ON THE WEYL-TITCHMARSH AND LIVŠIC FUNCTIONS

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Dedicated with great pleasure to our friend and colleague Fritz Gesztesy on the occasion of his 60th birthday anniversary

ABSTRACT. We establish a mutual relationship between main analytic objects for the dissipative extension theory of a symmetric operator $\hat{A}$ with deficiency indices $(1, 1)$. In particular, we introduce the Weyl-Titchmarsh function $M$ of a maximal dissipative extension $\hat{A}$ of the symmetric operator $\hat{A}$. Given a triple $(\hat{A}, A, \hat{A})$, with a reference self-adjoint extension $A$ of $\hat{A}$, we introduce a von Neumann parameter $\kappa$, $|\kappa| < 1$, characterizing the domain of the dissipative extension $\hat{A}$ against $\text{Dom}(A)$. We show that the pair $(\kappa, M)$ is a complete unitary invariant of the triple $(\hat{A}, A, \hat{A})$ unless $\kappa = 0$. As a by-product of our considerations we obtain a relevant functional model of a quasi-self-adjoint dissipative operator and get an analog of the formula of Krein for its resolvent.

1. INTRODUCTION

In 1946 M. Livšic obtained the following remarkable result [20, a part of Theorem 13] (for a textbook exposition see [3]).

**Theorem 1.1** ([20]). Suppose that $\hat{A}$ and $\hat{B}$ are closed prime $^1$ densely defined symmetric operators with deficiency indices $(1, 1)$. Then, $\hat{A}$ and $\hat{B}$ are unitarily equivalent if and only if for an appropriate choice of normalized deficiency elements

$$g_\pm \in \text{Ker}((\hat{A})^* \mp iI) \quad \text{and} \quad f_\pm \in \text{Ker}((\hat{B})^* \mp iI)$$

the following equality

$$\frac{(g_z, g_-)}{(g_z, g_+)} = \frac{(f_z, f_-)}{(f_z, f_+)}, \quad z \in \mathbb{C}_+,$$

1A symmetric operator $\hat{A}$ is prime if there does not exist a subspace invariant under $\hat{A}$ such that the restriction of $\hat{A}$ to this subspace is self-adjoint.
holds, where \( g_z \neq 0 \) and \( f_z \neq 0 \) are arbitrary deficiency elements of the symmetric operators \( \hat{A} \) and \( \hat{B} \) such that \( g_z \in \text{Ker}((\hat{A})^* - zI) \) and \( f_z \in \text{Ker}((\hat{B})^* - zI) \), respectively.

Livšic suggested to call the function

\[
s(z) = \frac{z - i}{z + i} \cdot \frac{(g_z, g_-)}{(g_z, g_+)} , \quad z \in \mathbb{C}_+,\]

the characteristic function of the symmetric operator \( \hat{A} \).

Theorem 1.2 identifies the function \( s(z) \) (modulo \( z \)-independent unimodular factor) with a complete unitary invariant of a prime symmetric operator with deficiency indices \( (1, 1) \) that determines the operator uniquely up to unitary equivalence.

Livšic also gave a criterion \([20, \text{Theorem 15}]\) (also see \([3]\)) for a contractive analytic mapping from the upper-half plane \( \mathbb{C}_+ \) to the unit disk \( \mathbb{D} \) to be the characteristic function of a densely defined symmetric operator with deficiency indices \( (1, 1) \).

**Theorem 1.2 \([20]\).** For an analytic mapping \( s \) from the upper-half plane to the unit disk to be the characteristic function of a densely defined symmetric operator with deficiency indices \( (1, 1) \) it is necessary and sufficient that

\[
(1.2) \quad s(i) = 0 \quad \text{and} \quad \lim_{z \to \infty} z(s(z) - e^{2i\alpha}) = \infty \quad \text{for all} \quad \alpha \in [0, \pi),
\]

\[
0 < \varepsilon \leq \arg(z) \leq \pi - \varepsilon.
\]

In the same article, Livšic also put forward a concept of a characteristic function of a quasi-self-adjoint dissipative extension of a symmetric operator with deficiency indices \( (1, 1) \).

Let us recall Livšic’s construction.

Suppose that \( \hat{A} \) is a symmetric operator with deficiency indices \( (1, 1) \) and that \( g_\pm \) are its normalized deficiency elements,

\[
g_\pm \in \text{Ker}((\hat{A})^* \mp iI), \quad \|g_\pm\| = 1.
\]

Suppose that \( \hat{A} \neq (\hat{A})^* \) is a quasi-self-adjoint maximal dissipative extension of \( \hat{A} \),

\[
\text{Im}(\hat{A}f, f) \geq 0, \quad f \in \text{Dom}(\hat{A}).
\]

Since \( \hat{A} \) is symmetric, its dissipative extension \( \hat{A} \) is automatically quasi-self-adjoint \([25], [27]\), that is,

\[
\hat{A} \subset \hat{A} \subset (\hat{A})^*,
\]

and hence

\[
(1.3) \quad g_+ - \kappa g_- \in \text{Dom}(\hat{A}) \quad \text{for some} \quad |\kappa| < 1.
\]
Based on the parameterization (1.3) of the domain of the quasi-self-adjoint extension $\hat{A}$, being an analog of the von Neumann formulae\(^2\), Livšic suggested to call the Möbius transformation

$$S(z) = \frac{s(z) - \kappa}{Z s(z) - 1}, \quad z \in \mathbb{C}_+,$$

where $s$ is given by (1.1), the characteristic function of the quasi-self-adjoint extension $\hat{A}$ [20].

A culminating point of Livšic’s considerations was the discovery of the following result [20, the remaining part of Theorem 13].

**Theorem 1.3** ([20]). Suppose that $\dot{A}$ and $\dot{B}$ are closed prime densely defined symmetric operators with deficiency indices $(1, 1)$. Assume, in addition, that $\hat{A}$ and $\hat{B}$ are their quasi-self-adjoint extensions, respectively.

Then, $\hat{A}$ and $\hat{B}$ are unitarily equivalent if and only if the corresponding characteristic functions coincide up to a unimodular constant factor.

In 1965 Donoghue [9] introduced a concept of the Weyl-Titchmarsh function $M(\hat{A}, A)$ associated with a pair $(\hat{A}, A)$ by

$$M(\hat{A}, A)(z) = ((Az + I)(A - zI)^{-1}g_+, g_+), \quad z \in \mathbb{C}_+,$$

$$g_+ \in \text{Ker}((\hat{A})^* - iI), \quad \|g_+\| = 1,$$

where $\hat{A}$ is a symmetric operator with deficiency indices $(1, 1)$, $\text{def}(\hat{A}) = (1, 1)$, and $A$ is its self-adjoint extension.\(^3\)

Furthermore, Donoghue showed that the Weyl-Titchmarsh function admits the following Herglotz-Nevanlinna representation

$$M(\hat{A}, A)(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu$$

for some infinite Borel measure $\mu$ with

$$\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1,$$

and discovered the following result (for a full presentation see [11]).

**Theorem 1.4** ([9], [11]). Suppose that $\hat{A}$ and $\hat{B}$ are closed prime symmetric operators with deficiency indices $(1, 1)$. Assume, in addition, that $A$ and $B$ are some self-adjoint extensions of $\hat{A}$ and $\hat{B}$, respectively.

Then,

\(^2\)Throughout this paper $\kappa$ will be called the von Neumann extension parameter.

\(^3\)The concept of the Weyl–Titchmarsh function in the general case where $\text{def}(\hat{A}) = (n, n)$, $n \in \mathbb{N} \cup \{\infty\}$ is due to Saakjan [26]. Different approaches to the concept can be found in [1], [7], [8], [10], [11], [14] and the bibliography therein.
(i) the pairs $(A, A)$ and $(B, B)$ are unitarily equivalent\(^4\) if and only if
the Weyl-Titchmarsh functions $M(A, A)$ and $M(B, B)$ coincide;
(ii) the pair $(A, A)$ is unitarily equivalent to the model pair $(B, B)$ in
the Hilbert space $L^2(\mathbb{R}; d\mu)$, where $B$ is the multiplication (self-
adjoint) operator by the independent variable, $\dot{B}$ is its restriction
on
\begin{equation}
\text{Dom}(\dot{B}) = \left\{ f \in \text{Dom}(B) \left| \int_{\mathbb{R}} f(\lambda)d\mu(\lambda) = 0 \right. \right\},
\end{equation}
and $\mu$ is the Borel measure from the Herglotz-Nevanlinna representation (1.5) for the Weyl-Titchmarsh function $M = M(A, A)$.

Theorem 1.4, on the one hand, recognizes the Weyl-Titchmarsh function $M$ as a (complete) unitary invariant of the pair of a symmetric operator with deficiency indices $(1, 1)$ and its self-adjoint extension that determines the operators uniquely up to unitary equivalence. On the other hand, this result provides the general model of a symmetric operator with deficiency indices $(1, 1)$ and its family of self-adjoint extensions.

