Extended Hilbert space approach to few-body problems

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A general formulation of the quantum scattering theory for a system of few particles, which have an internal structure, is given. Due to freezing out the internal degrees of freedom in the external channels, a certain class of energy-dependent potentials is generated. By means of potential theory, a modified Faddeev equation is derived both in external and internal channels. The Fredholmity of these equations is proven and this is what provides a sound basis for solving the addressed scattering problem.

I. INTRODUCTION

This paper is concerned with the treatment of low-energy quantum scattering for few particles with internal structure. Problems of this kind arise in describing hadron–hadron as well as nucleus–nucleus scattering1-5 and similarly in constructing NN potentials in the baglike approaches.6 There exist, however, no appropriate mathematical methods which may be applied for a rigorous study of the wavefunction properties. Already in the two-body problem there is no self-adjoint (s.a.) Hamiltonian, which could be generated by a time-dependent unitary group of operators responsible for the evolution of the system. In the three-body case, the original Faddeev equations are also not directly applicable due to the lack of an underlying s.a. Hamiltonian.

In the present paper, we overcome these difficulties by means of the extension theory using an auxiliary Hilbert space corresponding to the internal degrees of freedom.7-9 In the special extended Hilbert space we construct the total s.a. Hamiltonian. After eliminating the internal channels, we propose modified Faddeev equations for the components of external-channel Green functions. Using well-known methods,10,11 we prove that these equations represented in configuration space are of Fredholm type. Due to this property the equations provide the justification of treating the three-body scattering problem for particles interacting via energy-dependent potentials. Our modified Faddeev equations in differential form may also be used in an efficient manner for numerical calculations.

II. TWO-BODY PROBLEM

We will consider here the following special case of the general situation.7,9,12 Let us assume that the dynamics of the external degrees of freedom are given by the s.a. Hamiltonian \( h^e \), which is defined by

\[
h^e \psi = (-\Delta + V(x)) \psi
\]

in the Hilbert space \( \mathcal{H}^e = L^2(\mathbb{R}^3) \). The potential \( V(x) \) represents a so-called peripheral interaction (e.g., a meson-exchange potential) of strongly interacting particles and it will be assumed to decrease rapidly and be sufficiently smooth.

We shall also separate the two-body configuration space \( \mathbb{R}^3 \) into the two domains \( \Omega^- \) such that \( \mathbb{R}^3 = \Omega^- \cup \Omega^+ \). Let \( \Omega^- \) be the part of the space \( \mathbb{R}^3 \) where the coordinate \( x \) is bounded. Physically, the compact domain \( \Omega^- \) may be interpreted as the region of reaction (or where clusters overlap) and the domain \( \Omega^+ = \mathbb{R}^3 \setminus \Omega^- \) as the region where the particles move "asymptotically free." The common boundary \( \gamma \) of the domains \( \Omega^\pm \) will in this situation be a surface, where the phase transition between internal and external channels takes place.

In our model we shall restrict the s.a. Hamiltonian \( h^e \) to the symmetric operator \( h_0 \) with the domain \( \mathcal{D}(h_0) = C^\infty_0(\mathbb{R}^3) \), where \( C^\infty_0 \) is the class of infinitely differentiable functions, which vanish together with all derivative in the neighborhood of the surface \( \gamma \). Then the Hermitian conjugate operator \( h_0^* \) has a nontrivial boundary form \( J^e \), namely,

\[
J^e(u,v) = \langle h_0^* u, v \rangle - \langle u, h_0^* v \rangle = \lim_{\delta \to 0} \left[ \int_{\gamma^+} ds \left( \partial_n u \bar{w} - u \partial_n \bar{w} \right) - \int_{\gamma^-} ds \left( \partial_n \bar{u} v - \bar{u} \partial_n v \right) \right],
\]

where \( \partial_n \) is the normal derivative on the surfaces \( \gamma^\pm = \{ x \in \Omega^\pm : \text{dist}(x, \gamma^\pm) = \delta \} \).

Now we assume that the dynamics of the internal degrees of freedom without connection to the external channel \( \mathcal{H}^e \) is given by an arbitrary s.a. operator \( A \) acting in some Hilbert space \( \mathcal{H}^m \). In order to "switch on" the interaction between channels \( \mathcal{H}^e \) and \( \mathcal{H}^m \), one must restrict the operator \( A \) to some symmetric operator \( A_0 \) and construct all s.a. extensions of the operator \( h_0 \oplus A_0 \) in the direct sum \( \mathcal{H}^e \oplus \mathcal{H}^m \). The important question of the model is the following: How to construct the boundary form \( J^m \) for an arbitrary s.a. operator \( A \)? The general answer was obtained in Ref. 7. Namely, the symmetric restriction of the Hamiltonian should be made in terms of its Cayley transform \( U = (A - i)/(A + i) \). For this purpose let us consider the special isometric restriction \( U_0 = U | \mathcal{H}^m \Theta U^* \theta \), where \( \theta \) is a generative element6 of the operator \( A \). The symmetric restriction \( A_0 \) can be obtained as the inverse Cayley transformation of the isometry \( U_0 \). Hence, the operator \( A_0 \) has deficiency indices \( (1,1) \) and the domain \( \mathcal{D}(A^*_0) \) of its adjoint can be described in terms of von Neumann's theory:

\[
\mathcal{D}(A^*_0) = \mathcal{D}(A_0) + \mathcal{L}(\theta, U^* \theta); \quad \text{here } A_0 \text{ is the closure of}
\]

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$A_0$ and $\mathcal{L}(\theta, U \ast \theta)$ is the span of deficiency elements $\theta$ and $U \ast \theta$. It is convenient to introduce some new basis in $\mathcal{L}$:

$w^+ = \frac{1}{2}(U^* \theta + \theta), w^- = (1/2i)(U^* \theta - \theta)$. In accordance with von Neumann's representation, an arbitrary vector $u$ can be decomposed as

$u = \tilde{u} + \xi^+ w^+ + \xi^- w^-$, \quad $\text{in } \mathcal{L}(A^\delta), \quad \tilde{u} \in \mathcal{I}(A_0)$,

(3)

where $\xi^\pm(u)$ are the so-called boundary values of element $u$ \( (\text{Ref. } 7) \). In terms of $\xi^\pm$ the boundary form of the operator $A^\delta$ may be written as

$J^m(u,f) = \langle A^\delta u, f \rangle - \langle u, A^\delta f \rangle$

$= \xi^-(u) \xi^+(f) - \xi^+(u) \xi^-(f)$.

(4)

It should be noted that (4) is an abstract variant of (2).

After the preparation of the boundary forms $J^m$ and $J^e$ of the operators $h^m$ and $A^m$ the next step is to construct an s.a. extension $h$ of the operator $h_0 \ast A_0$, acting in the direct sum $\mathcal{H}^m \otimes \mathcal{H}^m$. In accordance with our general method one should impose on $\gamma$ such boundary conditions that make the sum of the boundary forms vanish, i.e., $J^m + J^e = 0$.

It can be shown that all such nullifying conditions may only be of two types, namely:

$\begin{pmatrix} u^+ \\
\partial_u u^- \end{pmatrix} = \begin{pmatrix} \varphi^+ \\
\varphi^- \end{pmatrix} \begin{pmatrix} \partial_u u^+ \\
\partial_u u^- \end{pmatrix}$

(5)

and

$\begin{pmatrix} \partial_u u^+ \\
\partial_u u^- \end{pmatrix} = \begin{pmatrix} \varphi^+ \\
\varphi^- \end{pmatrix} \begin{pmatrix} u^+ \\
\partial_u u^- \end{pmatrix}$

(6)

Here, $u^\pm$ and $\partial_u u^\pm$ are boundary values of $u$ and $\partial_u u$ on the bilateral surface $\gamma^\pm$ and the functions $\varphi^\pm \in L^2(\gamma^\pm)$ are parameters of the model; they generate functionals $\langle u, \varphi^\pm \rangle \equiv \int_{\gamma^\pm} dS u \varphi^\pm, \text{in } L^2(\gamma^\pm)$.

