Quantum scattering on a Cantor bar
Konstantin A. Makarov and Boris S. Pavlov

Citation: Journal of Mathematical Physics 35, 1522 (1994); doi: 10.1063/1.530604
View online: http://dx.doi.org/10.1063/1.530604
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/35/4?ver=pdfcov
Published by the AIP Publishing

Articles you may be interested in
EM localization for resonance frequency of pre-Cantor bar of higher stages

"Pathological" Cantor manifolds

Electronic excitations and correlations in quantum bars
Low Temp. Phys. 28, 539 (2002); 10.1063/1.1496664

A note on Bragg scattering of surface waves by sinusoidal bars

Wave interactions with generalized Cantor bar fractal multilayers
J. Appl. Phys. 70, 2500 (1991); 10.1063/1.349407
Quantum scattering on a Cantor bar

Konstantin A. Makarov
Fakultät für Mathematik, Ruhr Universität, D-4630 Bochum 1 Germany
and Department of Mathematical and Computational Physics,
St. Petersburg University, St. Petersburg, Russia

Boris S. Pavlov
Department of Mathematical and Computational Physics, St. Petersburg University,
St. Petersburg, Russia

(Received 25 July 1993; accepted for publication 3 December 1993)

A solvable model of the scattering of a one-dimensional particle by the familiar “middle-third” Cantor set is considered. It is shown that the presence of positive energy levels in such a model is typical. From the high-energy behavior of the scattering, data computation of the Hausdorff dimension of the scatterer is suggested.

I. INTRODUCTION

In recent years the class of solvable models in quantum mechanics has been essentially enlarged by means of extension theories. This approach seems to become the most advantageous tool for the investigation of the quantum systems with singular interactions, supported by thin manifolds or null-measure sets (with respect to the Lebesgue measure), see Refs. 3–5.

The other point of view on the subject is that the set of “singularities” of the interaction is regarded as a support of a family of singular measures or distributions in configuration space, which should “define” the interactions. Actually, an attractive class of models is generated by scattering problems by fractal sets. For so-called recursive (self-similar) fractals a natural homogeneous measure exists, thus these fractals are the most convenient for the aprobation of extension theory methods.

In this article we consider the simplest case of scattering by the Cantor bar, provided with the corresponding homogeneous singular continuous measure \( \mu \)

\[ \text{supp} \mu \subset \left[ -\frac{1}{2}, \frac{1}{2} \right]. \]

Specifically, let \( C \) be the ternary Cantor set which is constructed as follows. Let \( E_0 \) be the interval \([-\frac{1}{3}, \frac{1}{3}]\). Let \( E_1 \) be the set obtained by deleting the middle third of \( E_0 \), so that \( E_1 \) consists of the two intervals \([-\frac{1}{3}, -\frac{1}{9}] \) and \([-\frac{1}{3}, -\frac{1}{9}] \). Deleting the middle third of these intervals gives \( E_2 \). We continue in this way, with \( E_k \) obtained by deleting the middle third of each interval in \( E_{k-1} \). The middle-third Cantor set is the intersection \( C = \bigcap_{k=0}^{\infty} E_k \).

We construct a homogeneous measure \( \mu \) on the Cantor bar as follows. Split a unit mass so that the both left and right intervals of \( E_1 \) have mass \( \frac{1}{3} \). Divide the mass on each interval of \( E_1 \) between the two subintervals of \( E_2 \) in the ratio 1:1. Continue in this way, so that the mass on each interval of \( E_k \) is divided in the ratio 1:1 between its two subintervals in \( E_{k+1} \). This process defines a mass distribution \( \mu \) on the Cantor bar, which we call a homogeneous measure.

The Hamiltonian of the model is heuristically given by the rank-one perturbation of the one-dimensional Laplace operator
\[ H_\beta f = -f'' + \frac{1}{\beta} (f, \mu) \mu, \]

\(1/\beta\) being the coupling constant.