The main goal of this paper is
(i) to establish a relationship between the classical analytic objects of
the extension theory such as the characteristic function $s(z)$ of a
symmetric operator $A$, the characteristic function $S(z)$ of its quasi-
self-adjoint dissipative extensions $\hat{A}$, and the Weyl-Titchmarsh function $M(z)$ associated with the pair $(A, A)$;
(ii) to introduce the Weyl-Titchmarsh function of dissipative quasi-self-
adjoint extensions $\hat{A}$ of $A$, and
(iii) to provide a functional model for the triple $(A, \hat{A}, A)$ and prove an
analog of the formula of Krein [13] for the resolvents of $\hat{A}$ and $A$.

The paper is organized as follows.

In Sec. 2 we propose to consider the characteristic function of a symmetric operator $\hat{A}$ to be the one associated with a pair $(\hat{A}, A)$, with $A$ a special reference self-adjoint extension of $\hat{A}$ uniquely determined by the choice of the basis $\{g_\pm\}$ in the deficiency subspace (see, e.g., (2.1)). We call this function the Livšic function associated with the pair $(A, A)$. For different definitions of the characteristic functions in the unbounded case we refer to [12], [16], [23], [27] and [30].

We also show that the Weyl-Titchmarsh and the Livšic functions associated with the pair $(\hat{A}, A)$ are related by the Cayley transformation (see

\(^4\)We say that pairs of operators $(\hat{A}, A)$ and $(\dot{B}, B)$ in Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are unitarily equivalent if there is a unitary map $U$ from $\mathcal{H}_A$ onto $\mathcal{H}_B$ such that $\dot{B} = U\hat{A}U^{-1}$ and $B = UAU^{-1}$.
Theorem 2.2 (ii)). Based on this transformation law, we show that Livšic’s Theorem 1.2 and Donoghue’s Theorem 1.4 can be deduced from one another.

In Sec. 3 we introduce the Weyl-Titchmarsh function of a dissipative quasi-self-adjoint extension $\hat{A}$ of $\hat{A}$ and extend Theorem 2.2 to the dissipative case (see Theorem 3.2).

In Sec. 4 we introduce the characteristic function associated with the triple of operators $(\hat{A}, \hat{A}, A)$ and provide a functional model for such triples, continuing a list of various functional models for non-self-adjoint operators discovered by M. Livšic [20], [21], B. Sz.-Nagy and C. Foias [22], L. de Branges and J. Rovnyak [4], B. S. Pavlov [23], and others (also see [1], [16], [18], [19], [24], [27], [28], and references therein).

Based on the functional model, we obtain a refinement of Livšic’s uniqueness result of Theorem 1.3 (see Theorem 4.1).

In Sec. 5 we perform initial spectral analysis of a dissipative triple and obtain an analog of Krein’s resolvent formula in the rank-one dissipative extension theory (cf., [10], [11], [29]).

In Appendix A some spectral properties of the model symmetric operator are summarized.

Our exposition is based on the classical von Neumann extension theory. The proofs are straightforward and no knowledge of the rigged Hilbert space approach (see [1], [30]) and/or the boundary triplets theory (see [7], [17], [27]) is required.

2. The Livšic and the Weyl-Titchmarsh Functions

Livšic’s definition of a characteristic function of a symmetric operator (see, e.g., (1.1)) has some ambiguity related to the choice of the deficiency elements $g_\pm$. To avoid this ambiguity we proceed as follows.

Suppose that $A$ is a self-adjoint extension of a symmetric operator $\hat{A}$ with deficiency indices $(1,1)$. Let $g_\pm$ be deficiency elements $g_\pm \in \text{Ker}((\hat{A})^* \mp iI), \|g_+\| = 1$. Assume, in addition, that

$$g_+ - g_- \in \text{Dom}(A).$$

Introduce the Livšic function $s(\hat{A}, A)$ associated with the pair $(\hat{A}, A)$ by

$$s(z) = \frac{z - i}{z + i} \cdot \frac{(g_z, g_-)}{(g_z, g_+)}, \quad z \in \mathbb{C}_+,$$

where $0 \neq g_z \in \text{Ker}((\hat{A})^* - zI)$ is an arbitrary (deficiency) element.

The following result establishes a standard relationship between the Weyl-Titchmarsh and the Livšic functions associated with the pair $(\hat{A}, A)$ [1], [5], [6], [30] (also see [7], where a linear–fractional transformation between the
Livšic characteristic function of a symmetric operator and the Krein–Langer Q-function in the framework of boundary triplets theory was established).

**Theorem 2.1.** Denote by $M = M(\mathring{A}, A)$ and by $s = s(\mathring{A}, A)$ the Weyl-Titchmarsh function and the Livšic function associated with the pair $(\mathring{A}, A)$, respectively.

Then,

$$s(z) = \frac{M(z) - i}{M(z) + i}, \quad z \in \mathbb{C}_+.$$  

**Proof.** Let $\mu$ be the measure from the Herglotz-Nevanlinna representation (1.5) of the Weyl-Titchmarsh function $M$ associated with the pair $(\mathring{A}, A)$.

By splitting off the self-adjoint part of $\mathring{A}$, we may assume that $\mathring{A}$ is a prime symmetric operator and then, in accordance with Theorem 1.4, one may also assume without loss that $\mathring{A}$ and $A$ are already chosen in their model representation in the Hilbert space $L^2(\mathbb{R}; d\mu)$, with

$$g_z(\lambda) = \frac{1}{\lambda - z}, \quad \mu \text{-a.e.}, \quad \text{Im}(z) \neq 0,$$

and

$$g_{\pm}(\lambda) = g_{\pm i}(\lambda) = \frac{1}{\lambda \mp i}, \quad \mu \text{-a.e.}.$$  

Notice that $g_+ - g_- \in \text{Dom}(A)$ and hence

$$s(z) = \frac{z - i}{z + i} \cdot \frac{(g_z, g_-)}{(g_z, g_+)} = \frac{(z - i) \int_\mathbb{R} \frac{d\mu(\lambda)}{(\lambda - z)(\lambda - i)}}{(z + i) \int_\mathbb{R} \frac{d\mu(\lambda)}{(\lambda - z)(\lambda + i)}} = \frac{M(z) - i}{M(z) + i},$$

completing the proof.  

As a corollary we obtain the following analog of Theorem 1.4.

**Theorem 2.2.** Suppose that $\hat{A}$ and $\hat{B}$ are closed prime symmetric operators with deficiency indices $(1, 1)$. Assume, in addition, that $A$ and $B$ are some self-adjoint extensions of $\hat{A}$ and $\hat{B}$ respectively.

Then,

(i) the pairs $(\hat{A}, A)$ and $(\hat{B}, B)$ are unitarily equivalent if and only if the Livšic functions $s(\hat{A}, A)$ and $s(\hat{B}, B)$ coincide;

(ii) the pair $(\hat{A}, A)$ is unitarily equivalent to the model pair $(\hat{B}, B)$ in the Hilbert space $L^2(\mathbb{R}; d\mu)$, where $\mu$ is the representing measure for the Weyl-Titchmarsh function $M = M(\hat{A}, A)$.

In this case,

$$M(z) = \frac{1}{i} \cdot \frac{s(z) + 1}{s(z) - 1}, \quad z \in \mathbb{C}_+, \quad \text{with} \quad s = s(\hat{A}, A).$$
3. The Weyl-Titchmarsh Function in the Dissipative Case

The Weyl-Titchmarsh function can also be introduced in a more general context of dissipative extensions of a symmetric operator. We refer to [2] (also see [1]) where the concept of the Weyl-Titchmarsh function for bounded non-self-adjoint operators has been discussed.

To be more specific, suppose that $\dot{A}$ is a densely defined, closed, symmetric operator with deficiency indices $(1, 1)$ and $\hat{A}$ is its quasi-self-adjoint (maximal) dissipative extension, that is,

$$\dot{A} \subset \hat{A} \subset (\dot{A})^*$$

and

$$\text{Im}(\hat{A}f, f) \geq 0, \quad f \in \text{Dom}(\hat{A}).$$

Suppose that $g_+$ is a deficiency element $g_+ \in \text{Ker}((\dot{A})^* - iI)$ satisfying the normalization condition $\|g_+\| = 1$.

We introduce the Weyl-Titchmarsh function $M = M(\dot{A}, \hat{A})$ associated with the pair $(\dot{A}, \hat{A})$ by

$$M(z) = \left((\hat{A})^*z + I)((\hat{A})^* - zI)^{-1}g_+, g_+\right), \quad z \in \mathbb{C}_+.$$ 

We remark that in contradistinction to the self-adjoint case, the dependence of $M$ on the first argument, the symmetric operator $\dot{A}$, can be suppressed for, under our assumptions,

$$\dot{A} = \hat{A}_{|\text{Dom}(\dot{A}) \cap \text{Dom}((\dot{A})^*)}.$$ 

In other words, the symmetric operator $\dot{A}$ is uniquely determined by the quasi-self-adjoint extension $\hat{A}$ provided that $\hat{A}$ is not self-adjoint.

