**Proposition 2.1.** Let $A$ and $B$ be bounded self-adjoint operators and $\delta$ and $\Delta$ two Borel sets on the real axis $\mathbb{R}$. Then

$$\text{dist}(\delta, \Delta)\|E_A(\delta)E_B(\Delta)\| \leq \frac{\pi}{2}\|A - B\|.$$  

If, in addition, the convex hull of the set $\delta$ does not intersect the set $\Delta$, or the convex hull of the set $\Delta$ does not intersect the set $\delta$, then one has the stronger result

$$\text{dist}(\delta, \Delta)\|E_A(\delta)E_B(\Delta)\| \leq \|A - B\|.$$  

We split the proof of Theorem 1 into the following two lemmas.

**Lemma 2.2.** Assume Hypothesis 1. Assume, in addition, that (1.3) holds. Then

$$\|P - Q\| < 1.$$  

**Proof.** Clearly $\text{spec}(A + V) \subset U_{\|V\|}(\sigma \cup \Sigma)$, where the bar denotes the (usual) closure in $\mathbb{R}$, and then

$$Q^\perp = E_{A + V}(U_{\|V\|}(\Sigma)).$$  

By the first claim of Proposition 2.1

$$\|PQ^\perp\| \leq \frac{\pi}{2}\|V\| \text{dist}(\sigma, U_{\|V\|}(\Sigma)).$$  

The distance between the set $\sigma$ and the $\|V\|$-neighborhood of the set $\Sigma$ can be estimated from below as follows:

$$\text{dist}(\sigma, U_{\|V\|}(\Sigma)) \geq d - \|V\| > 0.$$  

Then (2.1) implies the inequality

$$\|PQ^\perp\| \leq \frac{\pi}{2\, d - \|V\|}\|V\|.$$  

Hence, from inequality (1.3) it follows that

$$\|PQ^\perp\| \leq \frac{\pi}{2\, d - \|V\|}\|V\| < 1.$$  

Interchanging the roles of $\sigma$ and $\Sigma$ one obtains the analogous inequality

$$\|P^\perp Q\| < 1.$$  

Since

$$\|P - Q\| = \max\{\|PQ^\perp\|, \|P^\perp Q\|\}$$  

(see, e.g., [2, Ch. III, Section 39]), inequalities (2.2) and (2.3) prove the assertion. \qed

Under additional assumptions on mutual disposition of the parts $\sigma$ and $\Sigma$ of the spectrum of $A$ one can relax the condition (1.3) on the norm of perturbation and replace it by the natural condition (1.1).
Lemma 2.3. Assume Hypothesis 1 and suppose that condition (1.1) holds.

(i) If either $\sigma \cap \text{conv.hull}(\Sigma) = \emptyset$ or $\text{conv.hull}(\sigma) \cap \Sigma = \emptyset$, then

$$
\|P - Q\| < 1.
$$

(ii) If in addition the sets $\sigma$ and $\Sigma$ are subordinated, that is,

$$
\text{conv.hull}(\sigma) \cap \text{conv.hull}(\Sigma) = \emptyset,
$$

then the following sharp estimate holds:

$$
\|P - Q\| < \frac{\sqrt{2}}{2}.
$$

Proof. (i) The proof follows that of Lemma 2.2. Applying the second assertion of Proposition 2.1 instead of inequality (2.1), one derives the estimates

$$
\|PQ\| \leq \frac{\|V\|}{\text{dist}(\sigma, U)} \leq \frac{\|V\|}{d - \|V\|} < 1,
$$

under hypothesis (1.4), and then the inequality $\|P^*Q\| < 1$, proving assertion (2.5) using (2.4).

(ii) First assume that $V$ is o-diagonal, that is,

$$
E_A(\sigma)V E_A(\sigma) = E_A(\sigma)^\perp V E_A(\sigma)^\perp = 0.
$$

Then the inequality $\|P - Q\| < \frac{\sqrt{2}}{2}$ follows from the tan $2\Theta$-Theorem proven first by C. Davis (see, e.g., [8])

$$
\|P - Q\| \leq \sin \left(\frac{1}{2} \arctan \frac{2\|V\|}{d}\right) < \frac{\sqrt{2}}{2}.
$$

A related result can be found in [1].

