A GENERALIZATION OF THE TAN 2Θ THEOREM

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Abstract. Let \( A \) be a bounded self-adjoint operator on a separable Hilbert space \( H \) and \( H_0 \subset H \) a closed invariant subspace of \( A \). Assuming that \( \sup \text{spec} (A_0) \leq \inf \text{spec} (A_1) \), where \( A_0 \) and \( A_1 \) are restrictions of \( A \) onto the subspaces \( H_0 \) and \( H_1 = H \ominus H_0 \), respectively, we study the variation of the invariant subspace \( H_0 \) under bounded self-adjoint perturbations \( V \) that are off-diagonal with respect to the decomposition \( H = H_0 \oplus H_1 \). We obtain sharp two-sided estimates on the norm of the difference of the orthogonal projections onto invariant subspaces of the operators \( A \) and \( B = A + V \). These results extend the celebrated Davis-Kahan tan 2Θ Theorem. On this basis we also prove new existence and uniqueness theorems for contractive solutions to the operator Riccati equation, thus, extending recent results of Adamyan, Langer, and Tretter.

1. Introduction

Given a self-adjoint bounded operator \( A \) and a closed invariant subspace \( H_0 \subset H \) of \( A \) we set \( A_i = A|_{H_i} \), \( i = 0, 1 \) with \( H_1 = H \ominus H_0 \). Assuming that the perturbation \( V \) is off-diagonal with respect to the orthogonal decomposition \( H = H_0 \oplus H_1 \) consider the \( 2 \times 2 \) self-adjoint operator matrix

\[
B = A + V = \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix},
\]

where \( V \) is a bounded operator from \( H_1 \) to \( H_0 \).

In the 1970 paper [8] Davis and Kahan proved that if

\[
\sup \text{spec} (A_0) < \inf \text{spec} (A_1),
\]

then the difference of the spectral projections

\[
P = E_A \left( (-\infty, \sup \text{spec} (A_0)) \right) \quad \text{and} \quad Q = E_B \left( (-\infty, \sup \text{spec} (A_0)) \right)
\]

for the operators \( A \) and \( B \), respectively, corresponding to the interval \( (-\infty, \sup \text{spec} (A_0)) \) admits the estimate

\[
\|P - Q\| \leq \sin \left( \frac{1}{2} \arctan \frac{2\|V\|}{d} \right) < \frac{\sqrt{2}}{2},
\]

where

\[
d = \text{dist}(\text{spec}(A_0), \text{spec}(A_1)).
\]
Estimate (1.2) can be equivalently expressed as the \( \tan 2 \Theta \) Theorem:
\[
\| \tan 2 \Theta \| \leq \frac{2\| V \|}{d}, \quad \text{spec}(\Theta) \subset [0, \pi/4),
\]
where \( \Theta \) is the operator angle between the subspaces \( \text{Ran} P \) and \( \text{Ran} Q \) (see, e.g., [10]).

By known results on graph subspaces (see, e.g., [3], [4], [6], [10]) estimate (1.2) in particular implies that the Riccati equation
\[
A_1X -XA_0 - XVX + V^* = 0
\]
has a contractive solution \( X : \mathcal{H}_0 \to \mathcal{H}_1 \) satisfying the norm estimate
\[
\| X \| = \frac{\| P - Q \|}{\sqrt{1 - \| P - Q \|^2}} \leq \tan \left( \frac{1}{2} \arctan \frac{2\| V \|}{d} \right) < 1.
\]
Moreover, the graph of \( X \), i.e., the subspace \( \mathcal{G}(\mathcal{H}_0, X) := \{ x \oplus Xx | x \in \mathcal{H}_0 \} \), coincides with the spectral subspace \( \text{Ran} \mathcal{E}_B((-\infty, \lambda)) \) of the operator \( B \).

Independently of the work of Davis and Kahan the existence of a unique contractive solution to the Riccati equation under condition (1.1) has been proven by Adamyan and Langer in [1], where the operators \( A_0 \) and \( A_1 \) were allowed to be semibounded. In a recent paper by Adamyan, Langer, and Tretter [2] the existence result has been extended to the case where the spectra of \( A_0 \) and \( A_1 \) intersect at one point \( \lambda \in \mathbb{R} \), that is,
\[
\sup \text{spec}(A_0) = \inf \text{spec}(A_1) = \lambda,
\]
provided that at least one of the following conditions
\[
\text{(1.5)} \quad \text{Ker}(A_0 - \lambda) = \{0\}, \quad \text{Ker}(A_1 - \lambda) \cap \text{Ker} V = \{0\}
\]
or
\[
\text{(1.6)} \quad \text{Ker}(A_1 - \lambda) = \{0\}, \quad \text{Ker}(A_0 - \lambda) \cap \text{Ker} V^* = \{0\}
\]
holds. In this case the Riccati equation (1.3) has been proven to have a unique contractive solution \( X \), which appears to be a strict contraction. The graph of \( X \), as above, coincides with the spectral subspace \( \text{Ran} \mathcal{E}_B((-\infty, \lambda)) \) of the operator \( B \).

Conditions (1.5) and (1.6) are rather restrictive. In particular, in this case \( \lambda \) may be an eigenvalue neither for both \( A_0 \) and \( A_1 \) nor for \( B \).

The main goal of the present article is to drop conditions (1.5) and (1.6) and to carry out the analysis under the only assumption that
\[
\text{(1.7)} \quad \sup \text{spec}(A_0) \leq \lambda \leq \inf \text{spec}(A_1).
\]
Below we will prove (see Theorem 2.4) that under hypothesis (1.7) the \( B \)-invariant subspace
\[
\Omega = \text{Ran} \mathcal{E}_B((-\infty, \lambda)) \oplus (\text{Ker}(A_0 - \lambda) \cap \text{Ker} V^*)
\]
is the graph of a contractive operator \( X : \mathcal{H}_0 \to \mathcal{H}_1 \). Moreover, the norm of the operator \( X \) satisfies the lower bound
\[
\| X \| \geq \frac{\delta}{\| V \|},
\]
where
\[ \delta = \max \{ \inf \text{spec}(A) - \inf \text{spec}(B), \sup \text{spec}(B) - \sup \text{spec}(A) \} \geq 0 \]
is the maximal shift of the edges of the spectrum of the operator \( A \) under the perturbation \( V \). These results can be stated equivalently as the two-sided estimate

\[ \frac{\delta}{\sqrt{\delta^2 + \|V\|^2}} \leq \|P - Q\| \leq \frac{\sqrt{2}}{2}, \]

where \( P \) and \( Q \) are orthogonal projections in \( \mathcal{H} \) onto the subspaces \( \mathcal{H}_0 \) and \( \mathcal{Q} \), respectively. Notice that for the subspace \( \mathcal{Q} \) to be a spectral subspace of \( B \) it is necessary and sufficient that

- either \( \text{Ker}(A_0 - \lambda) \cap \text{Ker} V^* = \{0\} \) or \( \text{Ker}(A_1 - \lambda) \cap \text{Ker} V = \{0\} \),

and then, necessarily,

- either \( \mathcal{Q} = \text{Ran} E_B((-\infty, \lambda)) \) or \( \mathcal{Q} = \text{Ran} E_B((-\infty, \lambda]) \).

The fact that the subspace \( \mathcal{Q} \) is a graph of a contractive operator \( X \) means that the Riccati equation (1.3) has a contractive solution. In contrast to the case studied in [2], the solution \( X \) is in general (under hypothesis (1.7)) neither necessarily strictly contractive nor unique in the set of all contractive solutions to the Riccati equation. Moreover, if the invariant subspace \( \mathcal{Q} \) is not a spectral subspace of the operator \( B \), then \( X \) is a non-isolated point (in the operator norm topology) of the set of all solutions of (1.3).

Below we will prove (see Theorem 4.1) that under assumption (1.7) the operator \( X \) is the unique solution to the Riccati equation (1.3) within the class of bounded linear operators from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \) satisfying the additional requirements that

\[ \text{Ker}(A_0 - \lambda) \cap \text{Ker} V^* \subset \text{Ker} X \quad \text{and} \quad \text{spec}(A_0 + VX) \subset (-\infty, \lambda]. \]

Furthermore, we formulate and prove necessary and sufficient conditions for a contractive solution to be a unique contractive solution (see Theorem 4.3 or/and Theorem 5.4). The solution \( X \) satisfying \( \mathcal{G}(\mathcal{H}_0, X) = \mathcal{Q} \) is shown to be the unique contractive solution to (1.3) if and only if it is strictly contractive (see Corollary 4.4). Note that in the case where the contractive solution is non-unique but its graph is a spectral subspace for the operator \( B \), a complete description of the set of all contractive solutions to the Riccati equation can be given by means of Theorem 6.2 in [10].

The technics developed in the present work to prove that the subspace \( \mathcal{Q} \) (1.8) is the graph of a contractive operator (Theorem 2.4), is an extension of the geometric ideas of Davis and Kahan [7], [8]. The existence and uniqueness results (Theorems 4.4, 5.3, and 5.4) are obtained in the framework of the geometric approach of our recent paper [10]. The previously known results by Davis and Kahan [7], [8], Adamyan and Langer [11], and Adamyan, Langer, and Tretter [2] appear to be their direct corollaries.

A few words about the notations used throughout the paper. Given a linear operator \( A \) on a Hilbert space \( \mathcal{H} \), by \( \text{spec}(A) \) we denote the spectrum of \( A \). If
not explicitly stated otherwise, $\mathcal{N} \perp$ denotes the orthogonal complement in $\mathcal{H}$ of a subspace $\mathcal{N} \subset \mathcal{H}$, i.e., $\mathcal{N} \perp = \mathcal{H} \ominus \mathcal{N}$. The identity operator on $\mathcal{H}$ is denoted by $I_\mathcal{H}$.

The notation $\mathcal{B}(\mathcal{H}, \mathcal{L})$ is used for the set of bounded operators from the Hilbert space $\mathcal{H}$ to the Hilbert space $\mathcal{L}$. Finally, we write $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$.

Acknowledgments. V. Kostrykin is grateful to V. Enss, A. Knauf, and R. Schrader for useful discussions. A. K. Motovilov acknowledges the great hospitality and financial support by the Department of Mathematics, University of Missouri–Columbia, MO, USA. He was also supported in part by the Russian Foundation for Basic Research within Project RFBR 01-01-00958.