In order to establish a relationship between the Livšic function associated with the pair $(\hat{A}, \dot{A})$ and the Weyl-Titchmarsh function of the dissipative quasi-self-adjoint extension $\hat{A}$ of $\dot{A}$ it is convenient to use an analog of the von Neumann parameterization of $\text{Dom}(\hat{A})$.

To set up the notation, introduce the following hypothesis.

**Hypothesis 3.1.** Suppose that $\hat{A} \neq (\dot{A})^*$ is a maximal dissipative extension of a symmetric operator $\dot{A}$ with deficiency indices $(1, 1)$. Assume, in addition, that $A$ is a self-adjoint extension of $\dot{A}$. Suppose, that the deficiency elements $g_\pm \in \text{Ker}((\dot{A})^* \mp iI)$ are normalized, $\|g_\pm\| = 1$, and chosen in such a way that

$$g_+ - g_- \in \text{Dom}(A) \quad \text{and} \quad g_+ - \varkappa g_- \in \text{Dom}(\hat{A}) \quad \text{for some} \quad |\varkappa| < 1.$$ 

The following result is an extension of Theorem 2.1 to the dissipative case.
**Theorem 3.2.** Assume Hypothesis 3.1. Denote by $s = s(\hat{A}, A)$ the Livšic function associated with the pair $(\hat{A}, A)$ and by $M = M(\hat{A})$ the Weyl-Titchmarsh function of the dissipative operator $\hat{A}$.

Then,

$$g(z) = \frac{M(z) - i}{M(z) + i}, \quad z \in \mathbb{C}_+.$$  

**Proof.** Since the quasi-self-adjoint extension $\hat{A}$ of $A$ is a restriction of $(\hat{A})^*$ on $\text{Dom}(\hat{A})$, one gets that

$$(\hat{A} - iI)(\hat{A} - zI)^{-1}g_+ \in \text{Ker}((\hat{A})^* - zI), \quad z \in \rho(\hat{A}),$$

where $\rho(\hat{A})$ denotes the resolvent set of $\hat{A}$.

Indeed,

$$(\hat{A} - iI)(\hat{A} - zI)^{-1}g_+ = g_+ + (z - i)(\hat{A} - zI)^{-1}g_+$$

and hence

$$( (\hat{A})^* - zI)(\hat{A} - iI)(\hat{A} - zI)^{-1}g_+ = ((\hat{A})^* - zI)g_+$$

$$+ (z - i)((\hat{A})^* - zI)(\hat{A} - zI)^{-1}g_+$$

$$= (i - z)g_+ + (z - i)(\hat{A} - zI)(\hat{A} - zI)^{-1}g_+ = 0$$

which proves (3.3).

From (3.3) one obtains that $(\hat{A} - iI)(\hat{A} + iI)^{-1}g_+ \in \text{Ker}((\hat{A})^* + iI)$ and hence $(\hat{A} - iI)(\hat{A} + iI)^{-1}g_+ = \alpha g_-$ for some $\alpha \in \mathbb{C}$.

On the other hand,

$$\alpha g_- = (\hat{A} - iI)(\hat{A} + iI)^{-1}g_+ = g_+ - 2i(\hat{A} + iI)^{-1}g_+$$

and therefore

$$\alpha g_- - g_+ = -2i(\hat{A} + iI)^{-1}g_+ \in \text{Dom}(\hat{A}).$$

Taking into account the characterization (3.1) of $\text{Dom}(\hat{A})$, one obtains that $\alpha = \kappa$ and therefore

$$\kappa g_- = (\hat{A} - iI)(\hat{A} + iI)^{-1}g_+,$$

as it follows from (3.4).

Introducing the elements

$$g_z = (\hat{A}^* - iI)(\hat{A}^* - zI)^{-1}g_+,$$

and taking into account that the adjoint operator $(\hat{A})^*$ is also a quasi-self-adjoint extension of $\hat{A}$, one concludes that

$$g_z \in \text{Ker}((\hat{A})^* - zI), \quad z \in \rho(\hat{A}),$$
which shows that the Livšic function \( s = s(\hat{A}, A) \) admits the representation (see definition (2.2))

\[
s(z) = \frac{z - i}{z + i} \cdot \frac{(g_z, g_-)}{(g_z, g_+)}, \quad z \in \mathbb{C}_+.
\]

Therefore, in view of (3.5) and (3.6), one computes

\[
\Re s(z) = \frac{z - i}{z + i} \cdot \frac{(g_z, \kappa g_-)}{(g_z, g_+)}
\]

\[
= \frac{z - i}{z + i} \cdot \frac{(\hat{A} - zI)^{-1}g_+, (\hat{A} + iI)(\hat{A} - zI)^{-1}g_+}{((\hat{A} - iI)(\hat{A} - zI)^{-1}g_+g_+)}
\]

\[
= \frac{z - i}{z + i} \cdot \frac{((\hat{A} - iI)(\hat{A} - zI)^{-1}g_+, g_+)}{((\hat{A} - iI)(\hat{A} - zI)^{-1}g_+g_+)}
\]

\[
= \frac{z - i}{z + i} \cdot \frac{(1 + (z + i)((\hat{A} - zI)^{-1}g_+, g_+))}{(1 + (z - i)((\hat{A} - zI)^{-1}g_+, g_+))}
\]

\[
= \frac{z - i}{z + i} \cdot \frac{(z^2 + 1)((\hat{A} - zI)^{-1}g_+, g_+)}{(z + i + (z^2 + 1)((\hat{A} - zI)^{-1}g_+, g_+)}
\]

\[
= \frac{M(z) - i}{M(z) + i}, \quad z \in \rho(\hat{A}^*)
\]

proving the claim.

\[
\square
\]

**Remark 3.3.** Combining Theorems 2.1 and 3.2, it is easy to see that under Hypothesis 3.1 the Weyl-Titchmarsh functions \( \mathcal{M} = \mathcal{M}(\hat{A}, A) \) and \( \mathcal{M}(\hat{A}) \) are related as follows

\[
\frac{\mathcal{M} - i}{\mathcal{M} + i} = \Re \cdot \frac{\mathcal{M} - i}{\mathcal{M} + i}.
\]

Our next result shows that the Weyl-Titchmarsh function of a dissipative operator admits the Herglotz-Nevanlinna representation with a (locally) absolutely continuous representing measure.

**Corollary 3.4.** Assume Hypothesis 3.1.

Then, the Weyl-Titchmarsh function \( \mathcal{M} = \mathcal{M}(\hat{A}) \) of the dissipative operator \( \hat{A} \) is an analytic function mapping the upper half-plane into the disk of radius \( \frac{2|\kappa|}{1 - |\kappa|^2} \) centered at the point \( \left(0, \frac{1 + |\kappa|^2}{1 - |\kappa|^2}\right) \) of the \( xy \)-plane.

Moreover, the function \( \mathcal{M} \) admits the Herglotz-Nevanlinna representation

\[
\mathcal{M}(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) f(\lambda) d\lambda, \quad z \in \mathbb{C}_+,
\]
where
\begin{equation}
\frac{1}{\pi} \frac{1 - |\kappa|}{1 + |\kappa|} \leq f(\lambda) \leq \frac{1}{\pi} \frac{1 + |\kappa|}{1 - |\kappa|} \quad \text{a.e.}
\end{equation}
and
\begin{equation}
\int_{\mathbb{R}} \frac{f(\lambda)}{1 + \lambda^2} d\lambda = 1.
\end{equation}

Proof. Since the Cayley transform
\[ z \mapsto \frac{z - i}{z + i} \]
maps the disk \( \frac{2|\kappa|}{1 - |\kappa|^2} \mathbb{D} + i \frac{1 + |\kappa|^2}{1 - |\kappa|^2} \) onto the disk \( |\kappa| \mathbb{D} \) and the Livšic function \( s(\hat{A}, A) \) is contractive in \( \mathbb{C}_+ \), from (3.2) follows that
\[ \text{Range} (M) \subset \frac{2|\kappa|}{1 - |\kappa|^2} \mathbb{D} + i \frac{1 + |\kappa|^2}{1 - |\kappa|^2}. \]
In particular,
\begin{equation}
\frac{1 - |\kappa|}{1 + |\kappa|} \leq \text{Im} \ M(z) \leq \frac{1 + |\kappa|}{1 - |\kappa|}, \quad z \in \mathbb{C}_+.
\end{equation}

Since \( M(i) = i \) and \( M \) has a bounded imaginary part in the upper half-plane, using Fatou’s Lemma, one proves the Herglotz-Nevanlinna representation
\[ M(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu(\lambda), \quad z \in \mathbb{C}_+, \]
where \( \mu \) is a (locally) absolutely continuous measure with the Radon-Nikodym density
\[ f(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \text{Im} \ M(\lambda + i\varepsilon), \quad \text{a.e.}, \]
which proves (3.8), using (3.10), and (3.9), since \( M(i) = i \). □

4. THE CHARACTERISTIC FUNCTION AND THE FUNCTIONAL MODEL OF A DISSIPATIVE OPERATOR

Under Hypothesis 3.1, we introduce the characteristic function \( S = S(\hat{A}, \widehat{A}, A) \) associated with the triple of operators \( (\hat{A}, \widehat{A}, A) \) as the Möbius transformation
\begin{equation}
S(z) = \frac{s(z) - \kappa}{\kappa s(z) - 1}, \quad z \in \mathbb{C}_+,
\end{equation}
of the Livšic function \( s = s(\hat{A}, A) \) associated with the pair \( (\hat{A}, A) \).
We remark that given a triple \((\hat{A}, \tilde{A}, A)\), one can always find a basis \(g_\pm, \|g_\pm\| = 1\), in the deficiency subspace \(\text{Ker}((\hat{A})^* - iI) + \text{Ker}((\hat{A})^* + iI)\) such that
\[
g_+ - g_- \in \text{Dom}(A) \quad \text{and} \quad g_+ - \kappa g_- \in \text{Dom}(\tilde{A}),
\]
and then, in this case,
\[
\kappa = S(\hat{A}, \tilde{A}, A)(i).
\]

Our next goal is to introduce a functional model of a prime dissipative triple\(^5\) parameterized by the characteristic function.