(7)

Finally, $\Box$ is a $2 \times 2$ arbitrary s.a. matrix given on the surface $\gamma$ and $\alpha_B$ is an arbitrary real number.

Let us denote by $u_0$, $\alpha = 0,1$ the external ($\alpha = 0$) and internal ($\alpha = 1$) channel wavefunctions. Then we can study their properties on the basis of the two-channel Schrödinger equation:

$(h-z)\psi = 0, \quad x \in \mathbb{R}^3 \backslash \gamma,$

(8)

$\psi = (u_0, u_1), \quad z = \langle u_0, u_1 \rangle$.

Let us suppose now for simplicity that the external channel wavefunctions $u_0$ are smooth on the surface $\gamma$, i.e., $u_0^+ = u_0^- = u_0$ and that the matrix $\Box$ has the special structure $\Box = \begin{pmatrix} I & 0 \\
0 & I \end{pmatrix}$; let us also put $\alpha_B = 0$. In this case the boundary conditions, which one should add to (8), are of the form

$[\partial_u u_0]_\gamma = -\varphi^-(u_1), \quad (9)$

where $[\partial_u u_0]_\gamma = \partial_u u^- - \partial_u u^+$ and $\varphi = \varphi^- = \varphi^+$.

We want to emphasize that the operator $\mathcal{H}$, which is the total Hamiltonian in the two-body system with internal structure, is an s.a. operator and hence the boundary value problem (8), (9), (10) is mathematically correctly defined. It should also be noted that in our model we are able to simulate an arbitrary complicated internal structure of particles due to the general nature of the internal s.a. operator $A$.

On the other hand, one can now operate in the external channel $\mathcal{H}^m$ only. For this purpose one must solve the boundary conditions (9), (10) by excluding the internal ingredients $\xi^\pm(u_1)$. This procedure is based on the following linear relations:\1

$\xi^- = \Delta(z) \xi^+$,

(11)

where $\Delta(z)$ is the Schwartz integral of the spectral measure of the s.a. operator $A$,

$\Delta(z) \equiv \langle (I + zA)(A-z)^{-1} \theta, \theta \rangle$.

(12)

Taking into account (11) we obtain from (9) and (11) the following energy-dependent boundary conditions in the external space $\mathcal{H}^m$:

$[\partial_u u_0]_\gamma = -\Delta(z) \varphi \langle u_0, \varphi \rangle$.

(13)

In accordance with (8) the external component $u_0$ obeys the equation

$(h - z)u_0 = 0, \quad x \in \mathbb{R}^3 \backslash \gamma$.

(14)

In order to obtain the differential equations in the whole configuration space $\mathbb{R}^3$ it is convenient to use the quasipotential approach (see, e.g., Ref. 11). Let us consider the quasipotential $w(x)$ acting on the function $u$ in accordance with the rule

$w(x)u = -\delta(z) \varphi \langle u, \varphi \rangle$.

(15)

Here, $\delta, \mu$ is the distribution, usually called the simple layer,\14 that acts on the set of sufficiently smooth functions $\varphi$ in the following way:

$\langle \delta, \mu, \varphi \rangle = \int_{\gamma} dS \mu \varphi$.

(16)

In terms of the quasipotential $w(z)$ the boundary-value problem (13) and (14) may be written as

$(h^m + w(z) - z)u_0 = 0$.

(17)

where the variable $x$ now runs over the whole configuration space $\mathbb{R}^3$.

One can show that (17) is equivalent to the boundary-value problem (13), (14).

At the end of this section we add the following remarks.\( \quad (1) \) As it was stated above, a mathematically correct formulation of the two-body problem with energy-dependent interactions can be achieved only in terms of an s.a. operator $\mathcal{H}$ acting in the sum $\mathcal{H}^m \otimes \mathcal{H}^m$ of internal and external channels. On the contrary, there exists no s.a. Hamiltonian corresponding to (17) or, what is the same, to the boundary-value problem (13), (14). This means that in
terms of the external space $\mathcal{H}^a$ only the well-defined total $S$-matrix cannot be constructed.

(2) As it follows from (12) and (15), the energy dependence of the potentials cannot be arbitrary. It is given by the Schwartz integral $\Delta (x)$, which is real on the real axis and is an analytical function on the half-plane \( \text{Im } z > 0 \) with the positive imaginary part $\text{Im } \Delta (z) > 0$. It can be shown that such kind of interactions ensure the analyticity and unitarity of the appropriate total scattering matrix.

(3) In our model the quasipotentials $w(x)$ are separable of rank 1. The generalization to any arbitrary rank of $w(x)$ is trivial. For this purpose one should increase the dimension of the deficiency subspaces $\mathcal{A} = \{0\}$, $\mathcal{A}^* = \{U^*0\}$ and change in a self-consistent way the functionals $\varphi$, $\langle \cdot, \varphi \rangle$ by arbitrary bounded operators $B, B^*$.

### III. THREE-BODY HAMILTONIAN

We consider in this section a system of three particles having a nontrivial internal structure. To describe this system, we use in the external configuration space $\mathbb{R}^6$ usual relative coordinates $x_\alpha$, $y_\alpha$, $\alpha = 1, 2, 3$, which we combine into the six-vector $X = x_\alpha \oplus y_\alpha$ (Ref. 15). Every pair $x_\alpha, y_\alpha$ fixes an orthogonal coordinate system in $\mathbb{R}^6$.

Let $\Gamma_\alpha = y_\alpha \times \mathbb{R}^5_{x_\alpha}$ be the cylinders in $\mathbb{R}^6$ and $\Gamma = \bigcup_\alpha \Gamma_\alpha$. An example of an external configuration space (for the one-dimensional case) is represented in Fig. 1.

A total s.a. Hamiltonian $H$ governing the dynamics of internal and external degrees of freedom will be an important object in the three-body analysis.

We start by considering the two-particle Hamiltonian $H_a$ in the six-dimensional external configuration space:

$$ H_a = \hat{h}_a \otimes I_{y_a} + I_{x_a} \otimes (-\Delta_{x_a}). $$

Here, $\hat{h}_a$ is the s.a. two-body Hamiltonian defined in Sec. II, $I_{y_a}$ and $I_{x_a}$ are the unit operators, in the spaces $L^2(\mathbb{R}^5_{x_a})$ and $\mathcal{H}^a = \mathcal{H}^x \otimes \mathcal{H}^y$, respectively, and $-\Delta_{x_a}$ is the Laplacian defined on its natural domain $W^2(\mathbb{R}^5_{x_a})$.

The operator $H_a$ is essentially s.a. on the domain

$$ \mathcal{D}(H_a) = \mathcal{D}(\hat{h}_a) \otimes W^2(\mathbb{R}^5_{x_a}). $$

The closure $\overline{H}_a$ of this operator is the s.a. operator, which will be denoted by the same symbol $H_a$.