In Sec. II we give a rigorous meaning to the formal operator \(H_\beta\) in \(p\) representation by means of the extension theory. In this case of rank-one perturbation by generalized vector \(\mu\) all the scattering data can be computed from the characteristic function of the positive homogeneous Cantor measure

\[ u(p) = \int e^{ipx} d\mu(x). \tag{1} \]

II. THE HAMILTONIAN AND STATIONARY SCATTERING THEORY

We start with the symmetric restriction \(h_0\) of the self-adjoint (s.a.) one-dimensional Laplacian \(h\) on the axis \(\mathbb{R}\)

\[ \mathcal{D}(h_0) = \left\{ f : f \in \mathcal{D}(h) = H^{2,2}(\mathbb{R}), \int f(x) d\mu(x) = 0 \right\}, \]

\[ h_0 f = -f'', f \in \mathcal{D}(h_0). \tag{2} \]

The operator \(h_0\) is easily seen to be the densely defined symmetric operator with deficiency indices \((1,1)\) and it has a one-parameter family of s.a. extension \(H_\beta, \beta \in \mathbb{R}\). The resolvent \(R_\beta(z) = (H_\beta - z)^{-1}\) of \(H_\beta\) (in \(p\) representation) is given by

\[ R_\beta(z) = \frac{g(p)}{p^2 - z} - \frac{u(p)}{p^2 - z} \left( \frac{g, u}{p^2 - z} \right) d_\beta(z), \quad \text{Im } z \neq 0, \tag{3} \]

where

\[ d_\beta(z) = \beta + \int \frac{|u(p)|^2}{p^2 - z} dp \tag{4} \]

and \(u(p)\) is the Fourier transform of the measure \(\mu\). Let us note that by replacing the Cantor measure by the Dirac distribution concentrated at the origin one obtains the standard point interaction.\(^1\)\(^7\) The corresponding Fourier transform of the Dirac distribution is equal to 1 and Eqs. (3), (4) give in this case the resolvent of the one-center point interaction Hamiltonian.\(^1\)

From Eqs. (3), (4) we conclude that the generalized \(t\) matrix of the problem has an integral kernel \(t_\beta(p,k,z)\) given by

\[ t_\beta(p,k,z) = \frac{u(p) \tilde{u}(k)}{d_\beta(z)}. \tag{5} \]

The corresponding generalized eigenfunctions associated with \(H_\beta\) can be expressed in terms of the \(t\) matrix (see Ref. 8) as

\[ \psi^\pm(x,k) = e^{\pm ikx} \int dp \ e^{ipx} \frac{t_\beta(p,k^2 \pm i0)}{p^2 - k^2 \mp i0} = e^{\pm ikx} \frac{\tilde{u}(k)}{d_\beta(k^2 \pm i0)} \int dp \ e^{ipx} \frac{u(p)}{p^2 - k^2 \mp i0}. \]

Here \(\pm\) corresponds to the left and right incidence. Using Lemma 2 (see below) we get the scattering wave functions in the \(x\) representation outside the Cantor bar for the left incidence
\[
\psi_+(k,x) = \begin{cases} 
  e^{ikx} - e^{-ikx} \frac{\pi |u(k)|^2}{ikd_\beta(k^2)}, & x < -\frac{1}{2} \\
  e^{ikx} \left(1 - \frac{\pi |u(k)|^2}{ikd_\beta(k^2)}\right), & x > \frac{1}{2}
\end{cases}
\]

and for the right incidence

\[
\psi_-(k,x) = \begin{cases} 
  e^{-ikx} \left(1 - \frac{\pi |u(k)|^2}{ikd_\beta(k^2)}\right), & x < -\frac{1}{2} \\
  e^{ikx} - e^{-ikx} \frac{\pi |u(k)|^2}{ikd_\beta(k^2)}, & x > \frac{1}{2}
\end{cases}.
\]

The corresponding reflection coefficient then reads

\[
r(k) = -\frac{\pi |u(k)|^2}{ikd_\beta(k^2)},
\]

where \(d_\beta(k^2) = d_\beta(x)\) denotes the perturbation determinant

\[
d_\beta(k^2) = \beta + \int_{-\infty}^{\infty} \frac{|u(p)|^2}{p^2 - k^2} \, dp = \beta + \text{VP} \frac{1}{k} \int_{-\infty}^{\infty} \frac{|u(p)|^2}{p - k} \, dp.
\]

Here "VP" means principal value.