Given a contractive analytic map \(S,\)
\[
S(z) = \frac{s(z) - \kappa}{\kappa s(z) - 1}, \quad z \in \mathbb{C}_+,
\]
where the analytic and contractive in \(\mathbb{C}_+\) function \(s\) satisfies the Livšic criterion \((1.2)\) and \(|\kappa| < 1\), introduce the function
\[
M(z) = \frac{1}{i} \cdot \frac{s(z) + 1}{s(z) - 1}, \quad z \in \mathbb{C}_+,
\]
so that
\[
M(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu(\lambda), \quad z \in \mathbb{C}_+,
\]
for some infinite Borel measure with
\[
\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1.
\]

In the Hilbert space \(L^2(\mathbb{R}; d\mu)\) introduce the multiplication (self-adjoint) operator by the independent variable \(B\) on
\[
\text{Dom}(B) = \left\{ f \in L^2(\mathbb{R}; d\mu) \left| \int_{\mathbb{R}} \lambda^2 |f(\lambda)|^2 d\mu(\lambda) < \infty \right. \right\},
\]
denote by \(\hat{B}\) its restriction on
\[
\text{Dom}(\hat{B}) = \left\{ f \in \text{Dom}(B) \left| \int_{\mathbb{R}} f(\lambda) d\mu(\lambda) = 0 \right. \right\},
\]
and let \(\tilde{B}\) be the dissipative restriction of the operator \((\hat{B})^*\) on
\[
\text{Dom}(\tilde{B}) = \text{Dom}(\hat{B}) + \text{lin span} \left\{ \frac{1}{i} - S(i) \frac{1}{i + i} \right\}.
\]

We will refer to the triple \((\hat{B}, \tilde{B}, B)\) as the model triple in the Hilbert space \(L^2(\mathbb{R}; d\mu)\).

\(^5\)We call a triple \((\hat{A}, \tilde{A}, A)\) a prime triple if \(\hat{A}\) is a prime symmetric operator.
Our next result shows that a triple \((\hat{A}, A, \hat{A})\) with the characteristic function \(S\) is unitarily equivalent to the model triple \((\hat{B}, \hat{B}, B)\) in the Hilbert space \(L^2(\mathbb{R}; d\mu)\) whenever the underlying symmetric operator \(\hat{A}\) is prime.

The triple \((\hat{B}, \hat{B}, B)\) will therefore be called the functional model for \((\hat{A}, A, \hat{A})\).

**Theorem 4.1.** Suppose that \(\hat{A}\) and \(\hat{B}\) are prime closed symmetric operators with deficiency indices \((1, 1)\). Assume, in addition, that \(A\) and \(B\) are some self-adjoint extensions of \(\hat{A}\) and \(\hat{B}\) and that \(\hat{A}\) and \(\hat{B}\) are quasi-self-adjoint dissipative extensions of \(\hat{A}\) and \(\hat{B}\), respectively.

Then,

(i) the triples \((\hat{A}, A, \hat{A})\) and \((\hat{B}, \hat{B}, B)\) are unitarily equivalent\(^6\) if and only if the characteristic functions \(S_A = S(\hat{A}, A, \hat{A})\) and \(S_B = S(\hat{B}, \hat{B}, B)\) of the triples coincide;

(ii) the triple \((\hat{A}, A, \hat{A})\) is unitarily equivalent to the model triple \((\hat{B}, \hat{B}, B)\) in the Hilbert space \(L^2(\mathbb{R}; d\mu)\), where \(\mu\) is the representing measure for the Weyl-Titchmarsh function \(M = M(\hat{A}, A)\) associated with the pair \((\hat{A}, A)\).

**Proof.** (i). Since \(S_A = S_B\) and

\[
s(\hat{A}, A) = \frac{S_A - S_A(i)}{S_A(i)S_A - 1} = \frac{S_B - S_B(i)}{S_B(i)S_B - 1} = s(\hat{B}, B),
\]

one concludes that the Livšic functions \(s(\hat{A}, A)\) and \(s(\hat{B}, B)\) coincide.

By Theorem 1.4 (i), there exists a unitary map \(\mathcal{U}\) such that

\[
\mathcal{U}\hat{A}\mathcal{U}^* = \hat{B} \quad \text{and} \quad \mathcal{U}A\mathcal{U}^* = B.
\]

Let \(g_+ \in \text{Ker}((\hat{A})^* \mp iI), \|g_+\| = 1, \) such that

\[
g_+ - g_- \in \text{Dom}(A).
\]

Set

\[
f_\pm = \mathcal{U}g_\pm.
\]

Equalities (4.7) yield \(f_+ \in \text{Ker}((\hat{B})^* \mp iI), \|f_+\| = 1, \) and

\[
f_+ - f_- \in \text{Dom}(B).
\]

In this case, since \(\varkappa = S_A(i) = S_B(i)\), the membership

\[
g_+ - \varkappa g_- \in \text{Dom}(\hat{\mathcal{A}})
\]

\(^6\) We say that triples of operators \((\hat{A}, \hat{A}, A)\) and \((\hat{B}, \hat{B}, B)\) in Hilbert spaces \(\mathcal{H}_A\) and \(\mathcal{H}_B\) are unitarily equivalent if there is a unitary map \(\mathcal{U}\) from \(\mathcal{H}_A\) onto \(\mathcal{H}_B\) such that \(B = \mathcal{U}\hat{A}\mathcal{U}^{-1}, \hat{B} = \mathcal{U}\hat{A}\mathcal{U}^{-1}, \) and \(B = \mathcal{U}A\mathcal{U}^{-1}.\)
means that $f_+ - \kappa f_- \in \text{Dom}(\hat{B})$. Thus,

$\mathcal{U} \text{Dom}(\hat{A}) = \text{Dom}(\hat{B})$,

and hence

$\mathcal{U} \hat{A} \mathcal{U}^* = \hat{B}$,

proving that the triples $(\hat{A}, \hat{A}, A)$ and $(\hat{B}, \hat{B}, B)$ are unitarily equivalent.

(ii). By Theorem 1.4 (ii) the Weyl-Titchmarsh function $M(\hat{B}, B)$ coincides with $M(\hat{A}, A)$ and hence the corresponding Livšic functions coincide, that is, $s(\hat{A}, A) = s(\hat{B}, B)$.

Since the deficiency elements

$$f_\pm(\lambda) = \frac{1}{\lambda \mp i}$$

are normalized by one, $f_+ - f_- \in \text{Dom}(B)$, and, by hypothesis,

$$f_+ - S(i)f_- \in \text{Dom}(\hat{B}),$$

one computes that

$$S_B = \frac{s(\hat{B}, B) - S_A(i)}{S_A(i)s(\hat{B}, B) - 1} = S_A.$$

Therefore, the triples $(\hat{A}, \hat{A}, A)$ and $(\hat{B}, \hat{B}, B)$ are unitarily equivalent by the first part of the proof.

The proof is complete. \qed

**Remark 4.2.** We remark that the Weyl-Titchmarsh function $M = M(\hat{A}, A)$ can be recovered from the characteristic function $S_A$ of the triple $(\hat{A}, \hat{A}, A)$ by the equation

$$M(z) = \frac{1}{i} \cdot \frac{s(z) + 1}{s(z) - 1} \quad \text{with} \quad s(z) = \frac{S_A(z) - S_A(i)}{S_A(i)S_A(z) - 1}, \quad z \in \mathbb{C}_+.$$

We conclude this section by showing that the characteristic function $S$ associated with the triple $(\hat{A}, \hat{A}, A)$ and the Weyl-Titchmarsh function $M(\hat{A})$ of the dissipative operator $\hat{A}$ are related by a linear transformation.