The domain $\mathcal{D}(H_a)$ also may be described in terms of boundary conditions. Namely, let $\mathcal{W} = \{u_0, u_a \in \mathcal{D}(H_a)\}$. Then the external component $u_0$ is a $W^2(\mathbb{R}^5_{x_a})$ smooth function outside the neighborhood of $\Gamma$ continued on $\Gamma$. The internal component $u_a \in \mathcal{W}^\infty = \mathcal{H}^a \otimes L^2(\mathbb{R}^5_{x_a})$ can be decomposed into the sum

$$ u_a = \tilde{u}_a + \tilde{x}_a \cdot (y_\alpha) w_+ + \tilde{x}_a \cdot (y_\alpha) w_-, \quad \tilde{u}_a \in \mathcal{D}(H^\infty), $$

where $w_\pm$ are the deficiency elements\(^{13}\) of the symmetric operator $A^\infty$, which is the restriction of the s.a. operator $A$ and

$$ A^\infty = A_x \otimes I_{y_a} + I_{x_a} \otimes (-\Delta_{x_a}). $$

The functions $\mathcal{W} \in \mathcal{D}(H_a)$ satisfy the boundary conditions

$$ [\partial_x u_0]_{r_a} = -\varphi_a (x_\alpha) \xi_+ (y_\alpha), $$

$$ \xi_+ (y_\alpha) = \langle u_0 \varphi_a \rangle (y_\alpha), $$

where

$$ \langle u_0 \varphi_a \rangle (y_\alpha) = \int_{r_a} d\alpha \cdot u_0 (x) \varphi_a (x_\alpha). $$

It should be noted that the boundary conditions (22), (23) are essentially of two-body character [see (9) and (10)]. The only difference is that the $\xi_\alpha^\pm (y_\alpha)$ are now functions of the variable $y_\alpha \in \mathbb{R}^3$.

We are now ready to construct the total three-body Hamiltonian $H$. Let us consider in the space

$$ \mathcal{W} = L^2(\mathbb{R}^6) \oplus \bigoplus_{\alpha=1}^3 \mathcal{W}^\infty $$

symmetric operator $H_0$,

$$ H_0 \mathcal{W} = \left\{ -\Delta_X + \sum_{\alpha=1}^3 u_\alpha (x_\alpha) u_0, \quad H^{\infty}_{\alpha} u_\alpha, \quad \alpha = 1, 2, 3, \right\} $$

on the domain

$$ \mathcal{D}(H_0) = C^\infty (\mathbb{R}^6 \setminus \Gamma) \oplus \bigoplus_{\alpha} \mathcal{D}(H^{\infty}_{\alpha}). $$

Any s.a. extension $H$ of the operator $H_0$ is a total three-body Hamiltonian describing the whole dynamics in both external and internal channels. In accordance with the von Neumann theory\(^{15}\) all such extensions can be obtained by the extension of the operator $H_0$ on its deficiency subspaces. So we shall extend the domain $\mathcal{D}(H_0)$ to the linear set $\mathcal{D}(\overline{H}_0)$ in the following manner:

\[ \text{FIG. 1. External configuration space for three identical particles.} \]
Here, \( R_0(z) = (H^{aa} - z)^{-1} \) is the resolvent of the s.a. operator \( H^{aa} = -\Delta_x + \Sigma_\alpha v_\alpha \) and \( \rho_\alpha \) are the densities of the simple layer potentials given on cylinders \( \Gamma_\alpha \), \( \alpha = 1,2,3 \). The operator \( \tilde{H}_0 \) on the domain \( \mathcal{D}(\tilde{H}_0) \) obtained in this way is the nonsymmetric restriction of the adjoint operator \( H^{aa} \). Now one must restrict the operator \( \tilde{H}_0 \) to some symmetric one. For this purpose it is sufficient to impose the boundary conditions (22), (23). In terms of the densities \( \rho_\alpha \) these conditions may be written as

\[
\rho_\alpha (X) = -\varphi_\alpha (x_\alpha) \xi^+ (y_\alpha), \tag{28}
\]

\[
\xi^+ (y_\alpha) = \left( \sum_{\beta=1}^{\alpha} R_\beta (-1) \rho_\beta \varphi_\alpha \right) (y_\alpha). \tag{29}
\]

As a result we restrict the domain \( \mathcal{D}(\tilde{H}_0) \) to the linear set \( \mathcal{D}(\tilde{H}) \) by means of conditions (28),(29). The symmetric operator that is such a restriction of \( \tilde{H}_0 \) on the domain \( \mathcal{D}(\tilde{H}) \) will be called \( \tilde{H} \).

Let us now collect the important facts about the operator \( \tilde{H} \).

**Theorem 1**: The operator \( \tilde{H} \) on the domain

\[ \mathcal{D}(\tilde{H}), \mathcal{D}(\tilde{H}) = \mathcal{S} = L^2(\mathbb{R}^6) \oplus \sum_\alpha \mathcal{S}_\alpha^a \]

is symmetric and bounded from below.

The proof of this statement is given in Appendix A.

The last step is now an extension of the symmetric operator \( \tilde{H} \) to the s.a. operator \( H \) obeying the following conditions:

(1) On the domain \( \mathcal{D}(H) \) the translation-invariant boundary conditions (22) and (23) must be kept.

(2) The Hamiltonian \( H \) must be bounded from below.

For this purpose we shall choose the Friedrichs extension \( H \) of the operator \( \tilde{H} \) (Ref. 13). On the domain \( \mathcal{D}(H) \), which can be described as usual, the action of \( H \) is given by

\[ H\varphi = \left\{ \begin{array}{l}
\Delta \varphi \star u_0, \\
-\Delta \varphi \star u_\alpha + A_\alpha \tilde{u}_\alpha - \xi^+ \star w^- + \xi^- \star w^+, \\
\varphi = (u_0 u_\alpha), \quad \alpha = 1,2,3,
\end{array} \right. \tag{30} \]

with the boundary conditions (22),(23).

**IV. RESOLVENT EQUATIONS**

This section deals with the Fredholm-type equations for the resolvent \( R(z) \) of the s.a. Hamiltonian \( H \). As in the case of energy-independent interactions, these equations provide the basis for the three-body scattering problem.

All the results of this section can be extended to the \( N \)-body case for arbitrary \( N \).

First we shall derive differential equations for the resolvent components \( R_{ab}(z) \) corresponding to the decomposition of \( \mathcal{S} \) into the sum (24),

\[ R(z) = \{ R_{ab}(z) \}, \quad a,b = 0,1,2,3. \tag{31} \]

Here the indices \( a,b \) stand for the external \( (a,b = 0) \) and internal \((a,b = 1,2,3)\) subspaces \( \mathcal{S}^a = L^2(\mathbb{R}^6) \) and \( \mathcal{S}_\alpha^a \), \( \alpha = 1,2,3 \).

Because \( R(z) \) is the resolvent of the s.a. operator \( H \) it satisfies the usual relation

\[ R_{ab}(z) = R_{ba}(z). \tag{32} \]

We shall introduce the following notations. Let \( F \) be an arbitrary element of \( \mathcal{S} \) and \( \mathcal{S} = R(z) F \), i.e., \( F = \{ f_\alpha \}, \quad a = 0,1,2,3, \)

\[ u_\alpha = \sum_\beta R_{ab}(z) f_\beta. \tag{33} \]

Then due to (27) and (33) one gets

\[ \xi^+ = \sum_{b=0}^{\alpha} \xi^+_b f_b, \quad \alpha = 1,2,3, \tag{34} \]

where \( \xi^+_b \) are the operators that act from \( \mathcal{S}^a \) in \( L^2(\mathbb{R}^6_\alpha) \) at \( b = 0 \) and from \( \mathcal{S}_\alpha^a \) in \( L^2(\mathbb{R}^6_\alpha) \) at \( b \neq 0 \). The relation (34) can be considered as the definition of these operators.