2. Upper Bound

Throughout the whole work we adopt the following hypothesis.

**Hypothesis 2.1.** Assume that the separable Hilbert space $\mathcal{H}$ is decomposed into the orthogonal sum of two subspaces

\[(2.1) \quad \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1.\]

Assume, in addition, that $\mathcal{B}$ is a self-adjoint operator on $\mathcal{H}$ represented with respect to the decomposition (2.1) as a $2 \times 2$ operator block matrix

\[\mathcal{B} = \begin{pmatrix} A_0 & V \\ V^* & A_1 \end{pmatrix},\]

where $A_i$ are bounded self-adjoint operators in $\mathcal{H}_i$, $i = 0, 1$, while $V$ is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_0$. More explicitly, $\mathcal{B} = \mathcal{A} + \mathcal{V}$, where $\mathcal{A}$ is the bounded diagonal self-adjoint operator,

\[\mathcal{A} = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix},\]

and the operator $\mathcal{V} = \mathcal{V}^*$ is an off-diagonal bounded operator

\[\mathcal{V} = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}.\]

Moreover, assume that

\[(2.2) \quad \sup \text{spec}(A_0) \leq \lambda \leq \inf \text{spec}(A_1)\]

for some $\lambda \in \mathbb{R}$.

If, under Hypothesis 2.1, $\lambda$ is a multiple eigenvalue of the operator $\mathcal{B}$, then $\mathcal{B}$ has infinitely many invariant subspaces $\mathcal{L}_B$ such that

\[(2.3) \quad \text{Ran}\, E_B((\lambda, \infty)) \subset \mathcal{L}_B \subset \text{Ran}\, E_B((\infty, \lambda])\]

that are necessarily not spectral subspaces.

A criterion for a $\mathcal{B}$-invariant subspace $\mathcal{L}_B$ satisfying (2.3) to be a graph subspace associated with the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is given by the following theorem.
Theorem 2.2. Assume Hypothesis \(2.1\). Then
\[
\text{(2.4)} \quad \text{Ker}(B - \lambda) = (\text{Ker}(A_0 - \lambda) \cap \text{Ker} V^*) \oplus (\text{Ker}(A_1 - \lambda) \cap \text{Ker} V).
\]
Moreover, given a subspace \(\mathfrak{N} \subset \text{Ker}(B - \lambda)\), the subspace
\[
\text{(2.5)} \quad \mathcal{L}_B := \text{Ran} E_B((-\infty, \lambda)) \oplus \mathfrak{N}
\]
is a graph subspace associated with the subspace \(\mathfrak{N}_0\) in the decomposition \(\mathfrak{N} = \mathfrak{N}_0 \oplus \mathfrak{N}_1\) if and only if \(\mathfrak{N}\) is a graph subspace associated with the subspace \(\text{Ker}(A_0 - \lambda) \cap \text{Ker} V^*\) in the decomposition \(\text{(2.4)}\).

Proof. Without loss of generality we may set \(\lambda = 0\). The inclusion
\[
\text{(2.6)} \quad (\text{Ker} A_0 \cap \text{Ker} V^*) \oplus (\text{Ker} A_1 \cap \text{Ker} V) \subset \text{Ker} B
\]
is obvious. In order to prove the opposite inclusion assume that \(\begin{pmatrix} x \\ y \end{pmatrix} \in \text{Ker} B\) with \(x \in \mathfrak{N}_0\) and \(y \in \mathfrak{N}_1\), i.e.,
\[
\text{(2.7)} \quad A_0 x + V y = 0 \quad \text{and} \quad V^* x + A_1 y = 0.
\]
Suppose that \(x \notin \text{Ker} A_0\). Then \((x, A_0 x) < 0\) and therefore \((x, V y) > 0\) using the first equation in \(\text{(2.7)}\). From the second of equations \(\text{(2.7)}\) it follows that \((y, A_1 y) < 0\), which is in a contradiction with \(\text{(2.2)}\). Thus \(x \in \text{Ker} A_0\). Similarly one proves that \(y \in \text{Ker} A_1\). By using \(\text{(2.7)}\) it follows that \(x \in \text{Ker} V^*\) and \(y \in \text{Ker} V\) which together with \(\text{(2.6)}\) proves \(\text{(2.2)}\).

In order to prove the second statement of the theorem notice that by Theorem \(\text{A.1}\) in the Appendix the subspace \(\mathfrak{N} \subset \text{Ker} B\) is a graph subspace associated with the subspace \(\text{Ker} A_0 \cap \text{Ker} V^*\) in the decomposition \(\text{(2.4)}\) (recall that we assumed that \(\lambda = 0\)) if and only if
\[
(\text{Ker} A_0 \cap \text{Ker} V^*) \cap (\text{Ker} B \oplus \mathfrak{N}) = (\text{Ker} A_1 \cap \text{Ker} V) \cap \mathfrak{N} = \{0\}.
\]
Again from Theorem \(\text{A.1}\) it follows that the subspace \(\text{Ran} E_B((-\infty, 0)) \oplus \mathfrak{N}\) is a graph subspace associated with the subspace \(\mathfrak{N}_0\) in the decomposition \(\mathfrak{N} = \mathfrak{N}_0 \oplus \mathfrak{N}_1\) if and only if
\[
\text{(2.8)} \quad \mathfrak{N}_0 \cap (\text{Ran} E_B((-\infty, 0)) \oplus \mathfrak{N}) = \mathfrak{N}_1 \cap (\text{Ran} E_B((-\infty, 0)) \oplus \mathfrak{N}) = \{0\}.
\]
Therefore, to complete the proof it is sufficient to establish the following equalities
\[
\text{(2.9)} \quad \mathfrak{N}_0 \cap (\text{Ran} E_B((-\infty, 0)) \oplus \mathfrak{N}) = (\text{Ker} A_0 \cap \text{Ker} V^*) \cap (\text{Ker} B \oplus \mathfrak{N})
\]
and
\[
\text{(2.10)} \quad \mathfrak{N}_1 \cap (\text{Ran} E_B((-\infty, 0)) \oplus \mathfrak{N}) = (\text{Ker} A_1 \cap \text{Ker} V) \cap \mathfrak{N}.
\]
First, we prove that the left-hand side of \(\text{(2.9)}\) is a subset of the right-hand side of \(\text{(2.9)}\), i.e.,
\[
\text{(2.11)} \quad \mathfrak{N}_0 \cap (\text{Ran} E_B((-\infty, 0)) \oplus \mathfrak{N}) \subset (\text{Ker} A_0 \cap \text{Ker} V^*) \cap (\text{Ker} B \oplus \mathfrak{N}).
\]
Let \(x \in \mathfrak{N}_0\) and \(x \perp \text{Ran} E_B((-\infty, 0)) \oplus \mathfrak{N}\). Clearly,
\[
(x, B x) \geq 0 \quad \text{for} \quad x \perp \text{Ran} E_B((-\infty, 0)) \oplus \mathfrak{N}.
\]
Moreover,
\[(x, Bx) = (x, Ax) \leq 0 \quad \text{for all} \quad x \in \mathcal{H}_0,
\]
since the operator matrix \(V\) is off-diagonal and the subspace \(\mathcal{H}_0\) is \(A\)-invariant. Hence,
\[(x, Ax) = (x, Bx) = 0 \quad \text{for} \quad x \in \mathcal{H}_0 \cap (\operatorname{Ran} E_B((\infty, 0)) \oplus \mathfrak{N})^\perp,
\]
which implies \(x \in \ker B \oplus \mathfrak{N}\) by the variational principle. Therefore, the inclusion (2.11) holds.

The opposite inclusion
\[\mathcal{H}_0 \cap (\operatorname{Ran} E_B((\infty, 0)) \oplus \mathfrak{N})^\perp \supset (\ker A_0 \cap \ker V^*) \cap (\ker B \oplus \mathfrak{N})\]
is obvious, which proves (2.9).

The equality (2.10) is proven in a similar way. □

Remark 2.3. As it follows from (2.4) the closed \(B\)-invariant subspace
\[\Omega = \operatorname{Ran} E_B((\infty, \lambda)) \oplus (\operatorname{Ran} E_B(\lambda) \cap \mathcal{H}_0) \subset \mathcal{H}.
\]
is a spectral subspace for the operator \(B\) if and only if
\begin{align*}
\text{either} & \quad \ker (A_0 - \lambda) \cap \ker V^* = \{0\} \quad \text{or} \quad \ker (A_1 - \lambda) \cap \ker V = \{0\}.
\end{align*}

The following theorem characterizes the subspace \(\Omega\) as the graph of some contractive operator \(X\) from \(\mathcal{H}_0\) to \(\mathcal{H}_1\), that is, \(\Omega = \mathcal{G}(\mathcal{H}_0, X)\) where
\[\mathcal{G}(\mathcal{H}_0, X) = \{x \oplus Xx | x \in \mathcal{H}_0\}.
\]

**Theorem 2.4.** Assume Hypothesis 2.1. Then:

(i) The \(B\)-invariant subspace \(\Omega\) is a graph subspace \(\mathcal{G}(\mathcal{H}_0, X)\) associated with the subspace \(\mathcal{H}_0\) in decomposition (2.1), where the operator \(X\) is a contraction, \(\|X\| \leq 1\), with the properties
\begin{align*}
\text{(i)} & \quad \ker (A_0 - \lambda) \cap \ker V^* \subset \ker X, \quad \ker (A_1 - \lambda) \cap \ker V \subset \ker X^*.
\end{align*}

(ii) If either \(\ker (A_0 - \lambda) = \{0\}\) or \(\ker (A_1 - \lambda) = \{0\}\), then \(X\) is a strict contraction, i.e.,
\[\|Xf\| < \|f\|, \quad f \neq 0.
\]

(iii) If
\[d = \operatorname{dist}(\text{spec}(A_0), \text{spec}(A_1)) > 0,
\]
then \(X\) is a uniform contraction satisfying the estimate
\[\|X\| \leq \tan \left(\frac{1}{2} \arctan \frac{2\|V\|}{d}\right) < 1.
\]

**Remark 2.5.** In Section 4 we will establish necessary and sufficient conditions guaranteeing that the operator \(X\) referred to in Theorem 2.4 is a strict contraction (see Corollary 4.4 below). The claim (ii) of Theorem 2.4 will then appear to be a corollary of this more general result.
Proof of Theorem 2.4. Without loss of generality we assume that \( \lambda = 0 \).