**Theorem 4.3.** Let $M$ be the Weyl-Titchmarsh function $M(\hat{A})$ of a dissipative operator $\hat{A}$ and $S = S(\hat{A}, \hat{A}, A)$ the characteristic function associated with the triple $(\hat{A}, \hat{A}, A)$.

Then,

$$s(z) = \frac{|\kappa|^2 - 1}{2i} M(z) + \frac{|\kappa|^2 + 1}{2}, \quad \text{with} \quad \kappa = S(i), \quad z \in \mathbb{C}_+. \quad (4.8)$$
Proof. By definition, the characteristic function $S$ and the Livšic function $s$ are related by the M"{o}bius transformation

$$S = \frac{s - \kappa}{\overline{s} - 1}.$$ 

By Theorem 3.2, one obtains that

$$\overline{s} = \frac{M - i}{M + i}$$

and therefore

$$\overline{S} = \overline{s} \cdot \frac{s - \kappa}{\overline{s} - 1} = \frac{\overline{s} - |\kappa|^2}{\overline{s} - 1} = \frac{M - i}{M + i} - |\kappa|^2$$

$$= \frac{i}{2} \left((1 - |\kappa|^2)M - i(1 + |\kappa|^2)\right)$$

which proves (4.8). \qed

Remark 4.4. Theorem 4.3 shows that in case when the von Neumann parameter $\kappa$ does not vanish, the characteristic function $S$ of the (prime) triple is uniquely determined by the pair $(\kappa, M)$. Therefore, along with the characteristic function $S$, the pair $(\kappa, M)$, with $\kappa \neq 0$, can also be considered to be a complete unitary invariant of a prime dissipative triple $(\hat{A}, \hat{A}, A)$.

However, if $\kappa = 0$, and therefore by (3.2), $M(z) = i$ for all $z \in \mathbb{C}_+$, the quasi-self-adjoint extension $\hat{A}$ coincides with the restriction of the adjoint operator $(\hat{A})^*$ on

$$\text{Dom}(\hat{A}) = \text{Dom}(\hat{A}) + \text{Ker}((\hat{A})^* - iI).$$

Hence, prime triples $(\hat{A}, \hat{A}, A)$ with $\kappa = 0$ are in one-to-one correspondence with the set of prime symmetric operators.

In this case, the characteristic function $S$ and the Livšic function $s$ coincide (up to a sign),

$$S(z) = -s(z), \quad z \in \mathbb{C}_+.$$ 

Therefore, for $S$ to be the characteristic function of a triple $(\hat{A}, \hat{A}, A)$ with $\kappa = 0$ and $M(z) = i$ for all $z$ in the upper half-plane it is necessary and sufficient that $S$ satisfy the Livšic criterion (1.2).

5. The Spectral Analysis of the Model Dissipative Operator

In the suggested functional model in the Hilbert space $L^2(\mathbb{R}; d\mu)$, the eigenfunctions of the model dissipative operator $\hat{B}$ from the triple $(\hat{B}, \hat{B}, B)$ look exceptionally simple.
Lemma 5.1. Suppose that \((\hat{\mathcal{B}}, \hat{\mathcal{B}}, \mathcal{B})\) is the model triple in \(L^2(\mathbb{R}; d\mu)\). Then a point \(z_0 \in \mathbb{C}_+\) is an eigenvalue of the dissipative operator \(\hat{\mathcal{B}}\) if and only if \(S(\hat{\mathcal{B}}, \hat{\mathcal{B}}, \mathcal{B})(z_0) = 0\).

In this case, the corresponding eigenfunction \(f\) is of the form

\[
f(\lambda) = \frac{1}{\lambda - z_0}, \quad \mu\text{-a.e.}
\]

Proof. Suppose that \(z_0 \in \mathbb{C}_+\) is an eigenvalue of \(\hat{\mathcal{B}}\) and that \(f, f \in L^2(\mathbb{R}; d\mu)\), is the corresponding eigenvector, that is,

\[
\hat{\mathcal{B}} f = z_0 f, \quad f \in \text{Dom}(\hat{\mathcal{B}}).
\]

Since \(f \in \text{Dom}(\hat{\mathcal{B}})\), the element \(f\) admits the representation

\[
f(\lambda) = f_0(\lambda) + K \left( \frac{1}{\lambda - i} - \frac{1}{\lambda + i} \right),
\]

where \(f_0 \in \text{Dom}(\hat{\mathcal{B}})\) and \(K\) is some constant. Then

\[
0 = (\hat{\mathcal{B}} - z_0 I) f(\lambda) = (\lambda - z_0) f_0(\lambda) + K \left( \frac{i - z_0}{\lambda - i} + \frac{i + z_0}{\lambda + i} \right)
\]

and hence

\[
f_0(\lambda) = -\frac{K}{\lambda - z_0} \left( \frac{i - z_0}{\lambda - i} + \frac{i + z_0}{\lambda + i} \right).
\]

Since \(f_0 \in \text{Dom}(\hat{\mathcal{B}})\),

\[
\int_{\mathbb{R}} f_0(\lambda) d\mu(\lambda) = 0
\]

and hence

\[
0 = \int_{\mathbb{R}} \frac{1}{\lambda - z_0} \left( \frac{i - z_0}{\lambda - i} + \frac{i + z_0}{\lambda + i} \right) d\mu(\lambda)
\]

\[
= -\int_{\mathbb{R}} \left( \frac{1}{\lambda - z_0} - \frac{1}{\lambda - i} \right) d\mu(\lambda) + \varkappa \int_{\mathbb{R}} \left( \frac{1}{\lambda - z_0} - \frac{1}{\lambda + i} \right) d\mu(\lambda)
\]

\[
= -M(z_0) + M(i) + \varkappa(M(z_0) - M(-i))
\]

\[
= -M(z_0) + i + \varkappa(M(z_0) + i).
\]

Therefore,

\[
\varkappa = \frac{M(z_0) - i}{M(z_0) + i} = s(\hat{\mathcal{B}}, \hat{\mathcal{B}})(z_0)
\]

and hence the characteristic function \(S(\hat{\mathcal{B}}, \hat{\mathcal{B}}, \mathcal{B})\) vanishes at the point \(z_0\),

\[
S(\hat{\mathcal{B}}, \hat{\mathcal{B}}, \mathcal{B})(z_0) = \frac{s(\hat{\mathcal{B}}, \hat{\mathcal{B}})(z_0) - \varkappa}{\varkappa s(\hat{\mathcal{B}}, \hat{\mathcal{B}})(z_0) - 1} = 0.
\]
In this case,
\[
f(\lambda) = K \left[ \left( \frac{1}{\lambda - i} - \frac{1}{\lambda + i} \right) - \frac{1}{\lambda - z_0} \left( \frac{i - z_0}{\lambda - i} + \frac{i + z_0}{\lambda + i} \right) \right] \\
= K \left[ \frac{1}{\lambda - i} \left( 1 - \frac{i - z_0}{\lambda - z_0} \right) - \frac{1}{\lambda + i} \left( 1 + \frac{i + z_0}{\lambda - z_0} \right) \right] \\
= K \frac{1 - \zeta}{\lambda - z_0}.
\]

So, we have shown that if \( z_0 \) is an eigenvalue of \( \hat{\mathcal{B}} \), then

\[
S(\hat{\mathcal{B}}, \mathcal{B})(z_0) = 0
\]

and that the corresponding eigenelement \( f \) is of the form

\[
(5.1) \quad f(\lambda) = \frac{1}{\lambda - z_0}.
\]

Repeating the same reasoning in the reverse order, one easily shows that if \( S(\mathcal{B}, \hat{\mathcal{B}}, \mathcal{B})(z_0) = 0 \), then the function \( f \) given by (5.1) belongs to \( \text{Dom}(\hat{\mathcal{B}}) \) and \( \hat{\mathcal{B}} f = z_0 f \).

For the resolvents of the model dissipative operator \( \hat{\mathcal{B}} \) and the self-adjoint (reference) operator \( \mathcal{B} \) from the model triple \( (\hat{\mathcal{B}}, \hat{\mathcal{B}}, \mathcal{B}) \) one gets the following resolvent formula.

**Theorem 5.2.** Suppose that \( (\hat{\mathcal{B}}, \hat{\mathcal{B}}, \mathcal{B}) \) is the model triple in the Hilbert space \( L^2(\mathbb{R}; d\mu) \).