Let \( \tilde{R}_{ab}(z) \) denote the operators

\[ \tilde{R}_{ab}(z) = (R_{ab} - w^+ \xi^+_b - w^- \xi^- B_{ab}) f_b, \quad \alpha = 1,2,3, \tag{35} \]

Then using the identity

\[ (H - z) R(z) F = F, \tag{36} \]

one can obtain the set of equations for the kernels of the operators \( R_{ab}(z) \) and \( \mathcal{S}^a_{\alpha b}(z) \):

\[ (H^{aa} - z) R_{ab}(z) = \delta_{ab} I_\alpha, \tag{37} \]

\[ A_\alpha \tilde{R}_{ab} - w^- \xi^+_b + w^+ \xi^- B_{ab} - (\Delta_\alpha + z) R_{ab}(z) = \delta_{ab} I_\alpha, \tag{38} \]

with the following boundary conditions:

\[ \left( \begin{array}{cc}
\delta_{\alpha b} & \xi^+ \\
\xi^- & \delta_{\alpha b}
\end{array} \right) = -\varphi_\alpha \mathcal{S}_{\alpha b}, \tag{39} \]

\[ \mathcal{S}_{\alpha b} = (R_{ab} \varphi_\alpha) \varphi_\alpha. \tag{40} \]

Equations (37) and (38) representing the set of differential equations for the external \( R_{ab} \) and internal \( \tilde{R}_{ab} \) components of the resolvent \( R(z) \) serve as background for the construction of the Faddeev equations.

We shall rewrite the conditions (39) and (40) in terms of the internal Hamiltonians \( A_\alpha \).

For this purpose we use the relation:

\[ \xi^+ = Q_\alpha(z) \xi^+ + \delta_{ab} ((A_\alpha - I)(H^a_\alpha - z)^{-1}, \theta_\alpha), \tag{41} \]

which can be obtained by arguments analogous to the two-body case" [see (11)] here.

\[ H^a_\alpha = A_\alpha \otimes I_\alpha + I_\alpha \otimes (-\Delta_\alpha) \tag{42} \]

and \( Q_\alpha(z) \) is the generalization of the Schwartz integral in the three-body configuration space:

\[ Q_\alpha(z) = (U + (\Delta_\alpha + z) A_\alpha)(H^a_\alpha - z)^{-1} \theta_\alpha, \theta_\alpha. \tag{43} \]

This operator, in accordance with (18), can be realized as the integral operator having the kernel
\[ Q_\alpha (y_\alpha - y'_\alpha, z) = \frac{1}{2\pi i} \int_{y_\alpha}^{y'_\alpha} d\lambda \Delta_\alpha (\lambda) r_\alpha^{(\alpha)} (y_\alpha - y'_\alpha, z - \lambda). \]  

(44)

Here, \( r_\alpha^{(\alpha)} (z) = (h_\alpha^{(\alpha)} - z)^{-1} \) is the resolvent of the two-body s.a. Hamiltonian \( h_\alpha^{(\alpha)} \); \( \Delta_\alpha (\lambda) \) is the two-body Schwartz integral, and the counter \( \Delta_\alpha \) encircles the spectrum of \( A_\alpha \).

The operators \( \mathcal{R}_\alpha \) can now be excluded from (39) and (40) by virtue of relation (41):

\[
[\partial_\alpha R_{0k}]_{\alpha} = -\varphi_\alpha \{Q_\alpha (z) \langle R_{0k}; \varphi_\alpha \rangle \\
+ \delta_{\alpha k} (A_\alpha - i)(h_\alpha^{(\alpha)} - z)^{-1} \varphi_\alpha \}. \tag{45}
\]

If the internal channel Hamiltonians \( A_\alpha \) have the point spectra \( \sigma (A_\alpha) = \{x_\alpha^\beta\} \) only, then the kernels \( Q_\alpha \) should be written in the form

\[
Q_\alpha (y_\alpha - y'_\alpha, z) = \sum_x (1 + (x_\alpha^\beta)^2) \{P_x^{\alpha} \varphi_\alpha, \varphi_\alpha \}
\times r_\alpha^{(\alpha)} (y_\alpha - y'_\alpha, z - x_\alpha^\beta), \tag{46}
\]

where \( P_x^{\alpha} \) are spectral projectors of the operators \( A_\alpha \).

Notice that such kind of internal Hamiltonians are used for describing internal channels, e.g., with quark confinement.5,6

V. FADDEEV EQUATIONS

The study of the total resolvent \( R(z) \) can be reduced to considering the external-channel component \( R_{00}(z) \) only. In order to see this, Eqs. (37)–(41) should be used. Namely, let the component \( R_{00}(z) \) be known. From (40) we can get \( \mathcal{R}_\alpha (z) \) for substitution into (41) to yield \( \mathcal{R}_\alpha (z) \). Then from (38) one can obtain \( \mathcal{R}_\alpha (z) \). It gives the components \( R_{00}(z) \), \( \alpha = 1,2,3 \). Then, in accordance with (32) the components \( R_{00}(z) \), \( \alpha = 1,2,3 \) will also be known. The diagram in Fig. 2 illustrates this procedure. Thus we shall now deal with \( R_{00}(z) \) which, for simplicity, is denoted by \( G(z) \). It should be noticed that \( G(z) \) is the so-called Krein’s quasiresolvent and it has corresponding properties.

By Eq. (45) the kernel \( G(X,X',z) \) of the quasi-resolvent \( G(z) \) obeys the boundary conditions

\[
[\partial_\alpha G(X,X',z)]_{\alpha} = -\varphi_\alpha (x_\alpha) Q_\alpha (z) \langle G(z); \varphi_\alpha \rangle. \tag{47}
\]

As in the two-body case, these conditions can be written in terms of quasipotentials

\[
W_\alpha (z) \varphi = \delta_{\Gamma_\alpha} V_\alpha (z) \varphi, \tag{48}
\]

where \( V_\alpha (z) \) is the integral operator in \( L^2(\Gamma_\alpha) \) with the kernel

\[
V_\alpha (X,X',z) = -\varphi_\alpha (x_\alpha) Q_\alpha (y_\alpha - y'_\alpha, z) \varphi_\alpha (x'_\alpha). \tag{49}
\]

In accordance with (37), (47), and (48) we obtain the following equation:

\[
\left( H^{X} + \sum_{\alpha=1}^{3} W_\alpha (z) - z \right) G(X,X',z) = \delta (X - X'). \tag{50}
\]

To derive an integral equation for Krein’s quasiresolvent one can use the usual procedure. Namely, applying the operator \( R_0(z) \) to (50) we obtain the resolvent identity for \( G(z) \):

\[
G(z) = R_0(z) - R_0(z) \sum_{\alpha=1}^{3} W_\alpha (z) G(z). \tag{51}
\]

In accordance with this equation of Lippman—Schwinger type, the operator \( G(z) \) may be represented explicitly in terms of generalized operators

\[
M_\alpha (z) = W_\alpha (z) G(z) \tag{52}
\]

by the relation

\[
G(z) = R_0(z) - R_0(z) \sum_{\alpha=1}^{3} M_\alpha (z). \tag{53}
\]

Thus we reduce the problem of investigating the quasi-resolvent \( G(z) \) to the study of the operators \( M_\alpha (z) \).

The next problem is to derive the Faddeev equations from Eq. (53). Applying the operators \( W_\alpha (z) \) to (53) one can write this equation in the form

\[
(I + W_\alpha R_0) M_\alpha = W_\alpha R_0 - W_\alpha R_0 \sum_{\beta \neq \alpha} M_\beta. \tag{54}
\]

Following Faddeev’s method we have to invert the operator \( I + W_\alpha R_0 \). This inversion may be done explicitly in terms of the two-body operator \( G_\alpha = (H\alpha - z)^{-1} \), which is the resolvent of the s.a. operator \( H_\alpha \). The following formula can be verified:

\[
(I + W_\alpha R_0) W_\alpha G_\alpha = W_\alpha R_0. \tag{55}
\]

This relation yields in a straightforward way the equations

\[
M_\alpha (z) = W_\alpha G_\alpha (z) - W_\alpha G_\alpha (z) \sum_{\beta \neq \alpha} M_\beta (z), \tag{56}
\]

which have the structure of Faddeev equations.

Nevertheless, to be convinced that these equations are the Faddeev ones, one must prove the following statement.