All the spectral properties of the considered Hamiltonian are determined by analytic properties of the function \(d_\beta(x)\), e.g., its roots correspond to the eigenvalues of the Hamiltonian.

**III. ANALYTIC PROPERTIES OF THE PERTURBATION DETERMINANT**

We start by the computation of the Fourier transform of the homogeneous Cantor measure \(\mu\) on the interval \([-\frac{1}{2}, \frac{1}{2}]\). For the sake of completeness of the treatment we note

**Lemma 1:** The Fourier transform of the Cantor measure \(\mu\) is given by the infinite product

\[
u(p) = \prod_{l=1}^{\infty} \cos \frac{p}{3^l},
\]

converging uniformly on every compact set of the complex plane and resulting in an even entire function of the exponential type \(\frac{1}{2}\) bounded by 1 on the real axis having real zeros \(p_{ln} = 3^l(n + \frac{1}{2})\), \(n \in \mathbb{Z}\), \(l = 1, 2, \ldots\).

**Proof:** Using the self-similarity property of the homogeneous measure \(\mu\) we have

\[
u(p) = e^{i(p/3)} \int_0^{1/3} e^{ipx} \, d\mu \left(\frac{x - 1/2}{2}\right) = e^{-i(p/3)} \left[ \int_0^{1/3} e^{ipx} \, d\mu \left(\frac{x - 1/2}{2}\right) + e^{i(p/3)} \int_0^{1/3} e^{ipx} \, d\mu \left(\frac{x - 1/2}{2}\right) \right]
\]

\[
e^{-i(p/3)} \cdot 2e^{i(p/3)} \cos \frac{p}{3} \int_0^{p/3} e^{ipx} \, d\mu \left(\frac{x - 1/2}{2}\right) = e^{-i(p/3)} \lim_{N \to \infty} e^{N(p/3 + (p/3)^2 + \cdots + (p/3)^N)}
\]

\[
\cdot 2^N \prod_{l=1}^{N} \cos \frac{p}{3^l} \cdot \int_0^{1/3} e^{ipx} \, d\mu \left(\frac{x - 1/2}{2}\right).
\]

Since, by definition \(\mu[-\frac{1}{2}, \frac{1}{2} + (1/3^N)] = 2^{-N}\) the following asymptotic is valid:
which proves Eq. (10).

Next, collect some explicit formulas concerning the Fourier transform of the generalized function $u(p)/(p^2-k^2+\text{i}0)$, which we have already used in Eq. (6).

**Lemma 2:** Outside the bar $x \in [-\frac{1}{2}, \frac{1}{2}]$ we have the following representation:

$$
\frac{1}{\pi i} \int e^{ipx} \frac{u(p)}{p^2-k^2+\text{i}0} = \frac{u(k)}{k} \begin{cases} 
\text{e}^{\pm ikx}, & x < -\frac{1}{2} \\
\text{e}^{\pm ikx}, & x > \frac{1}{2}.
\end{cases}
$$

**Proof:** The proof is an immediate consequence of the simple fact the the function $e^{ipx}u(p)$ exponentially decreases in the upper half plane (lower half plane) for $x > \frac{1}{2}$ ($x < -\frac{1}{2}$), since $u(p)$ is the Fourier transform of the finite measure with supp $\mu \subseteq [-\frac{1}{3}, \frac{1}{3}]$.

The explicit computation of the perturbation determinant uses similar ideas, but not in a such straightforward way.