Then the resolvent of the model dissipative operator \( \hat{\mathcal{B}} \) in \( L^2(\mathbb{R}; d\mu) \) has the form

\[
(\hat{\mathcal{B}} - zI)^{-1} = (\mathcal{B} - zI)^{-1} - p(z)(\cdot, g_z)g_z,
\]

with

\[
p(z) = \left( M(\hat{\mathcal{B}}, \mathcal{B})(z) + \frac{\zeta + 1}{\zeta - 1} \right)^{-1},
\]

\[z \in \rho(\hat{\mathcal{B}}) \cap \rho(\mathcal{B}).\]

Here \( M(\hat{\mathcal{B}}, \mathcal{B}) \) is the Weyl-Titchmarsh function associated with the pair \( (\hat{\mathcal{B}}, \mathcal{B}) \) continued to the lower half-plane by the Schwarz reflection principle, and the deficiency elements \( g_z \) are given by

\[
g_z(\lambda) = \frac{1}{\lambda - z}, \quad \mu\text{-a.e.}
\]

**Proof.** Given \( h \in L^2(\mathbb{R}; d\mu) \) and \( z \in \rho(\hat{\mathcal{B}}) \), suppose that

\[
(5.2) \quad (\hat{\mathcal{B}} - zI)f = h \quad \text{for some } f \in \text{Dom}(\hat{\mathcal{B}}).
\]
Since \( f \in \text{Dom}(\hat{B}) \), one gets the representation

\[
(5.3) \quad f(\lambda) = f_0(\lambda) + K \left( \frac{1}{\lambda - i} - \frac{1}{\lambda + i} \right)
\]

for some \( f_0 \in \text{Dom}(\hat{B}) \) and \( K \in \mathbb{C} \). Eq. (5.2) yields

\[
(\lambda - z)f_0(\lambda) + K \left( \frac{i - z}{\lambda - i} + \kappa \frac{i + z}{\lambda + i} \right) = h(\lambda)
\]

and hence

\[
(5.4) \quad f_0(\lambda) = \frac{h(\lambda)}{\lambda - z} - \frac{K}{\lambda - z} \left( \frac{i - z}{\lambda - i} + \kappa \frac{i + z}{\lambda + i} \right).
\]

Since \( f_0 \in \text{Dom}(A) \), and therefore

\[
\int_{\mathbb{R}} f_0(\lambda) d\mu(\lambda) = 0,
\]

integrating (5.4) against \( \mu \), one obtains that

\[
(5.5) \quad K \int_{\mathbb{R}} \frac{1}{\lambda - z} \left( \frac{i - z}{\lambda - i} + \kappa \frac{i + z}{\lambda + i} \right) d\mu(\lambda) = \int_{\mathbb{R}} \frac{h(\lambda)}{\lambda - z} d\mu(\lambda).
\]

Observing that

\[
\int_{\mathbb{R}} \frac{1}{\lambda - z} \left( \frac{i - z}{\lambda - i} + \kappa \frac{i + z}{\lambda + i} \right) d\mu(\lambda) = i - M(z) + \kappa(M(z) + i),
\]

with

\[
M(z) = M(\hat{B}, \hat{B})(z).
\]

Solving (5.5) for \( K \), one obtains

\[
K = \frac{\int_{\mathbb{R}} \frac{h(\lambda)}{\lambda - z} d\mu(\lambda)}{(\kappa - 1)M(z) + i(1 + \kappa)}.
\]

Combining (5.3) and (5.4), for the element \( f \) we have the representation

\[
 f(\lambda) = \frac{h(\lambda)}{\lambda - z} + \frac{K}{\lambda - z} \left( \frac{i - z}{\lambda - i} + \kappa \frac{i + z}{\lambda + i} \right) \left( \frac{\lambda - z}{\lambda - i} - \kappa \frac{\lambda - z}{\lambda + i} - \frac{\lambda - z}{\lambda - i} + \kappa \frac{i + z}{\lambda + i} \right)
\]

\[
= \frac{h(\lambda)}{\lambda - z} - K \frac{\lambda - z}{\lambda - z}
\]

\[
= \frac{h(\lambda)}{\lambda - z} - \left( M(z) + i \frac{\kappa + 1}{\kappa - 1} \right)^{-1} \frac{1}{\lambda - z} \int_{\mathbb{R}} \frac{h(\lambda)}{\lambda - z} d\mu(\lambda),
\]

\[ z \in \rho(\hat{B}) \cap \rho(B), \]

which proves the claim.
Remark 5.3. It is easy to see, using (2.3), that the poles of the function $p$ in the upper half-plane coincide with the roots of the equation

$$s(\hat{B}, B)(z) = \kappa, \quad z \in \mathbb{C}_+,$$

provided that $\kappa \neq 0$ and $M(z) \neq i$ identically in the upper half-plane. Therefore, the zeros of the characteristic function $S(\hat{B}, \hat{B}, B)$ in the upper half-pane determine the poles of the resolvent of the dissipative operator $\hat{B}$ (cf., Lemma 5.1).

We also remark that if $\kappa = 0$ and $M(z) = i$ for all $z \in \mathbb{C}_+$, then the point spectrum of the dissipative operator $\hat{B}$ fills in the whole open upper half-plane $\mathbb{C}_+$.

Given a triple $(\hat{A}, \hat{A}, A)$ satisfying Hypothesis 3.1, the following corollary provides an analog of the Krein formula for resolvents for all quasi-self-adjoint dissipative extensions of the symmetric operator $\hat{A}$ with deficiency indices $(1, 1)$ (cf., [17], where in the framework of the boundary triplets theory the general resolvent formula of dual pairs of linear relations was obtained).

Corollary 5.4. Under Hypothesis 3.1, the following resolvent formula

$$(\hat{A} - zI)^{-1} = (A - zI)^{-1} - p(z)(\cdot, g_z)g_z,$$

holds. Here

(a)$$p(z) = \left(M(\hat{A}, A)(z) + i\frac{\kappa + 1}{\kappa - 1}\right)^{-1},$$

(b)$$M(\hat{A}, A) \text{ and } s(\hat{A}, A) \text{ are the Weyl-Titchmarsh and the Livšic function of the pair } (\hat{A}, A), \text{ respectively};$$

(c) $g_z$ are deficiency elements of $\hat{A}$,

$$g_z \in \text{Ker}(\hat{A}^* - zI),$$

that satisfy the normalization condition

$$(5.9) \quad \|g_z\| = \left(\int_{\mathbb{R}} \frac{d\mu(\lambda)}{|\lambda - z|^2}\right)^{1/2}.$$
(the deficiency elements $g_z$ can be chosen to be analytic in $z \in \rho(\hat{A}) \cap \rho(A)$);

(d) $\mu$ is the measure from the Herglotz-Nevanlinna representation

$$M(\hat{A}, A)(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu;$$

and

(e) $\kappa$ is the von Neumann parameter characterizing the domain of the dissipative extension $\hat{A}$,

\begin{equation}
\label{5.10}
g_i - g_{-i} \in \text{Dom}(A) \quad \text{and} \quad g_i - \kappa g_{-i} \in \text{Dom}(\hat{A}).
\end{equation}

**Remark 5.5.** We would like to stress that

1. the value of the von Neumann parameter $\kappa$ from (5.10),
2. the Livšic function $s(\hat{A}, A)$, and
3. the Weyl-Titchmarsh function $M(\hat{A}, A)$,

can easily be recovered from the functional parameter of the (prime) triple, the characteristic function $S = S(\hat{A}, \hat{A}, A)$, which is a complete unitary invariant of $(\hat{A}, \hat{A}, A)$.

Indeed,

\begin{align*}
\kappa & = S(i), \\
s(\hat{A}, A)(z) & = \frac{S(z) - \kappa}{\kappa S(z) - 1}, \\
M(\hat{A}, A)(z) & = \frac{1}{i} \cdot \frac{s(\hat{A}, A)(z) + 1}{s(\hat{A}, A)(z) - 1},
\end{align*}

$z \in \mathbb{C}_+$,

with $M(\hat{A}, A)$ continued to the lower half-plane by the Schwarz reflection principle

$$M(\hat{A}, A)(z) = \overline{M(\hat{A}, A)(\overline{z})}, \quad z \in \mathbb{C}_-. $$

**Remark 5.6.** The resolvent formula (5.6)–(5.8) also holds if $|\kappa| = 1$ and hence $\hat{A}$ is self-adjoint. In this case, it provides the standard Krein resolvent formula for self-adjoint extensions of $\hat{A}$ via the von Neumann extension parameter $\kappa$ and the Weyl-Titchmarsh (Livšic) function.

Although it appears unlikely that the explicit normalization condition (5.9) has been missed in the literature, we were not able to locate an appropriate reference.
Remark 5.7. We also remark that if two triples $(\hat{A}, A, \hat{A}_1)$ and $(\hat{A}, A, \hat{A}_2)$ satisfy Hypothesis 3.1 with the von Neumann parameters $\varkappa_1$ and $\varkappa_2$, respectively, then one gets the following resolvent formula for the dissipative extensions $\hat{A}_1$ and $\hat{A}_2$ refining, in the rank one setting, a result in [15]:

$$(\hat{A}_2 - zI)^{-1} = (\hat{A}_1 - zI)^{-1} - q(z)(\cdot, g_z)g_z,$$

where $q(z) = p_2(z) - p_1(z)$ with

$$p_k(z) = \left( M(\hat{A}, A)(z) + i\varkappa_k + 1 \over \varkappa_k - 1 \right)^{-1},$$

$$= i \left( s(\hat{A}, A)(z) + 1 \over s(A, A)(z) - 1 \right)\left( \varkappa_k + 1 \over \varkappa_k - 1 \right)^{-1}, \quad k = 1, 2,$$

$$z \in \rho(A) \cap \rho(\hat{A}_1) \cap \rho(\hat{A}_2).$$

We recall that if $S_1$ and $S_2$ are the characteristic functions of the triples $(\hat{A}, A, \hat{A}_1)$ and $(\hat{A}, A, \hat{A}_2)$, respectively, then

$$s(\hat{A}, A) = \frac{S_k - \varkappa_k}{\varkappa_k S - 1}, \quad k = 1, 2.$$

APPENDIX A. THE SPECTRAL PROPERTIES OF THE MODEL SYMMETRIC OPERATOR

For completeness, we provide a summary of spectral properties of the model symmetric operator $\hat{B}$ in $L^2(\mathbb{R}; d\mu)$ (we refer to a relevant discussion in [11]) and obtain the characterization of the core of its spectrum in terms of the measure $\mu$.