Theorem 2: Let \( \mu_\alpha \) be the densities of the simple layer potentials \( M_\alpha (z) = \delta_{\Gamma_\alpha} \mu_\alpha \) and \( \mu = (\mu_1, \mu_2, \mu_3) \). Then the following can be proven:

(1) Equations (56) rewritten in terms of densities \( \mu_\alpha \):

\[
\mu(z) = \mu_\alpha(z) + B(z) \mu(z) \tag{57}
\]

are of Fredholm type and \( B^* \), \( n > N_{max} \) with sufficiently large \( N_{max} \) is a compact operator in an appropriate Banach space.

(2) Equations (56) or (57) are spectral equivalent to

FIG. 2. Diagram for the reconstruction of the resolvent components \( R_{\alpha \beta} \), \( \alpha, \beta = 1,2,3 \).
the original Schrödinger equation with the s.a. Hamiltonian $H$.

The proof of the first statement proceeds in a standard way. Nevertheless, for the reader’s convenience we sketch it in Appendix B.

The second statement of the theorem is much more delicate in contrast to the case of energy-independent interactions. Namely, we must show that the homogeneous equations (57) have a nontrivial solution, if and only if $2z\sigma_a(H)$, where $\sigma_a(H)$ is the point spectrum of the s.a. operator $H$.

Let $\mu$ be the solution of the homogeneous equations

$$\mu_a(z) = -V_a(z)G_a(z) \sum_{\beta \neq a} \mu_\beta(z). \quad (58)$$

Consider the function

$$u_0 = R_0(z) \sum_\beta \mu_\beta,$$

which is evidently the simple layer potential given on the hypersurface $\Gamma = U_a \Gamma_a$, and hence it satisfied the equation

$$(H^e - z)u_0(X) = 0, \quad X \in \Gamma. \quad (59)$$

To find the appropriate boundary conditions one must apply the operator $I + V_a R_0$ to Eq. (58). Taking into account (55) and the properties of simple layer potentials $[\partial_n u_0]_{\Gamma_a} = -\mu_a$, we find the boundary conditions

$$[\partial_n u_0]_{\Gamma_a} = V_a(z)u_0. \quad (60)$$

By means of iterations of (58) one can achieve that $\mu_\alpha \in W^2_{\text{loc}}(\mathbb{R}^5, \mathbb{C})$ both at $\text{Im} z \neq 0$ and at $z = E \pm i0, E \in \mathbb{R}$ and furthermore that $(u_0, \Phi_\alpha) \in W^2_{\text{loc}}(\mathbb{R}^5, \mathbb{C})$. Now we shall express the internal functions $\mu_a$ in terms of the external component $u_0$. To this end one must take into account the representation $u_0$ in the form (28) and relations (22) as well as (21), which state the connection between $\xi^\pm$ and $u_0$:

$$\xi^+ = (u_0, \Phi_\alpha), \quad (61)$$

$$\xi^- = Q_\alpha(z) \xi^+. \quad (62)$$

The functions $u_\alpha$ may then be found as the solution of the equation

$$( -\Delta_{y_\alpha} + A_{\alpha} - z )u_\alpha$$

$$= \xi^+(y_\alpha) u^- + \xi^-(y_\alpha) u^+$

$$+ ( \Delta_{y_\alpha} + z ) ( \xi^- (y_\alpha) u^- + \xi^+(y_\alpha) u^+ ). \quad (63)$$

By virtue of Eq. (61) the functions $\xi^\pm \in W^2_{\text{loc}}(\mathbb{R}^5)$ and hence $\xi^- \in W^2_{\text{loc}}(\mathbb{R}^5)$. This means that $\mathcal{H} = (u_\alpha, u_\alpha)$ $\in \mathcal{D}(H)$ and, in accordance with (59)-(63), $\mathcal{H}$ is an eigenvector of the s.a. operator $H$:

$$(H - z) \mathcal{H} = 0. \quad (64)$$

This equation implies that $\mathcal{H} = 0$ if $z \notin \sigma_a(H)$ and hence $u_\alpha = 0$. In other words we have proven that Eq. (57) has a unique solution, if $z \notin \sigma_a(H)$.

On the contrary let $\mathcal{H}$ be an eigenvector of the Hamiltonian $H$. Then one must repeat the derivation of (56) for densities $\mu_\alpha = -V_\alpha u_\alpha$ which obey Eqs. (58).

Hence we have proven that the Faddeev equations (58) are “spectral equivalent” to the original Schrödinger equation (64).

Consequently, the Fredholm alternative may be applied to (57) and the properties of densities $\mu_\alpha$ may be investigated and its completeness in the whole space $\mathcal{H}$ established using the methods of Refs. 19 and 20.

VI. INTERNAL-CHANNEL FADDEEV EQUATIONS

In this section, we shall derive Faddeev equations for the boundary values $\mathcal{F}^{\pm}_{ab}(z)$ of the resolvent components $R_{ab}(z)$.

First, one observes that due to the separability of the quasipotentials $V_{\alpha}(z)$, Eq. (56) can be written in terms of kernels of the operators $\mathcal{F}^{\pm}_{ab}(z)$,

$$\mathcal{F}^{\pm}_{ab}(y_\alpha, X', z) = \mathcal{Q}_a(y_\alpha - y'_\alpha, z) \langle G(z), \Phi_\alpha \rangle, \quad (65)$$

which by (40), (41), and (42) are the boundary values of $R_{ab}(z)$. In order to obtain equations for the operators $\mathcal{F}^{\pm}_{ab}(z)$, it is advantageous to pick out coefficients at the $\Phi_\alpha$ in (56) using the relations (48) and (49). Due to this procedure one gets

$$\mathcal{F}^{\pm}_{ab}(z) = D^{\pm}_{ab}(z) - \sum_{\beta \neq a} D^{\pm}_{a\beta}(z) \mathcal{F}^{\pm}_{ab}(z). \quad (66)$$

Here, the operators $D^{\pm}_{ab}(z)$ are average value of the resolvent

$$D^{\pm}_{ab}(z) = \langle G(z), \Phi_\alpha \rangle \quad (67)$$

and

$$D^{\pm}_{a\beta}(z) = \langle D^{\pm}_{a}(z), \Phi_\beta \rangle. \quad (68)$$

Equations (66) represent a set of integral equations in internal subspaces for Faddeev-type operators $\mathcal{F}^{\pm}_{ab}(z)$.

Let us note that there exists an additional effect, which can be taken into account. Namely, owing to the separability of the potentials $V_{\alpha}(z)$ Eqs. (66) are three dimensional in contrast to the five-dimensional equations (56). This property can be important for numerical calculations.

VII. DIFFERENTIAL FADDEEV EQUATIONS FOR COMPONENTS

The Faddeev differential equations are known to be useful for numerical calculations in nuclear physics. Let us discuss similar equations for three particles interacting via energy-dependent potentials. For their derivation we define the Faddeev components $G^{\alpha}(z)$ of the external-channel quasiresolvent $G(z)$ in the usual form:

$$G^{\alpha}(z) = \delta_{\alpha a} R_0(z) - R_0(z) W_{ab}(z) G(z). \quad (69)$$

By this definition the quasiresolvent $G(z)$ is the sum of its components

$$G(z) = \sum_{\alpha = 1}^{3} G^{\alpha}(z). \quad (70)$$

Applying the “operator” $H^e + W_{\alpha}(z) - z$ to Eq. (69) we obtain the differential equations

$$(H^e + W_{\alpha}(z) - z) G^{\alpha}(X, X', z)$$

$$= \delta_{a\alpha} \delta(X - X') - W_{ab}(z) \sum_{\beta \neq a} G^{\beta}(X, X', z) \quad (71)$$

to be fulfilled by the components $G^{\alpha}(z)$.