**Lemma 3:**

$$
\imath \Pi \int_{-\infty}^{\infty} \frac{u^2(p)}{p-k} dp = \imath \Pi \sum_{l=1}^{\infty} \frac{e^{2i(k/3^l)}}{2^l} u^2\left(\frac{k}{3^l}\right). 
$$

**Proof:** Using the functional equation for $u^2$ (here $|u|^2 = u^2$)

$$
u^2(p) = \cos^2 \frac{p}{3} u^2\left(\frac{p}{3}\right)
$$

we get

$$
F(k) = \imath \Pi \int_{-\infty}^{\infty} \frac{u^2(p)}{p-k} dp = \frac{1}{4} \imath \Pi \int_{-\infty}^{\infty} \left(\frac{e^{2i(p/3)} + e^{-2i(p/3)}}{p-k}\right) u^2\left(\frac{k}{3}\right) dp + \frac{1}{2} F\left(\frac{k}{3}\right).
$$

The integral term can be easily calculated, since the functions $e^{2i(p/3)}u^2(p/3)$ and $e^{-2i(p/3)}u^2(p/3)$ are exponentially decreasing in the upper and lower half planes, respectively,

$$
\frac{1}{4} \imath \Pi \int_{-\infty}^{\infty} \frac{e^{2i(p/3)} + e^{-2i(p/3)}}{p-k} u^2\left(\frac{P}{3}\right) dp = \frac{1}{2} e^{2i(k/3)} u^2\left(\frac{k}{3}\right).
$$

Since $\lim_{N \to \infty} 2^{-N} F(3^{-N}k) = 0$, we have

$$
F(k) = \frac{1}{2} \sum_{l=1}^{\infty} e^{2i(l/3^l)} \cdot \frac{1}{2} u^2\left(\frac{k}{3^l}\right).
$$

Using this representation we have the following series for the perturbation determinant:

$$
d_{\rho}(k^2) = \beta + \frac{i\pi}{k} \sum_{l=1}^{\infty} e^{2i(k/3^l)} \frac{1}{2} u^2\left(\frac{k}{3^l}\right).
$$

Separating the imaginary part, we get for real $k$, $k \in \mathbb{R}$
\[ \text{Im } d_\beta(k^2) = \pi \frac{k}{2} \sum_{l=1}^{\infty} \left( 2 \cos^2 \frac{k}{3^l} - 1 \right) \frac{1}{2} u^2 \left( \frac{k}{3^l} \right) \]
\[ = \pi \frac{k}{2} \left[ u^2(k) - \frac{1}{2} u^2 \left( \frac{k}{3} \right) \right] + \frac{1}{2} \left[ u^3 \left( \frac{k}{3} \right) - \frac{1}{2} u^2 \left( \frac{k}{3^2} \right) \right] + \cdots \]
\[ = \pi \frac{k}{2} u^2(k), \quad (11) \]

\[ \text{Re } d_\beta(k^2) = \beta - \pi \frac{k}{2} \sum_{l=1}^{\infty} \sin \frac{2k}{3^l} \cdot \frac{1}{2} u^2 \left( \frac{k}{3^l} \right) \equiv \beta - a(k). \quad (12) \]

Thus the reflection coefficient can be represented as
\[ r(k) = -\frac{\pi u^2(k)}{ikd_\beta(k^2)} = -\frac{\pi u^2(k)}{ik[\beta - a(k) + i(\pi/k)u^2(k)]}. \quad (13) \]

**IV. SPECTRAL PROPERTIES OF THE HAMILTONIAN**

Obviously the spectral properties of \( H_\beta \) are determined by the singularities of the scattering matrix \( S(k) \) on the spectral sheet \( \text{Im } k > 0 \). The cut \( z > 0 \) (the real axis in \( k \) plane) corresponds to the continuous spectrum of \( H_\beta \). Of interest is the position of the eigenvalues of \( H_\beta \).