Denote by $m$ the class of infinite Lebesgue-Stieltjes measures $\mu$ on the real axis, $\mu(\mathbb{R}) = \infty$, such that

$$\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1. \tag{A.1}$$

Our main goal is to show that every measure $\mu$ from the measure class $m$ gives rise to a prime symmetric operator in the Hilbert space $L^2(\mathbb{R}; d\mu)$ with deficiency indices $(1, 1)$ via the following construction.

Given $\mu \in m$, in the Hilbert space $L^2(\mathbb{R}; d\mu)$ introduce the multiplication (self-adjoint) operator $\mathcal{B}$ by the independent variable on

$$\text{Dom}(\mathcal{B}) = \left\{ f \in L^2(\mathbb{R}; d\mu) \left| \int_{\mathbb{R}} \lambda^2 |f(\lambda)|^2 d\mu(\lambda) < \infty \right. \right\} \tag{A.2}$$

and denote by $\hat{\mathcal{B}}$ its restriction on

$$\text{Dom}(\hat{\mathcal{B}}) = \left\{ f \in \text{Dom}(\mathcal{B}) \left| \int_{\mathbb{R}} f(\lambda)d\mu(\lambda) = 0 \right. \right\}. \tag{A.3}$$
We remark that the membership \( f \in \text{Dom}(\mathcal{B}) \) means that
\[
\int_{\mathbb{R}} (1 + \lambda^2)|f(\lambda)|^2d\mu(\lambda) < \infty.
\]
Therefore, using Cauchy-Schwarz,
\[
\int_{\mathbb{R}} |f(\lambda)|d\mu(\lambda) \leq \left( \int_{\mathbb{R}} (1 + \lambda^2)|f(\lambda)|^2d\mu(\lambda) \right)^{1/2} \left( \int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} \right)^{1/2} = \left( \int_{\mathbb{R}} (1 + \lambda^2)|f(\lambda)|^2d\mu(\lambda) \right)^{1/2} < \infty.
\]
(A.4)

Inequality (A.4) shows that the following unbounded functional \( \ell \) on \( \text{Dom}(\ell) = \text{Dom}(\mathcal{B}) \) given by
\[
\ell(f) = \int_{\mathbb{R}} f(\lambda)d\mu(\lambda), \quad f \in \text{Dom}(\ell)
\]
is well defined and hence the restriction \( \mathcal{B} \) on \( \text{Dom}(\mathcal{B}) \) given by (A.3) is well defined as well.

**Theorem A.1.** Suppose that \( (\mathcal{B}, \mathcal{B}) \) is the model pair in the Hilbert space \( L^2(\mathbb{R}; d\mu) \) as defined by (A.2) and (A.3). Then \( \mathcal{B} \) is a prime, densely defined, closed, symmetric operator with deficiency indices \((1, 1)\).

**Proof.** Since \( \mathcal{B} \) is a restriction of a self-adjoint operator, the operator \( \mathcal{B} \) is symmetric.

To show that \( \mathcal{B} \) is closed, assume that \( \{f_n\}_{n=1}^\infty \) is a sequence such that \( f_n \in \text{Dom}(\mathcal{B}) \) for all \( n \) and
\[
\lim_{n \to \infty} f_n = f \quad \text{and} \quad \lim_{n \to \infty} \mathcal{B}f_n = g
\]
for some \( f, g \in L^2(\mathbb{R}; d\mu) \). Since \( \mathcal{B} \) is a restriction of the self-adjoint operator \( \mathcal{B} \), one gets that
\[
\lim_{n \to \infty} f_n = f \quad \text{and} \quad \lim_{n \to \infty} \mathcal{B}f_n = g
\]
and hence \( f \in \text{Dom}(\mathcal{B}) \) and \( g = \mathcal{B}f \) for \( \mathcal{B} \) is a closed operator.

Since \( f_n \in \text{Dom}(\mathcal{B}) \), one gets that
\[
\int_{\mathbb{R}} f_n(\lambda)d\mu(\lambda) = \int_{\mathbb{R}} (\lambda - i)f_n(\lambda) \frac{1}{\lambda + i}d\mu(\lambda) = 0
\]
which means that \( (\mathcal{B} - iI)f_n \) is orthogonal to the element \( h \in L^2(\mathbb{R}; d\mu) \) given by
\[
h(\lambda) = \frac{1}{\lambda + i}.
\]
Therefore,
\[ \lim_{n \to \infty} ((\mathcal{B} - iI)f_n, h) = ((\mathcal{B} - iI)f, h) = 0 \]
which means that \( \int_{\mathbb{R}} f(\lambda) d\mu(\lambda) = 0 \) and hence \( f \in \text{Dom}(\hat{\mathcal{B}}) \).

To prove that \( \mathcal{B} \) is densely defined, we observe first, that the quotient space \( \text{Dom}(\mathcal{B})/\text{Dom}(\hat{\mathcal{B}}) \) is one-dimensional. More specifically,
\[ (A.5) \quad \text{Dom}(\mathcal{B}) = \text{Dom}(\hat{\mathcal{B}}) + \text{span}\{g\}, \]
where the function \( g \) is given by
\[ g(\lambda) = \frac{1}{1 + \lambda^2}, \quad \mu\text{-a.e.}. \]
Indeed, for any \( f \in \text{Dom}(\mathcal{B}) \) the function
\[ g(\lambda) = f(\lambda) - \frac{\int_{\mathbb{R}} f(s) d\mu(s)}{1 + \lambda^2} \]
belongs to \( \text{Dom}(\hat{\mathcal{B}}) \), since
\[ \int_{\mathbb{R}} g(\lambda) d\mu(\lambda) = \int_{\mathbb{R}} f(\lambda) d\mu(\lambda) - \int_{\mathbb{R}} f(s) d\mu(s) \int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 0, \]
due to the normalization condition
\[ \int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1. \]

To show that \( \text{Dom}(\hat{\mathcal{B}}) \) is dense in \( L^2(\mathbb{R}; d\mu) \), assume that \( h \perp \text{Dom}(\hat{\mathcal{B}}) \). Given \( n \in \mathbb{N} \), introduce the function
\[ h_n(\lambda) = h(\lambda) \chi_{(-n,n)}(\lambda), \]
where \( \chi_{(-n,n)}(\cdot) \) is the characteristic function of the interval \( (-n, n) \). The function
\[ g_n(\lambda, z) = \frac{h_n(\lambda)}{\lambda - z} - \frac{1}{1 + \lambda^2} \int_{\mathbb{R}} \frac{h_n(s)}{s - z} d\mu(s) \]
obviously belongs to \( \text{Dom}(\hat{\mathcal{B}}) \). In particular,
\[ g_n(\cdot, z) \perp h, \quad n \in \mathbb{N}, \]
which means that
\[ \int_{\mathbb{R}} \frac{h_n(\lambda) h(\lambda)}{\lambda - z} d\mu(\lambda) = \int_{\mathbb{R}} \frac{h(\lambda)}{1 + \lambda^2} d\mu(\lambda) \cdot \int_{\mathbb{R}} \frac{h_n(s)}{s - z} d\mu(s) \]
\[ = \int_{\mathbb{R}} \frac{h_n(\lambda) \overline{h}}{\lambda - z} d\mu(\lambda). \]
Here we used the notation
\[ m = \int_{\mathbb{R}} \frac{h(\lambda)}{1 + \lambda^2} d\mu(\lambda). \]

By the uniqueness theorem for Cauchy integrals, one gets that
\[ h_n(\lambda) h(\lambda) = |h(\lambda)|^2 = h(\lambda) m \quad \text{for } \mu\text{-a.e. } \lambda \in (-n, n). \]

Therefore,
\[ h(\lambda) = m \quad \text{for } \mu\text{-a.e. } \lambda \in (-n, n). \]

Since \( n \) is arbitrary, one concludes that
\[ h(\lambda) = m \quad \mu\text{-a.e. } . \]

According to the hypothesis, the measure \( \mu \) is infinite. Therefore, \( h \in L^2(\mathbb{R}; d\mu) \) only if the constant \( m \) is zero. That is, \( h = 0 \) which proves that \( \text{Dom}(\hat{B}) \) is dense in \( L^2(\mathbb{R}; d\mu) \), and hence, the operator \( \hat{B} \) is densely defined.