By Eq. (47) the Faddeev components $G^{\alpha}(z)$ obey the boundary conditions


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\[ [\partial_{\alpha} G^\alpha]_{r_a} = -\varphi^\alpha Q^\alpha (z) \sum_{\beta} \langle G^\beta (z), \varphi_{\beta} \rangle. \]  
(72)

Here we have used the following property of \( G^\alpha (z) \):
\[ [\partial_{\alpha} G^\alpha]_{r_a} = 0, \quad \alpha \neq \beta, \]  
(73)

which can be obtained by virtue of the usual properties of simple layer potentials.\(^{18}\)

If \( Z \) is real, i.e., \( Z = E + i0 \), the differential equations (71) together with the boundary conditions described in Refs. 19 and 20 define a unique solution for the wave functions.

The methods of Ref. 15 may be used to solve this boundary-value problem numerically.

At the end we would like to point out that all results and statements about equations for the Faddeev components \( G^\alpha (z) \) and for the operators \( M^\alpha (z) \) can be rigorously obtained only on the basis of equations for the total resolvent \( R(z) \) or, in other words, on the basis of the s.a. Hamiltonian \( H \).

VIII. DISCUSSION

In this paper, we have presented a new approach toward a mathematically correct study of the scattering theory for few-body systems with energy-dependent potentials. The main result is that the treating of such systems in usual configuration space is inconsistent from an operator point of view. We have demonstrated that an energy dependence of the potentials is generated by the internal structure of the interacting particles. This energy dependence, however, turns out not to be arbitrary, since it is given by some class of operator-valued \( R \) functions, including, in particular, Schwartz integrals as described above.

The main effect incorporated in our scheme is the possibility to separate the contributions from two-body and three-body forces. We remark that also many-body forces can be included into our consideration without a drastic change of the formulation.

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APPENDIX A: PROOF OF THEOREM 1

Let us prove that
\[ \mathcal{D}(H_{\alpha}) \subset \mathcal{D}(\tilde{H}) \subset \mathcal{D}(H_{\alpha}^\text{a}). \]  
(A1)

Let \( \xi^- (y_{\alpha}) \) be smooth functions with compact supports from \( L^2 (R^n_a) \) and define the densities \( \rho_{\alpha} \) and the functions \( \xi^+ (y_{\alpha}) \) by (28) and (29).

The representations (28) and (29) give us a possibility to reconstruct the components \( u_{\alpha}, \alpha = 0,1,2,3 \) with arbitrary elements \( \tilde{u}_{\alpha} \in \mathcal{D}(H_{\alpha}^\text{in}) \) and \( \bar{u}_{\alpha} \in C^\infty (R^n \setminus \Gamma) \). The elements \( \mathcal{D} \subset \{ u_{\alpha} \} \) defined in such manner, belongs to the domain \( \mathcal{D}(H_{\alpha}^\text{a}) \). To prove this it is sufficient to verify that the internal components \( u_{\alpha} \in \mathcal{D}(H_{\alpha}^\text{in}) \) or, what is the same, that the functions \( \xi^+ (y_{\alpha}) \in W^2_2 (R^n_a) \).

The simple layer potential \( R \rho_{\alpha} \) given as a function on the cylinder \( \Gamma_{\alpha} \), \( \alpha \neq \beta \) is not a smooth function in the neighborhood of the set \( \Gamma_{\alpha} = \cap_{\alpha}, \Gamma_{\beta} \). \( \alpha = \beta \) then the smoothness of \( R \rho_{\alpha} \) is defined by the smoothness of its density \( \rho_{\alpha} \). Nevertheless, the average \( \langle R \rho_{\alpha} (y_{\alpha}) \rangle \) on the section \( \{ y_{\alpha} = \text{const} \} \) of the cylinder \( \Gamma_{\alpha} \) is the \( W^2_2 \) smooth function. The proof of the last statement is based on the following local representation:

\[ R \rho_{\alpha} (x) = \frac{1}{2} \text{dist}(x, \Gamma_{\beta}) \rho_{\beta} (S^\alpha) + \tilde{R}_{\alpha} (x), \]

which takes place for the smooth \( \rho_{\beta} \). Here, \( S^\alpha \) is the projection of the point \( x \) on the cylinder \( \Gamma_{\beta} \) and \( R_{\alpha} (x) \) is a smooth function. In fact, one must verify the smoothness of the average \( \langle \text{dist} (., \Gamma_{\beta}) \rho_{\alpha} (y_{\alpha}) \rangle \) in the neighborhood of the set \( \Gamma_{\alpha} \), considered as a function of the distance between the plane \( \{ y_{\alpha} = \text{const} \} \) and the set \( \Gamma_{\beta} \). It can be done immediately. Thus we have proved that (A1) is true.

As a consequence of (A1), the closure of \( \mathcal{D}(\tilde{H}) \) coincides with the total Hilbert space \( \phi \).

To prove the symmetry of the operator \( \tilde{H} \), one must calculate the boundary form
\[ J(\varphi, \varphi') = \langle \tilde{H} \varphi, \varphi' \rangle - \langle \varphi, \tilde{H} \varphi' \rangle. \]  
(A2)

In order to estimate the contribution of the external channel operator into the total boundary form (A2), it is convenient to make such calculations for a system of smooth "parallel surfaces":

\[ \Gamma_{\delta}^1 = \{ x: \text{dist}(x, \Gamma_{\delta}) = \delta \}, \quad \delta > 0, \]
\[ \Gamma_{\delta}^0 \cap \{ x: \text{dist}(x, \Gamma_{\alpha}) > \delta \}, \quad \delta > 0, \]

in the limit \( \delta \to 0 \). The integration over the pieces \( \Gamma_{\delta}^1 \cap \{ x: \text{dist}(x, \Gamma_{\alpha}) > \delta \} \) gives the sum of the integrals over the cylinders \( \Gamma_{\alpha} \) when \( \delta \to 0 \). The contribution from the integration over \( \Gamma_{\delta}^0 \cap \{ x: \text{dist}(x, \Gamma_{\alpha}) > \delta \} \), \( \delta > 0 \), vanishes, if the simple layer potentials \( R \rho_{\alpha} \) were generated by smooth densities. It takes place because such simple layer potentials \( R \rho_{\alpha} \) have both uniformly bounded values and bounded normal derivatives on the surface \( \Gamma_{\delta}^0 \).

Thus the calculation of the boundary form corresponding to the external channel can be reduced to the evaluation of the contribution from every cylinder \( \Gamma_{\delta}^0 \):

\[ J(u_0, v_0) = \sum_{\alpha=1}^{3} \lim_{\delta \to 0} \int_{\Gamma_{\delta}^0} dS_{\alpha} (\partial_{\alpha} u_0 \bar{v}_0 - \partial_{\alpha} \bar{v}_0 u_0). \]

The external components \( u_0, v_0 \in \mathcal{D}(\tilde{H}) \) satisfy the boundary conditions (22), (23), in the form (28), (29). So, the contribution into the external boundary form can be written as
\[
\lim_{\delta \to 0} \int_{\Gamma_{\delta}^0} dS_{\alpha} \left( \left[ \partial_{\alpha} u_{\alpha} \right]_{r_{\alpha}a} \bar{v}_0 - u_{\alpha} \left[ \partial_{\alpha} \bar{v}_{\alpha} \right]_{r_{\alpha}a} \right) = \int_{\Gamma_{\delta}^0} dS_{\alpha} \left( \sum_{\beta} R_{\beta} \rho_{\beta} (u_{\alpha}) \varphi_{\beta}^{\alpha+} (u_{\alpha}) - \sum_{\beta} R_{\beta} \rho_{\beta} (u_{\alpha}) \xi^{\alpha-} (u_{\alpha}) \varphi_{\beta}^{\alpha+} (u_{\alpha}) - \sum_{\beta} R_{\beta} \rho_{\beta} (u_{\alpha}) \xi^{\alpha-} (u_{\alpha}) \varphi_{\beta}^{\alpha-} (u_{\alpha}) \right).
\]

This coincides with the contribution from the boundary form of the operator \( H_{\alpha}^\text{in} \) acting in the internal channel \( \mathcal{D}_{\alpha}^\text{in} \).