(i) Step 1. First we prove the assertion under the additional assumption that \( \text{Ker} \, B = \{0\} \).

By Theorem 2.2, the subspace \( \Omega \) is a graph subspace associated with the decomposition \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \), i.e.,

\[
\Omega = \mathcal{H}(\mathcal{H}_0, \mathcal{X}),
\]

where \( \mathcal{X} \) is a (possibly unbounded) densely defined closed operator from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \). Let \( X = U|X| \) be the polar decomposition for \( X \), where \( U : \mathcal{H}_0 \to \mathcal{H}_1 \) is a partial isometry with the initial subspace \( \text{Ker} \, X \) and the final subspace \( \text{Ran} \, X \) and \( |X| = (X^*X)^{1/2} \), the absolute value of \( X \).

First we show that

\[
|X|f = \mu f, \quad 0 \neq f \in \text{Dom}(|X|).
\]

By \( \text{spec}_{pp}(\mathcal{X}) \subset [0, 1] \),

\[
\text{spec}_{pp}(|X|) \subset [0, 1],
\]

where \( \text{spec}_{pp}(|X|) \) denotes the set of all eigenvalues of \( |X| \).

Let \( 0 \neq \mu \in \text{spec}_{pp}(\mathcal{X}) \) and \( f \) be an eigenvector of \( |X| \) corresponding to the eigenvalue \( \mu \), i.e.,

\[
|X|f = \mu f, \quad 0 \neq f \in \text{Dom}(|X|).
\]

By \( \text{spec}_{pp}(\mathcal{X}) \subset [0, 1] \),

\[
F = f \oplus Xf = f \oplus \mu Uf \in \Omega.
\]

using \( \text{spec}_{pp}(\mathcal{X}) \subset [0, 1] \). Since \( f \perp \text{Ker} \, X = \text{Ker} \, U \), the element \( f \) belongs to the initial subspace of the isometry \( U \). Moreover,

\[
Uf = \mu^{-1} U|X|f \in \text{Ran} \, X \subset \overline{\text{Ran} \, X},
\]

i.e., the element \( Uf \) belongs to the final subspace of \( U \) and hence \( U^*Uf = f \), which, in particular, proves that \( Uf \in \text{Dom}(X^*) \). Therefore,

\[
G = (-X^*Uf) \oplus Uf = (-\mu f) \oplus Uf \in \Omega^\perp.
\]

Using \( \text{Ker} \, B = \{0\} \) and the hypothesis \( \text{Ker} \, B = \{0\} \) one obtains the following two strict inequalities

\[
0 > (F, BF) = (f, A_0 f) + 2\mu \text{Re}(V^*f, Uf) + \mu^2 (Uf, A_1 Uf),
\]

\[
0 < (G, BG) = \mu^2 (f, A_0 f) - 2\mu \text{Re}(V^*f, Uf) + (Uf, A_1 Uf).
\]

If \( \mu > 0 \) satisfies \( \text{spec}_{pp}(\mathcal{X}) \subset [0, 1] \) and \( \text{spec}_{pp}(\mathcal{X}) \subset [0, 1] \), then necessarily \( \mu \leq 1 \). In order to see that we subtract \( 2.22 \) from \( 2.21 \) getting the inequality

\[
(1 - \mu^2) ((Uf, A_1 Uf) - (f, A_0 f)) > 4\mu \text{Re}(V^*f, Uf).
\]

Since \( A_0 \leq 0 \) and \( A_1 \geq 0 \), equation \( 2.23 \) implies \( \text{Re}(V^*f, Uf) < 0 \) for \( \mu > 1 \) which contradicts the orthogonality of the elements \( F \) and \( BG \):

\[
(F, BG) = \mu ((Uf, A_1 Uf) - (f, A_0 f)) + (1 - \mu^2) \text{Re}(V^*f, Uf) = 0.
\]

Hence, \( 2.17 \) is proven.

Our next goal is to prove that the operator \( X \) is a contraction.
Let \( \{ \mathcal{P}_n^{(0)} \}_{n \in \mathbb{N}} \) and \( \{ \mathcal{P}_n^{(1)} \}_{n \in \mathbb{N}} \) be two sequences of finite-dimensional orthogonal projections such that \( \text{Ran} \mathcal{P}_n^{(0)} \subseteq \mathcal{S}_0 \), \( \text{Ran} \mathcal{P}_n^{(1)} \subseteq \mathcal{S}_1 \), and

\[
\text{s-lim}_{n \to \infty} \mathcal{P}_n^{(0)} = P, \quad \text{s-lim}_{n \to \infty} \mathcal{P}_n^{(1)} = P^\perp,
\]

where \( P \) is the orthogonal projection from \( \mathcal{S}_0 \) onto \( \mathcal{S}_0 \) and

\[
\text{s-lim}_{n \to \infty} E_{A_n}(\{0\}) = E_A(\{0\}),
\]

\[
\text{s-lim}_{n \to \infty} E_{A_n}((-\infty, 0)) = E_A((-\infty, 0)),
\]

where

\[
A_n = \begin{pmatrix} \mathcal{P}_n^{(0)} A_0 \mathcal{P}_n^{(0)} & 0 \\ 0 & \mathcal{P}_n^{(1)} A_1 \mathcal{P}_n^{(1)} \end{pmatrix}
\]

are the corresponding finite-dimensional truncations of the operator \( A \). The existence of such sequences can easily be shown by splitting off the subspaces \( \text{Ker} A_0 \) and \( \text{Ker} A_1 \).

Introducing the finite rank operators

\[
V_n = \begin{pmatrix} 0 & \mathcal{P}_n^{(0)} V \mathcal{P}_n^{(1)} \\ \mathcal{P}_n^{(1)} V^* \mathcal{P}_n^{(0)} & 0 \end{pmatrix}
\]

one concludes (see, e.g., Theorem I.5.2 in [5]) that

\[
\text{s-lim}_{n \to \infty} (A_n + V_n) = B.
\]

Since \( \text{Ker} \mathcal{B} = \{0\} \), (2.28) implies (see, e.g., Theorem VIII.24 in [12])

\[
\text{s-lim}_{n \to \infty} E_{A_n + V_n}(\infty, 0) = E_B(\{0\}),
\]

and hence

\[
\text{s-lim}_{n \to \infty} E_{A_n + V_n}(0, \infty) = E_B(0, \infty),
\]

\[
\text{s-lim}_{n \to \infty} E_{A_n + V_n}(\{0\}) = E_B(\{0\}) = 0.
\]

Let \( \tilde{A}_n \) and \( \tilde{V}_n \) denote the parts of the operators \( A_n \) and \( V_n \) associated with their invariant finite dimensional subspace \( \mathcal{S}_n = \mathcal{S}_0^{(n)} \oplus \mathcal{S}_1^{(n)} \), where \( \mathcal{S}_0^{(n)} = \text{Ran} \mathcal{P}_n^{(0)} \) and \( \mathcal{S}_1^{(n)} = \text{Ran} \mathcal{P}_n^{(1)} \). By Theorem 2.2 the subspace (of the finite dimensional Hilbert space \( \tilde{\mathcal{S}}_n \))

\[
\text{Ran} E_{\tilde{A}_n + \tilde{V}_n}(\infty, 0) \oplus (\text{Ker} (\tilde{A}_n + \tilde{V}_n) \cap \mathcal{S}_0^{(n)}) \subset \mathcal{S}_n
\]

is a graph subspace

\[
\mathcal{G}(\text{Ran} E_{\tilde{A}_n}(\infty, 0) \oplus (\text{Ker} (\tilde{A}_n) \cap \mathcal{S}_0^{(n)}), X_n)
\]

for some \( X_n \in \mathcal{B}(\mathcal{S}_0^{(n)}, \mathcal{S}_1^{(n)}) \), \( n \in \mathbb{N} \). Since \( X_n \) is of finite rank, \( \|X_n\| \leq 1 \) by (2.17).
Applying Theorem A.2 in the Appendix one arrives at the inequality

\[
(2.31) \quad \|E_{\hat{A}_n + \hat{V}_n}((-\infty, 0)) + \hat{S}^{(n)} - E_{\hat{A}_n}((-\infty, 0)) + \hat{T}^{(n)}\| = \frac{\|X_n\|}{\sqrt{1 + \|X_n\|^2}} \leq \frac{\sqrt{2}}{2},
\]

where \(\hat{S}^{(n)}\) and \(\hat{T}^{(n)}\) are the orthogonal projections in \(\hat{\mathcal{H}}^{(n)}\) onto the subspaces \(\text{Ker}(\hat{A}_n + \hat{V}_n) \cap \hat{\mathcal{H}}_0^{(n)}\) and \(\text{Ker}(\hat{A}_n) \cap \hat{\mathcal{H}}_0^{(n)}\), respectively.

The subspaces \(\text{Ran}(\hat{S}^{(n)})\) and \(\text{Ran}(\hat{T}^{(n)})\) of the space \(\hat{\mathcal{H}}\) are naturally imbedded into the total Hilbert space \(\hat{\mathcal{H}}\). Denoting by \(\hat{S}^{(n)}\) and \(\hat{T}^{(n)}\) the corresponding orthogonal projections in \(\hat{\mathcal{H}}\) onto these subspaces, (2.31) yields the estimate

\[
(2.32) \quad \|E_{\hat{A}_n + \hat{V}_n}((-\infty, 0)) + \hat{S}^{(n)} - E_{\hat{A}_n}((-\infty, 0)) + \hat{T}^{(n)}\| \leq \frac{\sqrt{2}}{2}.
\]

From (2.26) it follows that

\[
(2.33) \quad \lim_{n \to \infty} \| \hat{T}^{(n)} \| = 0.
\]

Meanwhile, by (2.30)

\[
(2.34) \quad \lim_{n \to \infty} \| \hat{S}^{(n)} \| = 0.
\]

Combining (2.32) – (2.34) and passing to the limit \(n \to \infty\), by the lower semi-continuity of the spectrum (see, e.g., [9], Sec. VIII.1.2) one concludes that

\[
\|P - Q\| = \frac{\|X\|}{\sqrt{1 + \|X\|^2}} \leq \frac{\sqrt{2}}{2},
\]

where \(Q\) is the orthogonal projection in \(\mathcal{H}\) onto the subspace \(\Omega\) (2.12). This proves that the operator \(X\) is a contraction. The proof of (i) under the additional assumption that \(\text{Ker} B = \{0\}\) is complete.