Next, we prove that \( \hat{B} \) is a prime operator.

Assume that \( \mathcal{H}_0 \) reduces \( \hat{B} \) and that the part \( \hat{B}|_{\mathcal{H}_0} \) is a self-adjoint operator. Since the self-adjoint multiplication operator \( \mathcal{B} \) in \( \mathcal{H} = L^2(\mathbb{R}; d\mu) \) is an extension of the symmetric operator \( \hat{B} \), the subspace \( \mathcal{H}_0 \) also reduces the self-adjoint operator \( \mathcal{B} \) and hence
\[ \mathcal{H}_0 = \text{Ran} \ E_B(\delta) \quad \text{for some Borel set } \delta \subset \mathbb{R} \]

for \( \mathcal{B} \) has a simple spectrum. Here \( E_B(\cdot) \) denotes the projection-valued spectral measure of the self-adjoint operator \( \mathcal{B} \).

One observes that the function \( f(\cdot) = \chi_{\delta \cap (-n, n)}(\cdot) \) belongs to \( \text{Dom}(\mathcal{B}|_{\mathcal{H}_0}) \subset \text{Dom}(\hat{B}) \). In particular,
\[ \int_{(-n,n) \cap \delta} d\mu(\lambda) = \mu((-n, n) \cap \delta) = 0 \quad \text{for all } n \in \mathbb{N}. \]

Hence, \( \mu(\delta) = 0 \) and therefore, \( \mathcal{H}_0 = 0 \), proving that \( \hat{B} \) is a prime symmetric operator.

Finally, we prove that \( \hat{B} \) has deficiency indices \((1, 1)\).

We claim that
\[ \text{Ker}((\hat{B})^* \mp iI) = \text{span}\{g_\pm\}, \]

where
\[ g_\pm(\lambda) = \frac{1}{\lambda \mp i}, \quad \mu\text{-a.e. } . \]

First, we prove the inclusion
\[ \text{Ker}((\hat{B})^* \mp iI) \subset \text{span}\{g_\pm\}. \]
Indeed, it suffices to show that 
\((\hat{\mathcal{B}} \pm iI)f, g_\pm) = 0\) for all \(f \in \text{Dom}(\hat{\mathcal{B}})\).
One computes
\[
(\hat{\mathcal{B}} \pm iI)f, g_\pm) = \int_\mathbb{R} (\lambda \pm i)f(\lambda)\frac{1}{\lambda \mp i}d\mu(\lambda) = \int_\mathbb{R} f(\lambda)d\mu(\lambda) = 0,
\]
which shows that \(\hat{\mathcal{B}}\) has deficiency indices at least \((1, 1)\). Now the claim follows from (A.5) and the observation that
\[
\frac{1}{2i}(g_+(\lambda) - g_-(\lambda)) = \frac{1}{\lambda^2 + 1}, \quad \mu\text{-a.e.}.
\]

The following result characterizes the core of the spectrum of the symmetric operator \(\hat{\mathcal{B}}\) as the set of non-isolated points of the support of the measure \(\mu\).
Recall that a point \(\lambda \in \mathbb{C}\) is said to be quasi-regular for an operator \(T\) if \(T - \lambda I\) has a continuous inverse on \(\text{Ran}(T - \lambda I)\) and the core of the spectrum of \(T\) is defined to be the complement to the set of its quasi-regular points.

**Theorem A.2.** Assume that \(\hat{\mathcal{B}}\) is a symmetric operator from the model pair in the Hilbert space \(L^2(\mathbb{R}, d\mu)\).
Then a point \(\lambda_0, \lambda_0 \in \mathbb{R}\), is a quasi-regular point for the symmetric operator \(\hat{\mathcal{B}}\) if and only if there exists an \(\varepsilon > 0\) such that
\[
\mu((-\varepsilon + \lambda_0, \lambda_0) \cup (\lambda_0, \lambda_0 + \varepsilon)) = 0.
\]

**Proof.** “Only If” Part. Suppose that (A.6) holds true. If \(\mu(\{\lambda_0\}) = 0\), then the point \(\lambda_0\) belongs to the resolvent set of the self-adjoint operator \(\mathcal{B}\) and hence \(\lambda_0\) is automatically a quasi-regular point for the symmetric restriction \(\hat{\mathcal{B}}\) of \(\mathcal{B}\).
If \(\mu(\{\lambda_0\}) > 0\), one proceeds as follows. Suppose that
\[
\text{(A.7)} \quad u \in \text{Dom}(\hat{\mathcal{B}}).
\]
Then
\[
\|u\|^2 = \int_\mathbb{R} |u(\lambda)|^2d\mu(\lambda) = \int_{|\lambda - \lambda_0| \geq \varepsilon} |u(\lambda)|^2d\mu(\lambda) + |u(\lambda_0)|^2\mu(\{\lambda_0\})
\]
\[
\leq \left( \int_{|\lambda - \lambda_0| \geq \varepsilon} \frac{|u(\lambda)|^2}{(\lambda - \lambda_0)^2}d\mu(\lambda) \right)^{1/2} \left( \int_{|\lambda - \lambda_0| \geq \varepsilon} (\lambda - \lambda_0)^2|u(\lambda)|^2d\mu(\lambda) \right)^{1/2}
\]
\[
+ |u(\lambda_0)|^2\mu(\{\lambda_0\}) \leq \varepsilon^{-1}\|u\| \|\left(\hat{\mathcal{B}} - \lambda_0 I\right)u\| + |u(\lambda_0)|^2\mu(\{\lambda_0\}).
\]
Since (A.7) implies
\[ \int_{|\lambda - \lambda_0| \geq \varepsilon} u(\lambda) d\mu(\lambda) + u(\lambda_0) \mu(\{\lambda_0\}) = 0, \]
and hence
\[ u(\lambda_0) = -\left(\mu(\{\lambda_0\})\right)^{-1} \int_{|\lambda - \lambda_0| \geq \varepsilon} u(\lambda) d\mu(\lambda), \]
one gets the inequality
\[ \|u\|^2 \leq \varepsilon^{-1} \|u\| \|(\hat{B} - \lambda_0 I)u\| + \left(\mu(\{\lambda_0\})\right)^{-1} \left( \int_{|\lambda - \lambda_0| \geq \varepsilon} u(\lambda) d\mu(\lambda) \right)^2 \]
\[ \leq \varepsilon^{-1} \|u\| \|(\hat{B} - \lambda_0 I)u\| + \left(\mu(\{\lambda_0\})\right)^{-1} \int_{|\lambda - \lambda_0| \geq \varepsilon} \frac{d\mu(\lambda)}{(\lambda - \lambda_0)^2} \|(\hat{B} - \lambda_0 I)u\|^2. \]
Thus, the ratio
\[ r = \frac{\|u\|}{\|(\hat{B} - \lambda_0 I)u\|} \]
satisfies the inequality
\[ r^2 \leq \varepsilon^{-1} r + \left(\mu(\{\lambda_0\})\right)^{-1} \int_{|\lambda - \lambda_0| \geq \varepsilon} \frac{d\mu(\lambda)}{(\lambda - \lambda_0)^2}, \]
and therefore \( r \) is uniformly bounded with respect to \( \varepsilon \) which proves that the point \( \lambda_0 \) is a quasi-regular point.

"If" Part. If (A.6) does not hold, and therefore \( \lambda_0 \) is not an isolated point from the support of the measure \( \mu \), we proceed as follows. Take an \( \varepsilon > 0 \) and choose measurable sets \( \delta_{\pm} \) such that
\[ \delta_{\pm} \cup \delta_- = (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \setminus \{\lambda_0\}, \]
with \( \mu(\delta_{\pm}) > 0 \). Set
\[ u_\varepsilon(\lambda) = \left(\mu(\delta_{+})\right)^{-1} \chi_{\delta_{+}}(\lambda) - \left(\mu(\delta_-)\right)^{-1} \chi_{\delta_-}(\lambda). \]
Clearly, \( u_\varepsilon \in L^2(\mathbb{R}; d\mu) \) and
\[ \int_{\mathbb{R}} u_\varepsilon(\lambda) d\lambda = 0. \]
Thus, \( u_\varepsilon \in \text{Dom}(\hat{B}) \). One computes that
\[ \|u_\varepsilon\|^2 = \left(\mu(\delta_{+})\right)^{-1} + \left(\mu(\delta_-)\right)^{-1}. \]
However,
\[ \| (\hat{B} - \lambda_0 I)u_\varepsilon \|^2 \leq \varepsilon^2 \left( (\mu(\delta_+))^{-1} + (\mu(\delta_-))^{-1} \right) = \varepsilon^2 \| u_\varepsilon \|^2 \]
which proves that \( \lambda_0 \) belongs to the core of the spectrum of \( \hat{A} \) since \( \varepsilon \) can be chosen arbitrarily small.

The proof is complete. \( \square \)

REFERENCES


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