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Hence, the total boundary form (A2) is equal to zero. In other words, the operator \( \tilde{H} \) is the symmetric one.

To prove that the operator \( \tilde{H} \) is bounded from below consider its quadratic form on the domain \( \mathcal{D}(\tilde{H}) \) [We omit here, for simplicity, all energy-independent potentials \( v_x(x_a) \) in the external-channel Hamiltonian \( H^{xx} \)]:

\[
\langle \tilde{H} \varphi, \varphi \rangle = \langle -\Delta x v_{x0} u_0 \rangle + \sum_{a} \langle (A_{a}^{*} - \Delta_{x_{a}}) u_{a}, u_{a} \rangle.
\]

Integrating by parts

\[
\langle \tilde{H} \varphi, \varphi \rangle = \| \nabla u_0 \|_{L^2(W_4)}^2 - \sum_{a} \int_{\Gamma_{a}} dS_{a} [\partial_{n} u_{0}]_{\Gamma_{a}} \tilde{u}_{0}
\]

\[
+ \sum_{a} \left\{ \langle A_{a}^{*} u_{a}, u_{a} \rangle + \| \nabla u_{a} \|_{L^2(W_{4a})}^2 \right\},
\]

one can show that the boundary terms in (A3) can be estimated by the Dirichlet form of the operators \(-\Delta_{x}\) and \(-\Delta_{x_{a}}\), and also by the norms of the elements from the external and internal channels.

First, let us estimate the quadratic form of the operator \( A_{a}^{*} \). In the representation

\[
u_{a} = \tilde{u}_{a} + \xi_{a}^{\perp} w_{a}^{\perp} + \xi_{a}^{\perp} w_{a}^{\perp},
\]

the boundary values \( \xi_{a}^{\perp} \) are not arbitrary but are connected by the relation

\[
\xi_{a}^{\perp} (\partial_{\alpha} \partial_{\alpha}) = \langle (A_{a} - i) u_{a}, \partial_{\alpha} \partial_{\alpha} \rangle - \xi_{a}^{\perp} \langle A_{a} \partial_{\alpha} \partial_{\alpha} \rangle.
\]

This formula can be derived immediately from the condition \( \partial_{\alpha} (A_{a} - i) \tilde{u}_{a} \), where \( \partial_{\alpha} \) is the generative element of the Hamiltonian \( A_{a} \). By the relation (A5) one can estimate the \( L^2 \)-norm of the boundary vector \( \xi_{a}^{\perp} \):

\[
\| \xi_{a}^{\perp} \|_{L^2(W_{4a})} < C \left[ \| \xi_{a}^{\perp} \|_{L^2(W_{4a})} + \| u_{a} \|_{W_{1,0}} \right].
\]

By means of decomposition (A4) we have

\[
\| \tilde{u}_{a} \|_{W_{1,0}^{0}} < C \left[ \| u_{a} \|_{W_{1,0}^{0}} + \| \xi_{a}^{\perp} \|_{L^2(W_{4a})} + \| \xi_{a}^{\perp} \|_{L^2(W_{3a})} \right]
\]

\[
< C \left[ \| \xi_{a}^{\perp} \|_{L^2(W_{4a})} + \| u_{a} \|_{W_{1,0}^{0}} \right].
\]

By the equality

\[
A_{a}^{*} u_{a} = A_{a} \tilde{u}_{a} - \xi_{a}^{\perp} w_{a}^{\perp} + \xi_{a}^{\perp} w_{a}^{\perp},
\]

and under the assumption that the operator \( A_{a} \) is bounded, one has

\[
\| A_{a}^{*} u_{a} \|_{W_{1,0}^{0}} \leq C \left[ \| u_{a} \|_{W_{1,0}^{0}} + \| \xi_{a}^{\perp} \|_{L^2(W_{3a})} \right].
\]

Then, taking into account (23) one can estimate the quadratic form (A8) in terms of the external element

\[
\| \xi_{a}^{\perp} \|_{L^2(W_{3a})} < \| \varphi_{a} \|_{L^2(W_{4a})} \| u_{a} \|_{L^2(W_{3a})}.
\]

Therefore,

\[
\sum_{a} \langle A_{a}^{*} u_{a}, u_{a} \rangle \leq C \sum_{a} \left( \| u_{a} \|_{W_{1,0}^{0}}^2 + \| u_{a} \|_{L^2(W_{3a})}^2 \right).
\]

By the condition (22) and the relations (A6),(A9) we obtain

\[
\sum_{a} \langle A_{a}^{*} u_{a}, u_{a} \rangle \leq C \sum_{a} \left( \| u_{a} \|_{W_{1,0}^{0}}^2 + \| u_{a} \|_{L^2(W_{3a})}^2 \right).
\]

Using the embedding theorem

\[
\int_{\Gamma} |u_{a}|^2 dS < C \left( \delta \int |\nabla u_{0}|^2 dX + \frac{1}{\delta} \int |u_{0}|^2 dX \right),
\]

\[
\delta > 0,
\]

and collecting together (A10)–(A12) we have

\[
\langle \tilde{H} \varphi, \varphi \rangle \geq (1 - \delta C) \| u_{0} \|_{L^2(W_{4a})}^2
\]

\[
+ \sum_{a} \| \nabla u_{a} \|_{W_{1,0}^{0}}^2 - \frac{C}{\delta} \| u_{0} \|_{L^2(W_{3a})}^2
\]

\[
- C \sum_{a} \| u_{a} \|_{W_{1,0}^{0}}^2.
\]

If \( \delta C < 1 \) then

\[
\langle \tilde{H} \varphi, \varphi \rangle \geq \max \left\{ \frac{C}{\delta} C \right\} \left[ \| u_{0} \|_{W_{1,0}^{0}}^2 + \sum_{a} \| u_{a} \|_{W_{1,0}^{0}}^2 \right]
\]

\[
= - C \| u_{0} \|_{W_{1,0}^{0}}^2.
\]

It means that the operator \( \tilde{H} \) is bounded from below. So, Theorem 1 is completely proved.

Let us make some comments about essential points of the proof. The most important question concerns the presence of three-body forces in the model. From the geometrical point of view such kind forces are connected with the boundary conditions, which may be stated on the manifold \( \Gamma_{0} = \cap_{a} \Gamma_{a} \). The deficiency subspaces of the operator \( H_{0} \) corresponding to the manifold \( \Gamma_{0} \) are parametrized by simple layer densities belonging to the Sobolev class \( W_{2}^{-3/2} \). In order to conserve the pair character of the boundary conditions (28),(29) we do not include such deficiency elements into the domain \( \mathcal{D}(\tilde{H}) \).

**APPENDIX B: PROOF OF THEOREM 2**

Let us check the Fredholm nature of Eq. (57). Although the potentials \( W_{a}(z) \) are energy-dependent integral operators for variable \( y_{a} \) [see (48) and (49)] the kernels \( V_{a} G_{a}(S_{a}X'_{a}) \) have both standard analytical properties in the variable \( z \) and standard asymptotical behavior in the variables \( S_{a}X'_{a} \), which are typical for analogous kernels in the potential model \( ^{29} \) and in the boundary conditions model \( ^{10,11} \).