Step 2. Assume now that \(\text{Ker} B\) is not necessarily trivial. From Theorem 2.2 it follows that the subspace \(\text{Ker} B\) is \(A\)-invariant. Denote by \(\hat{A}\) and \(\hat{B}\) the corresponding parts of the operators \(A\) and \(B\) associated with the reducing subspace \(\hat{\mathcal{H}} = \text{Ran} E_B(\mathbb{R} \setminus \{0\})\). Clearly, the operator \(\hat{B}\) is an off-diagonal perturbation of the diagonal operator matrix \(\hat{A}\) with respect to the decomposition \(\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 \oplus \hat{\mathcal{H}}_1\), where

\[
\hat{\mathcal{H}}_0 = \text{Ran} E_B(\mathbb{R} \setminus \{0\}) \cap \hat{\mathcal{H}}_0,
\]

\[
\hat{\mathcal{H}}_1 = \text{Ran} E_B(\mathbb{R} \setminus \{0\}) \cap \hat{\mathcal{H}}_1,
\]

and \(\text{Ker} \hat{B} = \{0\}\). Moreover, Hypothesis 2.1 is satisfied with the replacements \(\hat{\mathcal{H}} \to \hat{\mathcal{H}}_0, \hat{\mathcal{H}}_0 \to \hat{\mathcal{H}}_0, \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_1, \text{and} A \to \hat{A}, B \to \hat{B}\).

By the first part of the proof the subspace \(\text{Ran} E_B((-\infty, 0))\), naturally imbedded into the Hilbert space \(\hat{\mathcal{H}}_1\) is a contraction \(\hat{X}\),

\[
(2.35) \quad \hat{X} : \hat{\mathcal{H}}_0 \to \hat{\mathcal{H}}_1.
\]

Clearly, the \(B\)-invariant subspace

\[
\text{Ran} E_B((-\infty, 0)) \oplus (\text{Ran} E_B(\{0\}) \cap \hat{\mathcal{H}}_0)
\]
is a graph subspace $G(H_0, X)$ associated with the subspace $H_0$ in decomposition (2.1) where the operator $X$ is given by

\[
X f = \begin{cases} \hat{X} f & \text{if } f \in \text{Ran} \mathcal{E}_B((-\infty, 0)) \text{ (naturally imbedded into } \hat{H}) , \\ 0 & \text{if } f \in \text{Ran} \mathcal{E}_B(\{0\}) \cap H_0 = \text{Ker} A_0 \cap \text{Ker} V^* . \end{cases}
\]

Since $\hat{X}$ is a contraction by hypothesis, the operator $X$ is also a contraction satisfying the properties (2.13) (with $\lambda = 0$):

\[
\text{Ker} A_0 \cap \text{Ker} V^* \subset \text{Ker} X \quad \text{and} \quad \text{Ran} \mathcal{E}_B(\{0\}) \cap H_0 = \text{Ker} A_0 \cap \text{Ker} V \subset \text{Ker} X^* .
\]

The proof of (i) is complete.

(ii) If at least one of the subspaces $\text{Ker} A_0$ or $\text{Ker} A_1$ is trivial, then (in the notations above) we have that

\[
(U f, A_1 U f) - (f, A_0 f) > 0,
\]

and, therefore, equality (2.24) cannot be satisfied for $\mu = 1$. Hence $\mu = 1$ is not a singular number of the contraction $X$ which proves that the operator $X$ is a strict contraction, i.e.,

\[
\|X f\| < \|f\| , \quad \text{for any } f \neq 0 .
\]

The proof of (ii) is complete.

(iii) Under Hypotheses 2.1 (with $\lambda = 0$) the fact that the spectra of the operators $A_0$ and $A_1$ are separated, i.e., $d = \text{dist} \{\text{spec} (A_0), \text{spec} (A_1)\} > 0$, means that at least one of the subspaces $\text{Ker} A_0$ and $\text{Ker} A_1$ is trivial. Therefore, the following estimate holds

\[
d \|f\|^2 < \left( (U f, A_1 U f) - (f, A_0 f) \right)
\]

and, hence, from (2.24) one derives the inequality

\[
d < \frac{\mu^2 - 1}{\mu} \text{Re}(V^* f, U f) \leq \frac{1 - \mu^2}{\mu} \|V\| ,
\]

which proves that the operator $X$ does not have singular values outside the interval $[0, \nu)$, where

\[
\nu = \tan \left( \frac{1}{2} \text{arctan} \frac{2 \|V\|}{d} \right) < 1 .
\]

Using the same strategy as in the proof of (ii) one arrives to the conclusion that $X$ is a uniform contraction satisfying the norm estimate

\[
\|X\| \leq \tan \left( \frac{1}{2} \text{arctan} \frac{2 \|V\|}{d} \right) < 1 ,
\]

which proves the upper bound (2.15).

**Remark 2.6.** The operator $X$ referred to in Theorem 2.4 is a contractive solution to the Riccati equation

\[
A_1 X - X A_0 - X V X + V^* = 0
\]

with the property that

\[
\text{spec} (A_0 + V X) \subset (-\infty, \lambda] ,
\]
since the operator $A_0 + VX$ is similar to the part of $B$ associated with the subspace $Q$ (see, e.g., [3]) and $\text{sup}(\text{spec}(B|_Q)) \leq \lambda$ by definition (2.12) of the invariant subspace $Q$. The similarity of $A_0 + VX$ and $B|_Q$ can also be seen directly from the identity

$$(x + Xx, B(x + Xx)) = (x, (A_0 + X^*V^*)(I + X^*X)x) = (x, (I + X^*X)(A_0 + VX)x)$$
valid for any $x \in \mathcal{F}_0$.

The result of Theorem A.2 in the Appendix shows that Theorem 2.4 admits the following equivalent formulation in terms of the corresponding spectral projections.

**Theorem 2.7.** Assume Hypothesis 2.1. Denote by $Q$ the orthogonal projection in $\mathcal{H}_0$ onto the subspace $Q$ (2.12) and by $P$ the orthogonal projection onto $\mathcal{H}_0$. Then:

(i) $\|P - Q\| \leq \sqrt{2}/2$,

(ii) If either $\text{Ker} (A_0 - \lambda) = \{0\}$ or $\text{Ker} (A_1 - \lambda) = \{0\}$, then

$$\pm \sqrt{\frac{d}{2}} \notin \text{spec}_{pp}(P - Q).$$

(iii) If

$$d = \text{dist}(\text{spec}(A_0), \text{spec}(A_1)) > 0,$$

then

$$\|P - Q\| \leq \sin \left( \frac{1}{2} \arctan \frac{2 \|V\|}{d} \right) < \sqrt{\frac{2}{2}}.$$

**Remark 2.8.** Note that if (2.40) holds, then the interval $(\text{sup} \text{spec}(A_0), \text{inf} \text{spec}(A_1))$ belongs to the resolvent set of the operator $B$. Although this fact is well known (see [11, 3]) we present a particularly simple and short alternative proof.

Given $\lambda \in (\text{sup} \text{spec}(A_0), \text{inf} \text{spec}(A_1))$, the subspace $\text{Ran}_B((-\infty, \lambda))$ is a graph subspace $\mathcal{G}(\mathcal{F}_0, X_\lambda)$ where $X_\lambda$ is a strictly contractive solution to the Riccati equation (2.38). By a uniqueness result (see Corollary 6.4 (i) in [10]) the solution $X_\lambda$ does not depend on

$\lambda \in (\text{sup} \text{spec}(A_0), \text{inf} \text{spec}(A_1)).$

Therefore, $E_B((\text{sup} \text{spec}(A_0), \text{inf} \text{spec}(A_1))) = 0$ which proves the claim.

As a by-product of our considerations we also get the following important properties of the subspaces $\text{Ker}(I_{\mathcal{F}_0} - X^*X)$ and $\text{Ker}(I_{\mathcal{F}_1} - XX^*)$. They will be used in Sections 4 and 5 below.

**Lemma 2.9.** Let $X$ be the operator referred to in Theorem 2.4. Then

$$\text{Ker}(I_{\mathcal{F}_0} - X^*X) \subset \text{Ker}(A_0 - \lambda),$$

$$\text{Ran}X|_{\text{Ker}(I_{\mathcal{F}_0} - X^*X)} \subset \text{Ker}(A_1 - \lambda),$$
and

$$\text{Ker}(I_{\mathcal{F}_0} - X^*X) \subset \text{Ker}(XVX - V^*).$$
Moreover, the subspace $\text{Ker} (I_{\delta_0} - X^*X)$ reduces both the operators $VX$ and $VV^*$. Similarly,

$$\text{Ker} (I_{\delta_1} - XX^*) \subset \text{Ker} (A_1 - \lambda),$$

$$\text{Ran} X^*|_{\text{Ker} (I_{\delta_1} - XX^*)} \subset \text{Ker} (A_0 - \lambda),$$

$$\text{Ker} (I_{\delta_1} - XX^*) \subset \text{Ker} (X^*V^*X^* - V),$$

and the subspace $\text{Ker} (I_{\delta_1} - XX^*)$ reduces both the operators $X^*V^*$ and $V^*V$.

**Proof.** Let $0 \neq f \in \text{Ker} (I_{\delta_0} - X^*X)$, that is, $|X|f = f$, where $|X| = (X^*X)^{1/2}$. Then the elements $f \oplus Uf$ and $B((-f) \oplus Uf)$ are orthogonal, where $U$ is the partial isometry from the polar decomposition $X = U|X|$ (see the proof of Theorem 2.4 part (i)). This means that

$$\langle Uf, (A_1 - \lambda)Uf \rangle = \langle f, (A_0 - \lambda)f \rangle,$$

which implies (2.42) and (2.43) (under Hypothesis 2.1).