The general case can be reduced to the o-diagonal one by the following trick. Assume that $V$ is not necessarily o-diagonal. Decomposing the perturbation $V$ into the diagonal $V_{\text{diag}}$ and o-diagonal $V_{\text{off}}$ parts with respect to the orthogonal decomposition $H = \text{Ran} E_A(\sigma) \oplus \text{Ran} E_A(\sigma)^\perp$ associated with the range of the projection $E_A(\sigma)$

$$
V = V_{\text{diag}} + V_{\text{off}},
$$

one concludes that

$$
E_{A + V_{\text{diag}}}(U_{d/2}(\sigma)) = E_A(\sigma).
$$

Moreover, the distance between the spectrum of the part of $A + V_{\text{diag}}$ associated with the invariant subspace $\text{Ran} E_{A + V_{\text{diag}}}(U_{d/2}(\sigma))$ and the remainder of the spectrum of $A + V_{\text{diag}}$ does not exceed $d - 2\|V_{\text{diag}}\| > 0$. Using the tan $2\Theta$-Theorem then yields

$$
\|P - Q\| \leq \sin \left(\frac{1}{2} \arctan \frac{2\|V_{\text{off}}\|}{d - 2\|V_{\text{diag}}\|}\right) \leq \sin \left(\frac{1}{2} \arctan \frac{2\|V\|}{d - 2\|V\|}\right) < \frac{\sqrt{2}}{2},
$$

completing the proof.
The sharpness of estimate (2.6) is shown by the following example.

**Example 2.4.** Let \( \mathcal{H} = \mathbb{C}^2 \). For an arbitrary \( \varepsilon \in (0, 3/4) \) consider the \( 2 \times 2 \) matrices
\[
A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1/2 - \varepsilon & \sqrt{\varepsilon}/2 \\ \sqrt{\varepsilon}/2 & -1/2 + \varepsilon \end{pmatrix}.
\]
Let \( \sigma = \{0\} \) and \( \Sigma = \{1\} \). Obviously, \( \text{dist}(\sigma, \Sigma) = 1 \). Since
\[
\|V\| = \frac{1}{2} \sqrt{1 - 3\varepsilon + 4\varepsilon^2} < \frac{1}{2},
\]
the perturbation \( V \) satisfies the hypotheses of Lemma 2.3. Simple calculations yield
\[
Q = E_{A+V}(U_{1/2}(\sigma)) = E_{A+V}(-1/2, 1/2)) = \frac{1}{1 + (2\sqrt{\varepsilon} + \sqrt{1+4\varepsilon})^2} \begin{pmatrix} (2\sqrt{\varepsilon} + \sqrt{1+4\varepsilon})^2 & -2\sqrt{\varepsilon} - \sqrt{1+4\varepsilon} \\ -2\sqrt{\varepsilon} - \sqrt{1+4\varepsilon} & 1 \end{pmatrix},
\]
and hence,
\[
\|P - Q\| = \left[1 + (2\sqrt{\varepsilon} + \sqrt{1+4\varepsilon})^2\right]^{-1/2} < \frac{\sqrt{2}}{2}.
\]
Taking \( \varepsilon \) sufficiently small, the norm \( \|P - Q\| \) can be made arbitrarily close to \( \sqrt{2}/2 \).

3. Proof of Theorem 2

**Lemma 3.1.** Assume Hypothesis \( \mathcal{H} \) and suppose, in addition, that \( V \) is a compact operator satisfying condition (1.1). Then there is a unitary \( W \) such that \( Q = WPW^* \) and \( W - I \) is compact.

**Proof.** Fix \( \varepsilon > 0 \) such that \((1 + \varepsilon)\|V\| < d/2\) and introduce the family of spectral projections
\[
\mathcal{P}(s) = E_{A+sV}(U_{d/2}(\sigma)), \quad s \in (-\varepsilon, 1 + \varepsilon).
\]
Clearly, \( \mathcal{P}(0) = P \) and \( \mathcal{P}(1) = Q \). From the analytical perturbation theory (see [10]) one concludes that the operator-valued function \( \mathcal{P}(s) \) is real-analytic on \((-\varepsilon, 1 + \varepsilon)\). Moreover (see [10], Section II.4.2),
\[
\mathcal{P}(s) = X(s)\mathcal{P}(0)X(s)^*, \quad s \in [0, 1],
\]
where \( X(s) \) is the unique unitary solution to the initial value problem
\[
X'(s) = H(s)X(s), \quad s \in [0, 1],
\]
\[
X(0) = I,
\]
with \( H(s) = \mathcal{P}'(s)\mathcal{P}(s) - \mathcal{P}(s)\mathcal{P}'(s) \).