The representation (18) of the operator \( H_{a} \) ensures validity of the relation

\[
V_{a} G_{a}(z) = \frac{1}{2\pi i} \int_{\gamma} d\lambda \ v_{a} G_{a}(\lambda) r_{a}^{(a)}(\lambda - z),
\]

where \( g(z) \) is the generalized Green's function of the "operator" from Eq. (17) and \( V(z) \) is the integral operator in the representation (15) such that

\[
w(z) = \delta \cdot v(z),
\]

\[
v(x,x',z) = - \varphi(x) \Delta(z) \varphi(x').
\]
discrete and continuous spectrum. We shall consider the situation when the Hamiltonian \( h \) has one bound state \( \chi \) of energy \(-\chi^2\). The corresponding decomposition of the kernel \( vG(z) \) looks as follows

\[
vG(x,x',z) = -\varphi(x)\chi(x')[(z + x^2)N_0]^{-1}
\]

\[
+ vG(x,x',z), \tag{B4}
\]

where

\[
N_0^2 = -\frac{d}{dz}[\Delta^{-1}(z) - \langle g_0(z)|\varphi,\varphi|\rangle]_{x = -\chi}, \tag{B5}
\]

and \( g_0(z) = (h^{ex} - z)^{-1} \) is the resolvent of the s.a. operator \( h^{ex} \).

The decomposition (B4) leads to the following representation of the kernel \( V_\alpha G_\alpha(z) \):

\[
V_\alpha G_\alpha = V_\alpha G^d_\alpha + V_\alpha G^c_\alpha, \tag{B6}
\]

where

\[
\begin{align*}
V_\alpha G^d_\alpha (S,X',z) &= -\varphi_\alpha(s_\alpha)\chi_\alpha(x'_\alpha)[N_0^{-1}S_\alpha^d]^\alpha \\
&\times (y_\alpha - y'_\alpha + x^2) \\
\end{align*}
\]

and \( S = \{s_\alpha,y_\alpha\}, S_\alpha^d \subseteq \mathbb{R}^3 \). The kernel \( V_\alpha G^c_\alpha \) describes the contribution from the continuous spectrum.

We shall need the asymptotic of the kernel \( V_\alpha G^c_\alpha (S,X',z) \) when \( x'_\alpha \to \infty \). This asymptotic can be obtained by the saddle-point method \(^{21}\) from the asymptotic of the corresponding kernel \( vG^c \) related to the two-body problem,

\[
vG^c(x,x',z) \sim -\varphi(x)\psi^{(+)}_{\alpha} \frac{\exp[i\sqrt{2}L_a]}{d(z)2\pi|x'|}. \tag{B8}
\]

Here, \( \psi^{(+)}_{\alpha} \) is the wavefunction of the continuous spectrum corresponding to the operator \( h^{ex} \), \( p = -\sqrt{2}\mathbf{x}' \) and

\[
d(z) = \Delta^{-1}(z) - \langle g_0(z)|\varphi,\varphi|\rangle. \tag{B9}
\]

Then the asymptotic of the kernel \( V_\alpha G^c_\alpha (S,X',z) \) when \( x'_\alpha \to \infty \) looks as follows:

\[
V_\alpha G^c_\alpha (S,X',z) \sim \varphi_\alpha(s_\alpha)C_0(z)
\times \exp[i\sqrt{2}L_{\alpha}](2\pi)^{-3/2} \mathcal{F}_\alpha
\times (z \cos^2 \omega_\alpha \hat{X}'). \tag{B10}
\]

Here, \( L_{\alpha} \) is the eikonal \(^{20}\) that corresponds to the propagation of the ray from the point \( \{0,y_{\alpha}\} \) to the point \( \{x'_\alpha,y_{\alpha}\} \):

\[
L_{\alpha}(y_{\alpha},X') = [x_{\alpha}^2 + (y_{\alpha} - y'_\alpha)^2]^{1/2}. \tag{B11}
\]

The function \( \mathcal{F}_\alpha \) looks like

\[
\mathcal{F}_\alpha(z,\hat{X}') = -d_{\alpha}^{-1}(z)(\psi^{(+)}_{\alpha}|\varphi_\alpha), \tag{B12}
\]

where \( p_\alpha = -\sqrt{2}\mathbf{x}'_\alpha \) and

\[
\begin{align*}
C_0(z) &= (1/4\pi)(i)\sqrt{2}/2\pi)^{1/2} \\
\cos \omega_\alpha &= |x_{\alpha}|/L_{\alpha}. \tag{B13}
\end{align*}
\]

Using the relations obtained above one can notice that the asymptotic and analytic properties of the kernels \( V_\alpha G_\alpha \) coincide with those of the usual three-body problem with energy-independent potentials. \(^{26}\) Hence, one can prove the Fredholm property for Eq. (57) using the techniques, proposed in Refs. 10 and 11. Namely, the kernels \( V_\alpha \) and \( G_\alpha \) have the singularities \( |y_{\alpha} - y'_\alpha|^{-1} \) and \( |S_{\alpha} - S'_{\alpha}|^{-4} \), respectively. Consequently, a restriction of the operators \( V_\alpha G_\alpha \) to any bounded part of the surface \( \Gamma = U_{\alpha} \Gamma_{\alpha} \) leads to a compact operator, because dim \( \Gamma_{\alpha} = 5 \). On the other hand, because of the slow decrease of the kernels \( V_\alpha G_\alpha (S,S',z) \) at infinity, the operator \( B(z) \) is not compact. However, due to the Faddeev structure of the operator \( B(z) \), the arguments of the kernels \( V_\alpha G_\alpha \) are located on different cylinders \( \Gamma_{\alpha} \) and \( \Gamma_{\alpha}, \alpha \neq \beta \) and during the iteration procedure the products of operators \( V_\alpha G_\alpha, V_\alpha G_\alpha, \cdots V_\alpha G_\alpha \) only occur and \( \alpha \neq \alpha_{\alpha + 1} \).

Using the representation (B6) one can pick up terms of the following types in the kernels of the operators \( B''(z) \):

1. The products \( V_\alpha G^c_\alpha \cdots V_\alpha G^d_\alpha \cdots V_\alpha G^c_\alpha \), which contain not less than one operator \( V_\alpha G^c_\alpha \) corresponding to the discrete spectrum of the two-body subsystem Hamiltonian

2. The product \( V_\alpha G^c_\alpha \cdots V_\alpha G^c_\alpha \), which contains operators corresponding to the continuous spectrum only.

Due to the existence of the exponentially decreasing eigenfunctions \( \chi_{\alpha}(x_{\alpha}') \) in the kernel \( V_\alpha G^c_\alpha \) and the asymptotics of the first-type kernels \( V_\alpha G^d_\alpha \), the phase of the integrand is defined by the sum of distances between the points located on the cylinders \( \Gamma_{\alpha}, \Gamma_{\alpha}, \cdots, \Gamma_{\alpha_{\alpha + 1}} \) axes \( \{0,y_{\alpha}\}, y_{\alpha} \in \mathbb{R}^3 \).

Minimization of such a sum in the framework of the stationary-phase method gives the length of the trajectory of the ray that goes from the point \( S_{\alpha}' = \{0,y_{\alpha}\} \) to the point \( S_{\alpha} = \{0,y_{\alpha}\} \) by reflecting on the cylinders \( \Gamma_{\alpha}, \cdots, \Gamma_{\alpha_{\alpha + 1}} \) axes. As a result one can obtain the description of the asymptotics in terms of the corresponding eikonal. \(^{20}\) If such a process is impossible, i.e., the corresponding minimization leads to a spherical eikonal \( |S| + |S'| \), the kernel \( V_\alpha G^c_\alpha \cdots V_\alpha G^c_\alpha \) asymptotically turns in a product of spherical waves in \( \mathbb{R}^6 \) in variables \( S \) and \( S' \) when \( S \) and \( S' \to \infty \). Maximal number \( N_{\text{max}} \) of possible reflections on the cylinders axes is defined by the angles between these axes. Thus the \( n \)th power \( B^n(z) \) of the operator \( B(z) \) is a compact operator in a proper Banach space when \( n > N_{\text{max}} \). Hence, the Fredholm alternative applies to Eq. (57). The first statement of Theorem 2 is proved.


14V. S. Vladimirov, Distributions in Mathematical Physics (Nauka, Moscow, 1979) (in Russian).