In order to prove (2.44) and that the subspace Ker$(I_{\delta_0} - X^*X)$ is both VX- and $VV^*$-invariant we proceed as follows.

By Remark 2.6 the operator $X$ solves the Riccati equation

$$A_1X - XA_0 - VX + V^* = 0$$

and, hence,

$$X^*A_1 - A_0X^* - X^*V^*X^* + V = 0,$$

which in particular implies that

$$X^*A_1X - A_0X^*X - X^*V^*X^*X + VX = 0.$$

For any $f \in \text{Ker} (I_{\delta_0} - X^*X)$ inclusions (2.42) and (2.43) yield

$$(A_1X - XA_0)f = (X^*A_1X - A_0)f = (X^*A_1X - A_0X^*X)f = 0.$$

Thus, from (2.49) and (2.51) it follows that

$$V^*f = VXf, \quad f \in \text{Ker} (I_{\delta_0} - X^*X),$$

which proves (2.44) and the representation

$$VXf = X^*V^*X^*Xf = X^*V^*f, \quad f \in \text{Ker} (I_{\delta_0} - X^*X).$$

Combining (2.52) and (2.53) proves that

$$(I_{\delta_0} - X^*X)VXf = VXf - X^*XVXf = VXf - X^*V^*f = 0$$

for any $f \in \text{Ker} (I_{\delta_0} - X^*X)$. That is, the subspace Ker$(I_{\delta_0} - X^*X)$ is VX-invariant. From (2.53) it follows that Ker$(I_{\delta_0} - X^*X)$ is also $X^*V^*$-invariant and, hence, Ker$(I_{\delta_0} - X^*X)$ reduces the operator VX.

Equality (2.54) implies that

$$VV^*f = VXVXf, \quad f \in \text{Ker} (I_{\delta_0} - X^*X).$$
which proves that \( \ker(I_{H_0} - X^*X) \) is \( VV^* \)-invariant, since \( \ker(I_{H_0} - X^*X) \) is already proven to be a \( VX \)-invariant subspace. Since \( VV^* \) is self-adjoint, the subspace \( \ker(I_{H_0} - X^*X) \) reduces \( VV^* \). The proof of (2.45), (2.46), and (2.47) is similar. \( \square \)

**Remark 2.10.** It follows from Lemma 2.9 that the multiplicity \( m \) of the singular value \( \mu = 1 \) of the operator \( X \) satisfies the inequality

\[
(2.55) \quad m \leq \min \{ \dim \ker(A_0 - \lambda), \dim \ker(A_1 - \lambda) \}.
\]

Equivalently,

\[
(2.56) \quad \dim \ker\left(Q - P - \frac{\sqrt{2}}{2}\right) = \dim \ker\left(Q - P + \frac{\sqrt{2}}{2}\right) \leq \min \{ \dim \ker(A_0 - \lambda), \dim \ker(A_1 - \lambda) \},
\]

where \( P \) and \( Q \) are orthogonal projections in \( H \) onto the subspaces \( H_0 \) and \( \Omega \), respectively. The subspaces \( \ker(I_{H_0} - XX^*) \) and \( \ker(I_{H_1} - XX^*) \) will be studied in Section 5 below.

3. Lower Bound

In this section we derive the lower bound on the norm of the difference of the orthogonal projections onto the \( A \)-invariant subspace \( H_0 \) and the \( B \)-invariant subspace \( \Omega \) given by (2.12).

**Theorem 3.1.** Assume Hypothesis 2.1. Let \( \delta_-(\delta_+) \) denote the shift of the bottom (top, respectively) of the spectrum of the operator \( A \) under the perturbation \( V \), i.e.,

\[
\delta_- = \inf \text{spec}(A) - \inf \text{spec}(B), \quad \delta_+ = \sup \text{spec}(B) - \sup \text{spec}(A).
\]

Then the solution \( X \) to the Riccati equation referred to in Theorem 2.4 satisfies the lower bound

\[
(3.1) \quad \|X\| \geq \frac{\delta}{\|V\|},
\]

where

\[
\delta = \max \{\delta_-, \delta_+\} \leq \|V\|.
\]

Equivalently,

\[
(3.2) \quad \|Q - P\| \geq \frac{\delta}{\sqrt{\delta^2 + \|V\|^2}},
\]

where \( P \) and \( Q \) are orthogonal projections onto the subspace \( H_0 \) and \( \Omega \), respectively.

**Remark 3.2.** From a general perturbation theory for off-diagonal perturbations it follows that \( \delta_- \geq 0 \) and \( \delta_+ \geq 0 \). For the proof of this fact we refer to [11].
Proof of Theorem 3.1. From Theorem 2.4 it follows that \( \text{Ran} \mathbf{E}_B ((-\infty, \lambda)) \oplus (\mathfrak{H}_0 \cap \text{Ker} V^*) \) is the graph subspace \( \mathcal{G}(H_0, X) \) associated with the subspace \( \mathfrak{H}_0 \) in the decomposition \( H = H_0 \oplus H_1 \) and by Remark 2.6 the operator \( X \) solves the Riccati equation (2.38). It is well known (see, e.g., [3]) that in this case

\[
\text{spec}(B) = \text{spec}(A_0 + VX) \cup \text{spec}(A_1 - X^* V^*)
\]

and, hence,

\[
\inf \text{spec}(B) \geq \inf \text{spec}(A) - \|V\|\|X\|
\]

and

\[
\sup \text{spec}(B) \leq \sup \text{spec}(A) + \|V\|\|X\|,
\]

which proves the lower bounds (3.1) and (3.2) using Theorem A.2 in the Appendix. \( \Box \)

Remark 3.3. Let \( \mathfrak{H}_0 = \mathfrak{H}_1 = \mathbb{C}, A_0 \) and \( A_1 \) are reals with \( A_1 > A_0 \), and \( V \in \mathbb{C} \). Then

\[
\text{spec}\left(\begin{array}{cc} A_0 & V \\ V^* & A_1 \end{array}\right) = \left\{(A_0 - \|V\| \tan \left( \frac{1}{2} \arctan \frac{2\|V\|}{A_1 - A_0} \right), A_1 + \|V\| \tan \left( \frac{1}{2} \arctan \frac{2\|V\|}{A_1 - A_0} \right) \right\},
\]

which can easily be seen by solving the characteristic equation

\[
(A_0 - \lambda)(A_1 - \lambda) - \|V\|^2 = 0.
\]

The upper bounds (2.15) and (2.41) give the exact value of the norms \( \|X\| \) and \( \|P - Q\| \), respectively, and, hence, the bounds (2.15) and (2.41) are sharp. In this case

\[
\delta_- = \delta_+ = \|V\| \tan \left( \frac{1}{2} \arctan \frac{2\|V\|}{A_1 - A_0} \right)
\]

and

\[
\|P - Q\| = \sin \left( \frac{1}{2} \arctan \frac{2\|V\|}{A_1 - A_0} \right),
\]

which shows that estimates (3.2) and (3.1) are also sharp.

4. Riccati Equation: Uniqueness

Under Hypothesis 2.1 Theorem 2.4 and Remark 2.6 guarantee the existence of a contractive solution to the Riccati equation

\[
A_1 X - XA_0 - XVX + V^* = 0
\]

with the properties

(i) \( \text{Ker} (A_0 - \lambda) \cap \text{Ker} V^* \subset \text{Ker} X \),

(ii) \( \text{spec} (A_0 + VX) \subset (-\infty, \lambda] \).

If, in addition,

\[
d = \text{dist}(\text{spec}(A_0), \text{spec}(A_1)) > 0,
\]

then \( X \) is a unique contractive solution to the Riccati equation (see Corollary 6.4 (i) in [10]; cf. [2]).
The following uniqueness result shows that there is no other solution to the Riccati equation (4.1) with properties (4.2).

**Theorem 4.1.** Assume Hypothesis 2.1. Then a contractive solution to the Riccati equation (4.1) satisfying the properties (4.2) is unique.

**Proof.** Exactly the same reasoning as in the proof of Theorem 2.4 (i) allows to conclude that without loss of generality one may assume that \( \ker (B - \lambda) = 0 \).

In this case the spectral subspace \( \text{Ran} \, E_B((\lambda, \infty)) \) is the graph of a contraction \( X \) which solves (4.1). Assume that \( \gamma \in \mathcal{B}(\mathcal{F}_0, \mathcal{F}_1) \) is a (not necessarily contractive) solution to (4.1) different from \( X \). Since the graph of \( X \) is a spectral subspace of \( B \), one concludes that the orthogonal projections onto the graphs \( \mathcal{G}(\mathcal{F}_0, X) \) and \( \mathcal{G}(\mathcal{F}_0, Y) \) of the operators \( X \) and \( Y \) commute.

We claim that \( \mathcal{G}(\mathcal{F}_0, X) \cap \mathcal{G}(\mathcal{F}_0, Y) \) is nontrivial. To show this we set

\[
\mathcal{L} := \mathcal{F}_0 \oplus \text{Ran} \, P|_{\mathcal{G}(\mathcal{F}_0, X) \cap \mathcal{G}(\mathcal{F}_0, Y)},
\]

where \( P \) is the orthogonal projection in \( \mathcal{F} \) onto the subspace \( \mathcal{F}_0 \). Any \( z \in \mathcal{G}(\mathcal{F}_0, X) \cap \mathcal{G}(\mathcal{F}_0, Y) \) admits the representations

\[
z = x \oplus Xx \quad \text{for some} \quad x \in \mathcal{L}^\perp,
\]

\[
z = y \oplus Yy \quad \text{for some} \quad y \in \mathcal{L}^\perp,
\]

where \( \mathcal{L}^\perp := \mathcal{F}_0 \ominus \mathcal{L} \). Obviously, \( x = y = Pz \), and, therefore, \( XPz = YPz \) for any \( z \in \mathcal{G}(\mathcal{F}_0, X) \cap \mathcal{G}(\mathcal{F}_0, Y) \). Thus, \( Y|_{\mathcal{L}^\perp} = X|_{\mathcal{L}^\perp} \) and

\[
\mathcal{G}(\mathcal{F}_0, X) \cap \mathcal{G}(\mathcal{F}_0, Y) = \{ x + Xx | x \in \mathcal{L}^\perp \}.
\]

Hence, \( X \neq Y \) if and only if the subspace \( \mathcal{L} \) is nontrivial.