The negative discrete spectrum \( E = -\kappa^2 \) is determined by zeros \( k = i\kappa, \kappa > 0 \), of the function \( d_\beta(k^2) \) in the upper half plane
\[ d_\beta(-\kappa^2) = \beta + \pi \kappa \sum_{l=1}^{\infty} e^{-2\pi/3^l} \frac{1}{2^l} \text{ch} \left( \frac{\kappa}{3^l} \right). \quad (14) \]

There are no eigenvalues for positive \( \beta \) and for negative \( \beta \) there is the only one of them. If \( \beta < 0, |\beta| > 1 \), then there exists a unique eigenvalue with asymptotic behavior \( -\kappa^2 \sim -(\pi/\beta)^2 \). The corresponding eigenfunction exponentially decreases at infinity.

As for the positive eigenvalues, they lie on the set of positive zeros of the function \( d_\beta(z) \).

**Lemma 4:** For each value of the coupling constant \( \beta \) there exists only a finite set of positive eigenvalues. The singular continuous spectrum is empty. Positive eigenvalues coincide with real zeros of the perturbation determinant \( d_\beta(k^2) \). The corresponding eigenfunctions vanish outside the bar.

**Proof:** Let \( \kappa \) be a real zero of \( d_\beta(k^2) \). Then \( u^2(\kappa) = 0 \), and \( a(\kappa) = \beta \). Being an entire function, \( u \) has only simple zeros or zeros of the higher multiplicity. Therefore in this case the formal solution
\[ \bar{f}(p) = \frac{u(p)}{p^2 - \kappa^2} \]
of the homogeneous equation \( H_\beta \bar{f} = \kappa^2 \bar{f} \) belongs to \( L_2(\mathbb{R}) \), being an eigenfunction of \( H_\beta \) corresponding to the eigenvalue \( \kappa^2 > 0 \). Being an entire function of the order \( \frac{1}{2} \) it has the original
\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(p)e^{-ipx} \, dp, \]
which vanishes outside the bar.
Let $\tilde{f}(p)$ be an eigenfunction of $H_\beta$ corresponding to the eigenvalue $\kappa^2 > 0$. Then it has a form

$$\tilde{f}(p) = -\frac{u(p)}{p^2 - \kappa^2} \frac{1}{\beta}.$$ (15)

Multiplying Eq. (15) by $u$ and integrating over the real axis we also get

$$\left( \beta + \int \frac{|u(p)|^2}{p^2 - \kappa^2} \, dp \right) (\tilde{f}, u) = 0.$$ (16)

Since $\tilde{f}$ belongs to $L_2(\mathbb{R})$, from Eq. (16) we infer that $u(\pm \kappa) = 0$ and

$$\beta + \int \frac{|u(p)|^2}{p^2 - \kappa^2} \, dp = \beta + \frac{1}{\kappa} \int_{-\infty}^{\infty} \frac{|u(p)|^2}{p - \kappa} \, dp = \beta + \frac{1}{\kappa} \int_{-\infty}^{\infty} \frac{|u(p)|^2}{p - \kappa} \, dp = \text{Re} \, d_\beta(\kappa) = 0.$$ (17)

Therefore $d(\kappa^2) = 0$. Note, that it is necessary that $(\tilde{f}, u) \neq 0$ for the eigenfunctions of the discrete spectrum. If not, then the function $f$ would also be an eigenfunction of the discrete spectrum of the unperturbed operator, which cannot take place.

**Remark 1:** It is clear, that the eigenfunction $\varphi_{\kappa^2}$ corresponding to the positive eigenvalue $\kappa^2$ of the Hamiltonian $H_\beta$ is some linear combination of exponents $\exp(\pm ikx)$ outside the Cantor set on the gaps $x \in (\Delta_-, \Delta_+)$

$$\varphi_{\kappa^2}(x) = \frac{1}{2\pi} \int \frac{u(p)}{p^2 - \kappa^2} e^{-ipx} \, dp = -\frac{1}{2ik} \int e^{ik|x-s|} \, d\mu(s) - \frac{1}{2ik} e^{ikx} \int_{\Delta_-}^{\Delta_+} e^{-iks} \, d\mu(s).$$

Being an element of $L_2(\mathbb{R})$, this eigenfunction vanishes outside the bar for sure.