Let \( \Gamma \) be a Jordan counterclockwise oriented contour encircling \( U_{d/2}(\sigma) \) in a way such that no point of \( U_{d/2}(\Sigma) \) lies within \( \Gamma \). Then
\[
\mathcal{P}(s) = -\frac{1}{2\pi i} \int_{\Gamma} (A + sV - z)^{-1}dz, \quad s \in [0, 1],
\]
and hence,
\[
\mathcal{P}'(s) = \frac{1}{2\pi i} \int_{\Gamma} (A + sV - z)^{-1}V(A + sV - z)^{-1}dz, \quad s \in [0, 1].
\]
By the hypothesis \( V \) is compact, and hence, \( \mathcal{P}'(s), s \in [0, 1] \), is also compact, which implies that \( H(s) \) is a compact operator for \( s \in [0, 1] \).
Applying the successive approximation method
\[ X_n(s) = I + \int_0^s H(t)X_{n-1}(t)dt, \quad X_0(s) = I, \]
yields that \( X_n(s) \) converges to \( X(s) \), \( s \in [0, 1] \), in the norm topology and \( X_n(s) - I \) is compact for all \( n \in \mathbb{N} \). Thus, \( X(s) - I \) is a compact operator for all \( s \in [0, 1] \). Taking \( W = X(1) \) yields \( Q = WPW^* \), completing the proof.

Lemma 3.1 implies that the operator \( PWP \) viewed as a map from \( \text{Ran} \, P \) to \( \text{Ran} \, P \) is Fredholm with zero index. By Theorem 5.2 of [3] it follows that the pair \( (P, Q) \) is Fredholm and \( \text{index}(P, Q) = \text{index}(PW|_{\text{Ran} \, P}) = 0 \), proving Theorem 2.

4. Overcritical Perturbations

If the perturbation \( V \) closes a gap between the separated parts \( \sigma \) and \( \Sigma \) of the spectrum of the unperturbed operator \( A \), then, necessarily, we are dealing with the case \( \|V\| \geq d/2 \). In this case one encounters a new phenomenon: It may happen that any invariant subspace of the operator \( A + V \) contains a nontrivial element orthogonal to \( \text{Ran} \, P = \text{Ran} \, E_A(\sigma) \).

To illustrate this phenomenon we need the following abstract result.

Lemma 4.1. Let \( A \) and \( V \) be bounded self-adjoint operators and \( \sigma \neq \emptyset \) be a finite set consisting of isolated eigenvalues of \( A \) of finite multiplicity. Assume that the spectrum of the operator \( A + V \) has no pure point component. Then for the orthogonal projection \( Q \) onto an arbitrary invariant subspace of the operator \( A + V \), the subspace \( \text{Ker}(P^+Q - I) \), where \( P = E_A(\sigma) \), is infinite-dimensional. In particular,
\[ \|P - Q\| = 1. \]

Proof. Since \( A + V \) has no eigenvalues, \( \text{Ran} \, Q \) is an infinite-dimensional subspace. By hypothesis, \( \text{Ran} \, P \) is a finite-dimensional subspace. Thus, there exists an orthonormal system \( \{f_n\}_{n \in \mathbb{N}} \) in \( \text{Ran} \, Q \) such that \( f_n \) is orthogonal to \( \text{Ran} \, P \) for any \( n \in \mathbb{N} \) and hence \( P^+Qf_n = f_n \), \( n \in \mathbb{N} \), proving \( \dim(\text{Ker}(P^+Q - I)) = \infty \). Now equality (4.1) follows from representation (2.4).

The next lemma shows that an isolated eigenvalue of the unperturbed operator \( A \) separated from the remainder of the spectrum of \( A \) by a gap of length 1 may “dissolve” in the essential spectrum of the perturbed operator \( A + V \) turning into a “resonance”, with the norm of the perturbation being larger but arbitrarily close to 1/2.