Note that

\[
(x_0 \oplus Yx_0, x_0 \oplus Yx_0) = 0, \quad x := (I_{\mathcal{F}_0} + Y^*Y)^{-1}y
\]

for any \( x_0 \in \mathcal{L}^\perp \) and any \( y \in \mathcal{L} \). Hence,

\[
(4.3) \quad \mathcal{G}(\mathcal{F}_0, Y) \ominus (\mathcal{G}(\mathcal{F}_0, X) \cap \mathcal{G}(\mathcal{F}_0, Y)) = \{ x + Yx | x = (I_{\mathcal{F}_0} + Y^*Y)^{-1}y, y \in \mathcal{L} \}.
\]

Since the orthogonal projections onto the graph subspaces \( \mathcal{G}(\mathcal{F}_0, X) \) and \( \mathcal{G}(\mathcal{F}_0, Y) \) commute, by (4.3) we conclude that the subspace

\[
\mathcal{G}(\mathcal{F}_0, X)^\perp \cap \mathcal{G}(\mathcal{F}_0, Y) = \mathcal{G}(\mathcal{F}_0, Y) \ominus (\mathcal{G}(\mathcal{F}_0, X) \cap \mathcal{G}(\mathcal{F}_0, Y))
\]

is nontrivial.

For any \( z \in \mathcal{F}, z \neq 0 \) such that \( z \in \mathcal{G}(\mathcal{F}_0, Y) \) and \( z \perp \mathcal{G}(\mathcal{F}_0, X) \) we have

\[
(4.4) \quad (z, Bz) > \lambda.
\]

Therefore, for the operator \( A_0 + Y \gamma \) the condition (ii) does not hold, since the spectrum of \( A_0 + Y \gamma \) coincides with that of the restriction of \( B \) onto its invariant subspace \( \mathcal{G}(\mathcal{F}_0, Y) \) and by (4.4) the operator \( B|_{\mathcal{G}(\mathcal{F}_0, Y)} \) has points of the spectrum to the right of the point \( \lambda \). The proof is complete. \( \square \)
Remark 4.2. If \( \lambda \in \mathbb{R} \) is a multiple eigenvalue of the operator \( B \) and both \( \text{Ker} (B - \lambda) \cap \mathcal{S}_0 \neq \{0\} \) and \( \text{Ker} (B - \lambda) \cap \mathcal{S}_1 \neq \{0\} \), it follows from Theorem 2.2 that the Riccati equation (4.1) has uncountably many bounded solutions (even if \( \mathcal{S} \) is finite-dimensional). This can also be seen directly. Let

\[
T : \text{Ker} (A_0 - \lambda) \cap \text{Ker} V^* \rightarrow \text{Ker} (A_1 - \lambda) \cap \text{Ker} V
\]

be any bounded operator acting from \( \text{Ker} (A_0 - \lambda) \cap \text{Ker} V^* \) to \( \text{Ker} (A_1 - \lambda) \cap \text{Ker} V \). The bounded operator \( X \in \mathcal{B}(\mathcal{S}_0, \mathcal{S}_1) \) defined by

\[
(4.5) \quad \tilde{X} f = \begin{cases} T f & \text{if } f \in \text{Ker} (A_0 - \lambda) \cap \text{Ker} V^* \\ 0 & \text{if } f \in \text{Ker} (A_0 - \lambda) \cap (\text{Ker} (A_0 - \lambda) \cap \text{Ker} V^*) \end{cases}
\]

satisfies the equation

\[
(4.6) \quad ((A_1 - \lambda \mathcal{I}_{\mathcal{S}_0}) \tilde{X} - \tilde{X} (A_0 - \lambda \mathcal{I}_{\mathcal{S}_0}) - \tilde{X} V \tilde{X}^* + V^*) f = 0 \quad \text{for all } f \in \mathcal{S}_0,
\]

and, thus, it is also a solution to (4.1). If \( \dim \text{Ker} (A_0 - \lambda) \cap \text{Ker} V^* = \infty, \dim \text{Ker} (A_1 - \lambda) \cap \text{Ker} V = \infty, \) and \( T \) is a closed densely defined unbounded operator from \( \text{Ker} (A_0 - \lambda) \cap \text{Ker} V^* \) to \( \text{Ker} (A_1 - \lambda) \cap \text{Ker} V \), then the operator \( \tilde{X} \) defined by (4.5) is an unbounded solution to the Riccati equation (4.1) in the sense of Definition 4.2 in [10].

Our next result is the following uniqueness criterion.

Theorem 4.3. Assume Hypothesis 2.7. A contractive solution \( X \) to the Riccati equation (4.1) is a unique contractive solution if and only if the graph of \( X \) is a spectral subspace of the operator \( B \) and \( \mu = 1 \) is not an eigenvalue of the operator \( |X| \). In this case, \( X \) is a strict contraction and

\[
e_{B}((\infty, \lambda)) = \mathcal{S} (\mathcal{S}_0, X) \quad \text{or} \quad e_{B}((\lambda, +\infty)) = \mathcal{S} (\mathcal{S}_1, -X^*).
\]

Proof: “If Part”. Since \( \mu = 1 \) is not an eigenvalue of the contraction \( X \), it follows that \( X \) is a strict contraction. Then the claim follows from Corollary 6.4 of [10].

“Only If Part”. Assume that \( X \) is the unique contractive solution to the Riccati equation. Then \( X \) coincides with the operator referred to in Theorem 2.4. We need to prove that the graph of \( X \) is a spectral subspace of \( B \) and that \( \text{Ker} (I_{\mathcal{S}_0} - X^*X) = \{0\} \).

We will prove these statements by reduction to contradiction. If the graph of \( X \) is not a spectral subspace of \( B \), then by Remark 2.3 and Theorem 2.4 both \( \text{Ker} (A_0 - \lambda) \cap \text{Ker} V^* \) and \( \text{Ker} (A_1 - \lambda) \cap \text{Ker} V \) are nontrivial. Let \( T \) be any contractive operator from \( \text{Ker} (A_0 - \lambda) \cap \text{Ker} V^* \) to \( \text{Ker} (A_1 - \lambda) \cap \text{Ker} V \) with \( T \neq \{0\} \). Then the operator \( \tilde{X} \) defined by (4.5) is also a contractive solution to the Riccati equation (see Remark 4.2). Since by Theorem 2.4 \( \text{Ker} (A_0 - \lambda) \cap \text{Ker} V^* \subset \text{Ker} X \), where \( X \) is the contractive solution to the Riccati equation referred to in Theorem 2.4, the contractive solution \( \tilde{X} \) to the Riccati equation is different from \( X \) by construction. A contradiction.

Assume now that \( \mu = 1 \) is an eigenvalue of \( |X| \), that is, \( \text{Ker} (I_{\mathcal{S}_0} - X^*X) \) is nontrivial. By Lemma 2.9

\[
(4.7) \quad \text{Ker} (I_{\mathcal{S}_0} - X^*X) \subseteq \text{Ker} (VXX^* - V^*)
\]
and \( \ker(I_{\delta_0} - X^*X) \) reduces both \( A_0 \) and \( VX \). Applying Theorem 6.2 in [10] we conclude that the Riccati equation (4.1) has a contractive solution \( Y \) such that \( \ker(I_{\delta_0} - X^*X) = \ker(I_{\delta_0} + Y^*X) \). This solution necessarily differs from \( X \) which contradicts the hypothesis that \( X \) is a unique contractive solution.

From Theorem 2.2 it follows now that the graph of \( X \) is the spectral subspace of the operator \( B \) and

\[
\mathcal{G}(\delta_0, X) = \begin{cases} 
\mathcal{E}_B((\infty, \lambda)) & \text{if } \ker(A_0 - \lambda) \cap \ker V^* = \{0\} \\
\mathcal{E}_B((0, \lambda)) & \text{if } \ker(A_1 - \lambda) \cap \ker V = \{0\}
\end{cases}
\]

which proves the remaining statement of the theorem. \( \blacksquare \)

As an immediate consequence of Theorem 4.3 we get the following results.

**Corollary 4.4.** Let \( X : \mathcal{H}_0 \rightarrow \mathcal{H}_1 \) be the solution to the Riccati equation referred to in Theorem 2.4. It is the unique contractive solution if and only if it is strictly contractive.

**Remark 4.5.** Let \( X \) be an arbitrary strictly contractive solution to the Riccati equation (4.1). In general this solution has not to be the unique contractive solution or to be isolated point of the set of all solutions (cf. Remark 4.2).

**Corollary 4.6.** Let \( X : \mathcal{H}_0 \rightarrow \mathcal{H}_1 \) be the solution to the Riccati equation referred to in Theorem 2.4. It is an isolated point (in the operator norm topology) in the set of all solutions to the Riccati equation (4.1) if and only if either \( \ker(A_0 - \lambda) \cap \ker V^* = \{0\} \) or \( \ker(A_1 - \lambda) \cap \ker V = \{0\} \).

**Proof.** By Remark 2.3 and Theorem 2.4 the graph \( \mathcal{G}(\delta_0, X) \) is associated with a spectral subspace of the operator \( B \) if and only if either \( \ker(A_0 - \lambda) \cap \ker V^* = \{0\} \) or \( \ker(A_1 - \lambda) \cap \ker V = \{0\} \). Now the claim follows from Theorem 5.3 in [10]. \( \blacksquare \)

**Remark 4.7.** If either \( \ker(A_0 - \lambda) \cap \ker V^* = \{0\} \) or \( \ker(A_1 - \lambda) \cap \ker V = \{0\} \) holds, Theorem 6.2 in [10] allows to construct all contractive solutions to the Riccati equation from that referred to in Theorem 2.4.