**Remark 2:** The critical set $\mathcal{B}$ of the inverse coupling constants $\beta$ for values of those positive eigenvalues does exist, and coincides with the set of the values of $\{a(k)\}$ on the set of zeros of the function $u$: $\{k_n = 3^{l/(n+1/2)}\}$. It is interesting, that this set is also self-similar

$$\frac{1}{3} \mathcal{B} \subset \mathcal{B}.$$ (18)

Really, the function $a$ has the property

$$a(3k_n) = 3a(k_n).$$

Thus, if $\beta \in \mathcal{B}$, then $\beta/3 \in \mathcal{B}$, $l=1,2,...$ It means that for $\beta$ small enough the existence of positive bound states "living" only on the fractal is rather typical.

**V. COMPUTATION OF HAUSDORFF DIMENSION FROM SCATTERING DATA**

Despite the fact that the perturbation determinant tends to $\beta$ like $o(k^{-1})$, the behavior of the reflection coefficient [see Eq. (18)]
is rather irregular at infinity, $k \to \infty$. Actually, a renormalized reflection coefficient
\[ r_n(k) = -\frac{ik\beta}{\pi} r(k) - u^2(k) \]
shows rather a chaotic behavior. Note, that
\[ u^2(3^N k) = \prod_{i=0}^{N-1} \cos^2 3^i k \cdot u^2(k). \]
Here, being an entire function, $u^2$ is rather smooth on every bounded interval of the real axis. But the factor $\prod_{i=0}^{N-1} \cos^2 3^i k \cdot u^2(k)$ has properties, which are similar to those of noise, if $N$ is sufficiently large (see, i.e., Ref. 9). Nevertheless, the averaged function
\[ F(R) = \frac{1}{R} \int_0^R |u(k)|^2 dk \]
has a regular behavior at infinity, $R \to \infty$
\[ F(R) \sim R^{-\kappa}, \quad (18) \]
where $\kappa = \log 2 / \log 3$ is the Hausdorff dimension of the Cantor set. Here we give the rigorous proof of this fact (see also Ref. 10 on a related topics).

**Theorem 1:** For $R$ large enough we have the following bounds:
\[ C_1 < R^{1-\kappa} \int_0^R u^2(p) dp < C_2, \]
with appropriate positive constants $C_i$, $i = 1, 2$.

**Proof:** We start with inequalities
\[ \left( \frac{3}{2} \right)^N u^2(\pi) \pi \lesssim \int_{3^N \pi}^{3^{N+1} \pi} u^2(p) dp \lesssim \left( \frac{3}{2} \right)^N 2\pi. \]
Using the functional equation for $u^2(p)$ we get
\[ \int_{3^N \pi}^{3^{N+1} \pi} u^2(p) dp = 3^N \int_\pi^{3\pi} u^2(3^N p) dp \]
\[ = 3^N \int_\pi^{3\pi} \prod_{k=0}^{N-1} \cos^2(3^k p) u^2(p) dp \]
\[ \lesssim 3^N \int_\pi^{3\pi} \prod_{k=0}^{N-1} \cos^2(3^k p) dp \]
\[ \lesssim 3^N \int_\pi^{3\pi} \prod_{k=0}^{N-1} \cos^2(3^k p) dp \]
\[
\left(\frac{3}{2}\right)^N \int_\pi^{3\pi} \prod_{k=0}^{N-1} (1 + \cos 2 \cdot 3^k p) \, dp
\]
\[
= (\frac{3}{2})^N 2\pi,
\]
which proves the right inequality.