Lemma 4.2. Let \( \varepsilon > 0 \). Let \( A \) and \( V \) be \( 2 \times 2 \) operator matrices in \( \mathcal{H} = L^2(0, 1) \oplus \mathbb{C} \),
\[ A = \begin{pmatrix} M & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} -\left( \frac{1}{2} + \varepsilon \right) I_{L^2(0, 1)} & \sqrt{\varepsilon}v \\ \sqrt{\varepsilon}v^* & \left( \frac{1}{2} + \varepsilon \right) I_{\mathbb{C}} \end{pmatrix} \]
with respect to the decomposition \( \mathcal{H} = L^2(0, 1) \oplus \mathbb{C} \). Here \( M \) denotes the multiplication operator in \( L^2(0, 1) \),
\[ (Mf)(\mu) = \mu f(\mu), \quad 0 < \mu < 1, \quad f \in L^2(0, 1), \]
and \( v \in \mathcal{B}(\mathbb{C}, L^2(0, 1)) \),
\[ (vg)(\mu) = w(\mu)g, \quad \mu \in (0, 1), \quad g \in \mathbb{C}, \quad w(\mu) = \sqrt{\mu(1 - \mu)}. \]
If \( \varepsilon < 2/5 \), then the operator \( A + V \) has no eigenvalues.
Proof. Assume to the contrary that $\lambda \in \mathbb{R}$ is an eigenvalue of the perturbed operator $A + V$, that is,

$$(\mu - 1/2 - \varepsilon)f(\mu) + \sqrt{\varepsilon}w(\mu)g = \lambda f(\mu) \quad \text{a.e. } \mu \in (0, 1)$$

and

$$\sqrt{\varepsilon} \int_0^1 d\mu f(\mu)w(\mu) + (-1/2 + \varepsilon)g = \lambda g$$

for some $f \in L^2(0, 1)$ and $g \in \mathbb{C}$. In particular,

$$f(\mu) = \frac{\sqrt{\varepsilon}w(\mu)}{\lambda - (\mu - 1/2 - \varepsilon)}g,$$

and hence $f \notin L^2(0, 1)$ whenever $\lambda \in [-1/2 - \varepsilon, 1/2 - \varepsilon]$ (unless $f = 0$ and $g = 0$). Thus, the interval $[-1/2 - \varepsilon, 1/2 - \varepsilon]$ does not intersect the point spectrum of $A + V$. Moreover, $\lambda \in (-\infty, -1/2 - \varepsilon) \cup (1/2 - \varepsilon, \infty)$ is an eigenvalue of $A + V$ if and only if

$$\lambda + \frac{1}{2} - \varepsilon + \varepsilon \int_0^1 d\mu \frac{\mu(1 - \mu)}{\mu - 1/2 - \varepsilon - \lambda} = 0. \tag{4.2}$$

Elementary analysis of the graph of the function on the left-hand side of (4.2) then yields that under the condition $0 < \varepsilon < 2/5$ there is no solution of equation (4.2) in $(-\infty, -1/2 - \varepsilon) \cup (1/2 - \varepsilon, \infty)$. Thus, the point spectrum of $A + V$ is empty. \hfill \square

Remark 4.3. We note that $\text{spec}(A) = \{-1\} \cup [0, 1]$ and hence $\text{spec}(A)$ has two components separated by a gap of length one, and the norm of the perturbation $V$ may be arbitrarily close to $1/2$ (from above):

$$\|V\| = \sqrt{\left(\frac{1}{2} + \varepsilon\right)^2 + \frac{1}{6} \varepsilon} = \frac{1}{2} + \frac{7}{6} \varepsilon + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0. \tag{4.3}$$

Using scaling arguments, Remark 4.3 combined with the result of Lemma 4.1 shows that given $d > 0$, for any $\varepsilon > 0$ one can find a self-adjoint operator $A$ satisfying Hypothesis 4.1 and a self-adjoint perturbation $V$ with $\|V\| = d/2 + \varepsilon$ such that

$$\|E_A(\sigma) - Q\| = 1$$

for the orthogonal projection $Q$ onto an arbitrary invariant subspace of the operator $A + V$.

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References


Fraunhofer-Institut für Lasertechnik, Steinbachstrasse 15, D-52074, Aachen, Germany

E-mail address: kostrykin@ilt.fhg.de

E-mail address: kostrykin@t-online.de

Department of Mathematics, University of Missouri, Columbia, Missouri 65211

E-mail address: makarov@math.missouri.edu

Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russia

E-mail address: motovilov@thsun1.jinr.ru

Current address: Department of Mathematics, University of Missouri, Columbia, Missouri 65211