## 5. More on the Subspaces \( \ker(I_{\delta_0} - X^*X) \) and \( \ker(I_{\delta_1} - XX^*) \)

The main goal of this section is to prove the fact that the subspace \( \ker(I_{\delta_0} - X^*X) \) associated with the operator \( X \) referred to in Theorem 2.4 admits an intrinsic description as the maximal \( VV^* \)-invariant subspace \( \mathcal{R}_0 \subset \mathcal{H}_0 \) with the properties

\[
\mathcal{R}_0 \subset \ker(A_0 - \lambda) \cap \mathrm{ran} V,
\]

(5.1)

\[
\mathrm{ran} V^*|_{\mathcal{R}_0} \subset \ker(A_1 - \lambda)
\]

(see Theorem 5.3 below). Similarly, the subspace \( \ker(I_{\delta_1} - XX^*) \) can be characterized as the maximal \( V^*V \)-invariant subspace with the properties

\[
\mathcal{R}_1 \subset \ker(A_1 - \lambda) \cap \mathrm{ran} V^*,
\]

(5.2)

\[
\mathrm{ran} V|_{\mathcal{R}_1} \subset \ker(A_0 - \lambda).
\]
We start with the observation that the maximal subspaces $\mathcal{R}_0$ and $\mathcal{R}_1$ with indicated properties do exist and admit a constructive description.

**Lemma 5.1.** The subspaces

(5.3)  
$$\mathcal{R}_0 := \text{closure}\{ x \in \text{Ker} (A_0 - \lambda) \cap \text{Ran} V | V^n (V^* V)^n x \in \text{Ker} (A_1 - \lambda), (V^* V)^n x \in \text{Ker} (A_0 - \lambda) \text{ for any } n \in \mathbb{N}_0\}$$

and

(5.4)  
$$\mathcal{R}_1 := \text{closure}\{ x \in \text{Ker} (A_1 - \lambda) \cap \text{Ran} V^* | V (V^* V)^n x \in \text{Ker} (A_0 - \lambda), (V^* V)^n x \in \text{Ker} (A_1 - \lambda) \text{ for any } n \in \mathbb{N}_0\}$$

reduce the operators $V V^*$ and $V^* V$, respectively. Moreover, the subspaces $\mathcal{R}_0$ and $\mathcal{R}_1$ satisfy the properties (5.1) and (5.2), respectively.

The subspace $\mathcal{R}_0$ is maximal in the sense that if $\mathcal{L}_0$ is any other $V V^*$-invariant subspace with the properties (5.1), then $\mathcal{L}_0 \subset \mathcal{R}_0$. Analogously, the subspace $\mathcal{R}_1$ is maximal in the sense that if $\mathcal{L}_1$ is any other $V^* V$-invariant subspace with the properties (5.2), then $\mathcal{L}_1 \subset \mathcal{R}_1$.

**Proof.** Clearly, the subspace $\mathcal{R}_0$ is invariant under the operator $V V^*$ and, therefore, $\mathcal{R}_0$ reduces $V V^*$, since $V V^*$ is self-adjoint. It follows from (5.3) that (5.1) holds.

Now, let $\mathcal{L}_0$ be an arbitrary closed subspace of $\text{Ker} (A_0 - \lambda) \cap \text{Ran} V$ invariant under $V V^*$ such that $V^* \mathcal{L} \subset \text{Ker} (A_1 - \lambda)$. Then $\mathcal{L} \subset \mathcal{R}_0$. Indeed, since $\mathcal{L}$ is invariant under $V V^*$, we have $(V V^*)^n \mathcal{L} \subset \mathcal{L} \subset \text{Ker} (A_0 - \lambda)$ for any $n \in \mathbb{N}$. Hence, $V^* (V V^*)^n \mathcal{L} \subset V^* \mathcal{L} \subset \text{Ker} (A_1 - \lambda)$ for any $n \in \mathbb{N}_0$ and one concludes that $\mathcal{L} \subset \mathcal{R}_0$.

The maximality of the subspace $\mathcal{R}_1$ is proven in a similar way.  

**Lemma 5.2.** The subspaces $\mathcal{R}_0$ and $\mathcal{R}_1$ satisfy the properties that

(5.5)  
$$\overline{\text{Ran} V^* |_{\mathcal{R}_0}} = \mathcal{R}_1,$$

(5.6)  
$$\text{Ran} V^* |_{\mathcal{R}_0 \cap \mathcal{R}_1} \subset \mathcal{R}_1 \subset \mathcal{R}_0 \cap \mathcal{R}_1,$$

$$\text{Ran} V^* |_{\mathcal{R}_0 \cap \mathcal{R}_1} \subset \mathcal{R}_0 \cap \mathcal{R}_1.$$

**Proof.** Equations (5.5) follow from the explicit description (5.3) and (5.4) of the subspaces $\mathcal{R}_0$ and $\mathcal{R}_1$, respectively.

Let $x \in \mathcal{R}_0 \cap \mathcal{R}_1$ be arbitrary. Choose an arbitrary $y \in \mathcal{R}_1$ and consider

$$(y, V^* x) = (V^* y, x).$$

Since, by (5.5), $V y \in \mathcal{R}_0$ we have $(V y, x) = 0$. Thus, $V^* x \in \mathcal{R}_1 \subset \mathcal{R}_0 \cap \mathcal{R}_1$ which proves the first inclusion in (5.6). The second inclusion in (5.6) is proven similarly.

**Theorem 5.3.** Assume Hypothesis 2.7. Let $X : \mathcal{R}_0 \rightarrow \mathcal{R}_1$ be the solution to the Riccati equation (2.1) referred to in Theorem 2.7. Then

(5.7)  
$$\text{Ker} (I_{\mathcal{R}_0} - X^* X) = \mathcal{R}_0 \quad \text{and} \quad \text{Ker} (I_{\mathcal{R}_1} - XX^*) = \mathcal{R}_1.$$

Moreover, $\overline{\text{Ran} X |_{\mathcal{R}_0}} = \mathcal{R}_1$ and

$$X |_{\mathcal{R}_0} = -\hat{S}, \quad \hat{S} : \mathcal{R}_0 \rightarrow \mathcal{R}_1,$$
where $\hat{S} = S|_{\mathcal{S}_0}$ with $S : \mathcal{S}_0 \to \mathcal{S}_1$ being the partial isometry with initial space $\text{Ran} V$ and final space $\text{Ran} V^*$ defined by the polar decomposition $V^* = S(VV^*)^{1/2}$.

In particular, $\mathcal{S}_1 = \mathcal{S}(\overline{\mathcal{S}_0}, \hat{S})$.

**Proof.** Without loss of generality we assume that $\lambda = 0$.

First, we will prove the inclusion

$$(5.8) \quad \text{Ker} (I_{\mathcal{S}_0} - X^*X) \subseteq \mathcal{S}_0. \tag{5.8}$$

It is sufficient to establish that

(a) $\text{Ker} (I_{\mathcal{S}_0} - X^*X)$ reduces $VV^*$,

(b) $\text{Ker} (I_{\mathcal{S}_0} - X^*X) \subseteq \text{Ran} V^* \cap \text{Ker} A_0$,

(c) $V^* \text{Ker} (I_{\mathcal{S}_0} - X^*X) \subseteq \text{Ker} A_1$.

The statement (a) follows from Lemma 2.9.

In order to see that (b) holds note that if $z \in \text{Ker} V^* \cap \text{Ker} A_0$, then $Xz = 0$, since $\text{Ker} V^* \cap \text{Ker} A_0 \subseteq \text{Ker} X$ by Theorem 2.4. Therefore, $\text{Ker} V^* \cap \text{Ker} A_0 \subseteq \text{Ker} (I_{\mathcal{S}_0} - X^*X)$ since

$$(z, x) = (z, X^*Xx) = (Xz, Xx) = 0 \quad \text{for any} \quad x \in \text{Ker} (I_{\mathcal{S}_0} - X^*X),$$

which proves (b) taking into account that $\text{Ker} (I_{\mathcal{S}_0} - X^*X) \subseteq \text{Ker} A_0$ (see Lemma 2.9).

To prove (c) we proceed as follows. If $x \in \text{Ker} (I_{\mathcal{S}_0} - X^*X)$, then $A_0x = 0$, since $\text{Ker} (I_{\mathcal{S}_0} - X^*X) \subseteq \text{Ker} A_0$. The same reasoning as in the proof of Lemma 2.9 shows that (see (2.52))

$$(V^*x = VXVx \quad \text{and} \quad VX = X^*V^*x).$$

Therefore, $(I_{\mathcal{S}_1} - XX^*)V^*x = V^*x - XX^*V^*x = V^*x - VXVx = 0$, that is, $V^*x \in \text{Ker} (I_{\mathcal{S}_1} - XX^*)$. By Lemma 2.9, $\text{Ker} (I_{\mathcal{S}_1} - XX^*) \subseteq \text{Ker} A_1$, which completes the proof of (c).

The inclusion $\text{Ker} (I_{\mathcal{S}_1} - XX^*) \subseteq \mathcal{S}_1$ is proven similarly. Hence, we have established that

$$(5.9) \quad \text{Ker} (I_{\mathcal{S}_0} - X^*X) \subseteq \mathcal{S}_0 \quad \text{and} \quad \text{Ker} (I_{\mathcal{S}_1} - XX^*) \subseteq \mathcal{S}_1. \tag{5.9}$$

Now we turn to the proof of the opposite inclusions

$$(5.10) \quad \mathcal{S}_0 \subseteq \text{Ker} (I_{\mathcal{S}_0} - X^*X) \quad \text{and} \quad \mathcal{S}_1 \subseteq \text{Ker} (I_{\mathcal{S}_1} - XX^*).$$

Clearly, the subspaces $\hat{\mathcal{S}}_0 = \mathcal{S}_0 \ominus \mathcal{S}_0$ and $\hat{\mathcal{S}}_1 = \mathcal{S}_1 \ominus \mathcal{S}_1$ reduce the operators $A_0$ and $A_1$, respectively, since $\mathcal{S}_0 \subseteq \text{Ker} A_0$, $\mathcal{S}_1 \subseteq \text{Ker} A_1$, and $A_0$, $A_1$ are self-adjoint. Denote by $\hat{A}_0$ and $\hat{A}_1$ the corresponding parts of the operators $A_0$ and $A_1$ associated with these subspaces:

$$\hat{A}_0 = A_0|_{\hat{\mathcal{S}}_0} \quad \text{and} \quad \hat{A}_1 = A_1|_{\hat{\mathcal{S}}_1}.$$ 

Since by Lemma 5.2, $\text{Ran} V|_{\hat{\mathcal{S}}_1} \subseteq \hat{\mathcal{S}}_0$, the restriction $\hat{V}$ of the operator $V$ onto $\hat{\mathcal{S}}_1$ is a map from $\hat{\mathcal{S}}_1$ to $\hat{\mathcal{S}}_0$. By Theorem 4.1, the Riccati equation

$$(5.11) \quad \hat{A}_1 \hat{X} - \hat{X} \hat{A}_0 - \hat{X} \hat{V} \hat{X} + \hat{V}^* = 0$$

would imply the inclusion $\hat{\mathcal{S}}_0 \subseteq \text{Ran} \hat{V}$. Hence, if $x \in \hat{\mathcal{S}}_0$, then $\hat{V}x \in \hat{\mathcal{S}}_1$, and

$$(5.12) \quad \hat{V}x \in \hat{\mathcal{S}}_1 \quad \text{for any} \quad x \in \hat{\mathcal{S}}_0.$$
has a unique solution $\hat{X}$ satisfying $\text{Ker} \hat{A}_0 \cap \text{Ker} \hat{V}^* \subset \text{Ker} \hat{X}$ and $\text{spec}(\hat{A}_0 + \hat{V} \hat{X}) \subset (-\infty, 0]$.