State the lower bound. Again, using the functional equation, we have
\[
\int_\pi^{3\pi} u^2(p) \, dp = 3^N \int_\pi^{3\pi} \prod_{k=0}^{N-1} \cos^2(3^k p) \frac{2}{3} u^2 \left(\frac{p}{3}\right).
\]

Since \(u(p/3)\) decreases on \([\pi, 3\pi]\), we infer that
\[
\int_\pi^{3\pi} u^2(p) \, dp \leq 3^N \int_\pi^{3\pi} \prod_{k=0}^{N-1} \cos^2(3^k p) \frac{2}{3} \, dp.
\]

Moreover
\[
u^2(\pi) \leq \int_0^\pi u^2(p) \, dp < \pi.
\]

Let \(R \in [3^N \pi, 3^{N+1} \pi]\), then
\[
u^2(\pi) \left[1 + \frac{3}{2} + \cdots + \left(\frac{3}{2}\right)^N\right] < \int_0^R u^2(p) \, dp < \pi \left[1 + 2 \left(1 + \frac{3}{2} + \cdots + \left(\frac{3}{2}\right)^N\right)\right].
\]

Therefore, by a simple computation we infer that the following inequalities hold:
\[
\pi \nu^2(\pi) \frac{2(3/2)^N - 1}{3(N+1)(1-\kappa)} < R^{\kappa-1} \int_0^R \nu^2(p) \, dp < \pi \frac{4(3/2)^{N+1} - 3}{3N(1-\kappa)}.
\]

Since the lower bound behaves as \(\pi \nu^2(\pi) (\frac{3}{2} + o(1))\) and the upper one as \(\pi (6 + o(1))\), the theorem is proven.

Thus the reflection coefficient (considered as a function of the energy) integrated over a sufficiently large interval of energy has quite a regular behavior at infinity
\[
\int_0^E r(E) \, dE \sim E^{(1-\kappa)/2}.
\]
Note that for scattering by the $\delta$ potential, the analogous integral behaves like $E^{1/2}$ and for the usual potential scattering [by a smooth (absolute continuous) measure] it converges. Moreover, this formula shows a way for the determination of the Hausdorff dimension $\kappa$ of the scatterer from the high-energy behavior of the scattering data.

VI. DISCUSSION AND GENERALIZATIONS

The program described above is easy to fulfill for Hamiltonians extended from the domains corresponding to any finite deficiency indices, i.e., submitted to several conditions of orthogonality, e.g.,

$$\int f(x) q(x) d\mu(x) = 0,$$

with some finite-dimensional set of densities $q(x)$, given in a form of trigonometric polynomials

$$q(x) = \sum_{k=1}^{N} a_k e^{i k x}.$$

Then instead of one function $u(p)$ it appears as a family of shifted functions

$$u_q(p) = \int e^{ipx} q(x) d\mu(x) = \sum_{k=1}^{n} a_k u(p - \lambda_k).$$

Next, the matrix,

$$||d_p(z)||_{ij} = \beta_{ij} + \int \frac{u(p + \lambda_i) u^*(p + \lambda_j)}{p^2 - z} dp,$$

plays the role of the perturbation determinant, which determines the corresponding $t$ matrix

$$t(p,k,z) = \sum u(p + \lambda_i) d_p^{-1}(z) u^*(k + \lambda_j).$$

The suggested scheme could be generalized to the $n$-dimensional case. The only restriction here is that the fractal should not be too "thin," just to keep the interaction on it. If we deal with the Laplace operator, the homogeneous measure should belong to the Sobolev class $H_{-2}$. Moreover, if it belongs to $H_{-1}$, then the scheme is entirely valid. In particular, the representations for the resolvents (3), (4) hold without changes, the only difference is that the variable $p$ [see Eq. (3)] should be regarded as an $n$-dimensional vector. If $p \in H_{-2}$ only, then the scheme needs some special regularization procedure, which is similar to those used in the case of thin manifolds (see Refs. 3, 4).

Finally, let us note that for the described scheme it is not essential to demand that the fractal measure be a purely singular continuous one with respect to the Lebesgue measure. It may have absolutely continuous or singular discrete parts which can be separated and treated in the usual way.

ACKNOWLEDGMENTS

We are grateful to Dr. Kostrykin and Dr. S. Yakovlev for active discussions. One of the authors (B.S.P.) is indebted to Professor S. Albeverio for hospitality during the stay at the Institute of Mathematics of Ruhr University Bochum, where this work was finished.

The financial support of the Alexander von Humboldt Stiftung is also gratefully acknowledged.