Let $S : \mathcal{H}_0 \to \mathcal{H}_1$ be a partial isometry with the initial subspace $\text{Ran} \hat{V}$ and final subspace $\text{Ran} \hat{V}^*$ defined by the polar decomposition $V^* = S(VV^*)^{1/2}$. From Lemma 5.2 it follows that $\text{Ran} \hat{S}|_{\mathcal{H}_0} = \mathcal{H}_1$.

Let $\hat{S} : \mathcal{H}_0 \to \mathcal{H}_1$ be the restriction of $S$ onto $\mathcal{H}_0$: $\hat{S} = S|_{\mathcal{H}_0}$. Define the operator $Y : \mathcal{H}_0 \to \mathcal{H}_1$ by the following rule

$$ Yx = \begin{cases} \hat{X}x, & x \in \mathcal{H}_0 \\ -\hat{S}x, & x \in \mathcal{H}_0 \\ \end{cases}. $$

Since $\hat{S}$ maps $\mathcal{H}_0$ onto $\mathcal{H}_1$ isometrically, one immediately concludes that

$$ \mathcal{H}_0 \subset \text{Ker}(I - \hat{Y}^* \hat{Y}) \quad \text{and} \quad \mathcal{H}_1 \subset \text{Ker}(I - \hat{Y} \hat{Y}^*). $$

We claim that the operator $Y$ solves the Riccati equation

$$ (5.13) \quad A_1 Y - YA_0 - YVV^* + V^* = 0. $$

Indeed, if $x \in \mathcal{H}_0$ then

$$ (5.14) \quad (A_1 Y - YA_0 - YVV^* + V^*)x = 0 $$

as a consequence of (5.11). If $x \in \mathcal{H}_0$, then $A_0 x = 0$ (recall that we assumed that $\lambda = 0$). Moreover, $\hat{S}x \in \mathcal{H}_1$ and hence $A_1 \hat{S}x = 0$ resulting in

$$ (A_1 Y - YA_0 - YVV^* + V^*)x = (-YVV^* + V^*)x = (-\hat{S}V\hat{S} + V^*)x = 0, $$

where we have used the fact that $V\hat{S}x \in \mathcal{H}_0$ and the equality

$$ \hat{S}V\hat{S}x = \hat{S}(VV^*)^{1/2}\hat{S}x = \hat{S}(VV^*)^{1/2}x = V^*x. $$

Therefore, $Y$ solves the Riccati equation (5.14).

Our next claim is that

$$ \text{Ker} A_0 \cap \text{Ker} V^* \subset \text{Ker} Y. $$

Since $\mathcal{H}_0 \subset \text{Ker} A_0$, by Lemma 5.2 one concludes that the subspace $\text{Ker} \hat{A}_0 \cap \text{Ker} \hat{V}^*$, naturally imbedded into $\mathcal{H}_0$, coincides with $\text{Ker} A_0 \cap \text{Ker} V^*$. One also concludes that the subspace $\text{Ker} \hat{X}$ naturally imbedded into $\mathcal{H}_0$ coincides with $\text{Ker} Y$ by the definition of the operator $Y$. Therefore, (5.15) follows from the inclusion $\text{Ker} \hat{A}_0 \cap \text{Ker} \hat{V}^* \subset \text{Ker} \hat{X}$, proving (5.15).

Finally, observe that

$$ VS|_{\mathcal{H}_0} = (VV^*)^{1/2}S|_{\mathcal{H}_0} = (VV^*)^{1/2}|_{\mathcal{H}_0} \geq 0 $$

and

$$ \text{spec}(\hat{A}_0 + \hat{V} \hat{X}) \subset (-\infty, 0]. $$

Since the operator $A_0 + YY$ is diagonal with respect to the decomposition $\mathcal{H}_0 = \mathcal{H}_0 \oplus \mathcal{H}_0$,

$$ A_0 + YY = (\hat{A}_0 + \hat{V} \hat{X}) \oplus (-VS|_{\mathcal{H}_0}), $$

one infers that

$$ \text{spec}(A_0 + YY) \subset (-\infty, 0]. $$
Combining (5.13), (5.15), and (5.16) proves that the operator $Y$ coincides with $X$ using the uniqueness result of Theorem 4.1. Thus, (5.10) follows from (5.12).

Combining (5.10) and (5.9) proves (5.7).

The remaining statement of the theorem follows from the definition of the operator $Y$ and the fact that $X = Y$. □

By Theorem 5.3 the uniqueness criterion (Theorem 4.3) admits the following equivalent purely geometric formulation.

**Theorem 5.4.** Assume Hypothesis 2.1 and let $X$ be the solution to the Riccati equation

$$A_1X - AX_0 - XVX + V^* = 0$$

referred to in Theorem 2.2. Let $\mathcal{R}_0$ and $\mathcal{R}_1$ be the subspaces given by (5.3) and (5.4), respectively. Then $X$ is the unique contractive solution if and only if

(i) either $\text{Ker}(A_0 - \lambda) \cap \text{Ker} V^*$ or $\text{Ker}(A_1 - \lambda) \cap \text{Ker} V$ are trivial and

(ii) either $\mathcal{R}_0$ or $\mathcal{R}_1$ (and hence both) are trivial.

The solution $X$ is strictly contractive.

### APPENDIX A. TWO SUBSPACES

Here we collect some facts about two closed subspaces of a separable Hilbert space which are used in the body of the paper. Their comprehensive presentation with proofs as well as some further results and the history of the problem can be found in [10].

Let $(P, Q)$ be an ordered pair of orthogonal projections in the separable Hilbert space $\mathcal{H}$. Denote

$$\mathcal{M}_{pq} := \{ f \in \mathcal{H} | Pf = pf, Qf = qf \}, \quad p, q = 0, 1,$$

$$\mathcal{M}_0 := \text{Ran} P \oplus (\mathcal{M}_{10} \oplus \mathcal{M}_{11}),$$

$$\mathcal{M}_1 := \text{Ran} P^\perp \oplus (\mathcal{M}_{00} \oplus \mathcal{M}_{01}),$$

$$\mathcal{M}' := \mathcal{M}'_0 \oplus \mathcal{M}'_1,$$

$$P' := P|_{\mathcal{M}'},$$

$$Q' := Q|_{\mathcal{M}'}. $$

The space $\mathcal{H}$ admits the canonical orthogonal decomposition

(A.1) $$\mathcal{H} = \mathcal{M}_{00} \oplus \mathcal{M}_{01} \oplus \mathcal{M}_{10} \oplus \mathcal{M}_{11} \oplus \mathcal{M}' .$$

The following theorem provides a criterion for the subspace $\text{Ran} Q$ to be a graph subspace associated with the subspace $\text{Ran} P$.

**Theorem A.1.** Let $P$ and $Q$ be orthogonal projections in a Hilbert space $\mathcal{H}$. The subspace $\text{Ran} Q$ is a graph subspace $\mathcal{G}(\text{Ran} P, X)$ associated with some closed densely defined (possibly unbounded) operator $X : \text{Ran} P \to \text{Ran} P^\perp$ with $\text{Dom}(X) \subset \text{Ran} P$. 

Ran \( P \) if and only if the subspaces \( M_{01}(P, Q) \) and \( M_{10}(P, Q) \) in the canonical decomposition (A.1) of the Hilbert space \( \mathcal{H} \) are trivial, i.e.,

\[
M_{01}(P, Q) = M_{10}(P, Q) = \{0\}.
\]

For a given orthogonal projection \( P \) the correspondence between the closed subspaces \( \text{Ran} \, Q \) satisfying (A.2) and closed densely defined operators \( X : \text{Ran} \to \text{Ran} \, P^\perp \) is one-to-one.

The subspaces \( M_{11} \) and \( M_{00} \) have a simple description in terms of the operator \( X \):

\[
M_{11} = \ker X \quad \text{and} \quad M_{00} = \ker X^*.
\]

Note that \( M_{01}(P, Q) = M_{10}(P, Q) = \{0\} \) if \( \|P - Q\| < 1 \). Moreover, Theorem A.1 has the following corollary.

**Theorem A.2.** Let \( P \) and \( Q \) be orthogonal projections in a Hilbert space \( \mathcal{H} \). Then the inequality \( \|P - Q\| < 1 \) holds true if and only if \( \text{Ran} \, Q \) is a graph subspace associated with the subspace \( \text{Ran} \, P \) and some bounded operator \( X \in \mathcal{B}(\text{Ran} \, P, \text{Ran} \, P^\perp) \), that is, \( \text{Ran} \, Q = G(\text{Ran} \, P, X) \). In this case

\[
\|X\| = \frac{\|P - Q\|}{\sqrt{1 - \|P - Q\|^2}}
\]

and

\[
\|P - Q\| = \frac{\|X\|}{\sqrt{1 + \|X\|^2}}.
\]

**References**


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