Boundary Layer of Eigenfunctions of a Diffusion Operator

K. A. MAKAROV

Abstract. An asymptotics of the boundary layer type for eigenfunctions of the "small diffusion" operator with a potential drift field that correspond to exponentially small eigenvalues is investigated.

1. The paper deals with the investigation of formal asymptotics in the boundary layer approximation of "junior" eigenfunctions of the diffusion operator with a small parameter at the derivatives of the highest order,

$$\mathcal{L}_\varepsilon u = -\varepsilon \Delta u + \nabla \varphi \nabla u,$$

acting in the space $L_2(\Omega)$ of square integrable functions on the unit sphere $\Omega$. Here $\varphi$ is the potential of the drift field

$$\mathbf{b} = -\nabla \varphi, \quad \varphi(x) = \frac{1}{2} - \sum_{i \neq k} \langle x, e_i \rangle^2 \langle x, e_k \rangle^2.$$

By the symbols $e_k$ we denote the standard unit basis vectors (the form of the potential is of the physical origin [1]). The operator $\mathcal{L}_\varepsilon$ has a series $0 = \lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots, \lambda_q(\varepsilon)$ of exponentially small with respect to $\varepsilon$ eigenvalues [2]; the asymptotics of this series as $\varepsilon \to 0$ is closely connected with the values of the potential $\varphi$ on the unit sphere. Since the drift field $\mathbf{b}$ is a potential, the operator $\mathcal{L}_\varepsilon$ is a selfadjoint operator in the weighted space $L^2_{w_0}(\Omega), \quad W_0(x) = \exp(-\varphi(x)/\varepsilon)$, and the proof that such a series $\{\lambda_i(\varepsilon)\}_{i=1}^q$ exists can be obtained by means of the variational principle.

In the paper [3] it was shown that the eigenfunctions of the operator $\mathcal{L}_\varepsilon$ corresponding to the exponentially small eigenvalues $\lambda = O(e^{-K/\varepsilon})$, $K > 0$, are practically constant outside the boundaries of the domains of attraction $\Omega_\varepsilon$ of the dynamical system $\dot{x} = -\nabla \varphi$. In a $\sqrt{\varepsilon}$-neighborhood of common
boundaries \( \Gamma_{st} \) of such domains there can appear a boundary layer. Moreover, the leading term of this asymptotics was obtained in that paper. As a formal asymptotics of the corresponding eigenfunctions, the solution of the homogeneous equation

\[
\mathcal{L}_\varepsilon v = 0, \quad v|_{\Gamma_{st}} = 0,
\]

is considered; this solution has the properties of a boundary layer in a \( \sqrt{\varepsilon} \)-neighborhood of \( \Gamma_{st} \).

This paper is an expanded presentation of a part of the author’s paper [2] and is devoted to the construction of subsequent terms of the asymptotics of the boundary layer in a tubular neighborhood of the saddle point lying on \( \Gamma_{st} \). A detailed investigation of the boundary layer permits one to carry out “with authority” the construction of trial functions and to obtain, by means of these functions and using the variational principle, rigorous and effective estimates for the lowest eigenvalues of the operator \( \mathcal{L}_\varepsilon \) [2].

2. The equators and meridians of the unit sphere that pass through the maximum points of the potential \( \varphi \) divide the sphere into eight spherical triangles \( \Omega_i \), which are the domains of attraction of the dynamical system \( \dot{x} = -\nabla \varphi \). The symmetry of the problem enables us to reduce the investigation of eigenfunctions of the operator \( \mathcal{L}_\varepsilon \) to the solution of the following boundary value problems in one of the spherical triangles, say, in \( \Omega_1 \):

\[
\mathcal{L}_\varepsilon v = \lambda v,
\]

(1)

\( v = 0 \) or \( \partial v/\partial n = 0 \) on the sides of the triangle \( \Omega_1 \). Since the eigenfunctions of problem (1) are smoothly extended by symmetry to the whole sphere, they turn out to be eigenfunctions of the operator \( \mathcal{L}_\varepsilon \) (on the entire sphere). The boundary layer is formed only in a neighborhood of that side of the spherical triangle \( \Omega_i \), where we have the Dirichlet condition, and turns out to be insensitive to the type of boundary conditions imposed on the remaining sides.

We shall construct a complete expansion of the boundary layer type for the solution of the following equation:

\[
\varepsilon \Delta v - \nabla \varphi \nabla v = 0, \quad v|_l = 0,
\]

(2)

in a tubular neighborhood of the saddle point of the potential \( \varphi \) that lies on a part of the equator \( l \) outside a neighborhood of the peaks (i.e., maximum points of the potential \( \varphi \)). In a neighborhood of the peaks the asymptotics constructed ceases to be valid and the answer has a different form.

Let us rewrite problem (2) in the semigeodesic system of coordinates \((s, n)\), where \( s \) is the natural parameter on the equator \( l \); we shall assume that \( s = 0 \) at the saddle point of the potential \( \varphi \) on \( l \). We introduce the notation

\[
\varphi_n = -nb_1(s, n), \quad \cos^{-2} n \varphi_s = -b_2(s, n),
\]

\[
B(s) = b_1(s, 0), \quad C(s) = b_2(s, 0).
\]
Then
\[ B(s) > 0, \quad C(s)_{s=0} = 0, \quad C(s) < 0 \quad \text{for} \quad s > 0. \]
Here we have used the fact that the contour \( l \) is the saddle contour of the function \( \varphi \) \( (\partial \varphi / \partial n)_{l} = 0 \), and that the point \( s = 0, \ n = 0 \) is the saddle point. The function \( C(s) \) vanishes only at the saddle points, i.e., the points of the maximum of \( \varphi \). The smoothness of the functions \( b_1, b_2 \) enables us to expand them in power series in \( \varepsilon \) up to any order,
\[
\begin{align*}
b_1(s, n) &= B(s) + b_2^{(1)}(s)n^2 + b_4^{(1)}n^4 + \ldots + b_{2k}^{(1)}n^{2k} + O(n^{2k+2}), \\
b_2(s, n) &= C(s) + b_2^{(2)}(s)n^2 + b_4^{(2)}n^4 + \ldots + b_{2k}^{(2)}n^{2k} + O(n^{2k+2}).
\end{align*}
\]

We introduce a new independent variable \( \nu = \varepsilon^{-1/2}n \). In the variables \( (s, \nu) \), equation (2) takes the form
\[
v_{\nu\nu} + \mathcal{E}v_{\nu} + \mathcal{F}v_{\nu} + \varepsilon \mathcal{G}v_{ss} = 0, \quad (3)
\]
where
\[
\mathcal{E} = (\nu b_1(s, \sqrt{\nu}) - \sqrt{\nu} \tan \sqrt{\nu}), \quad \mathcal{F} = b_2(s, \sqrt{\nu}), \quad \mathcal{G} = \cos^{-2} \sqrt{\nu}.
\]
The functions \( \mathcal{E}, \mathcal{F}, \mathcal{G} \) can be expanded in series in powers of \( \varepsilon \):
\[
\begin{align*}
\mathcal{E} &= \nu B(s) + \varepsilon (b_2^{(1)}(s)\nu^3 - \nu) + \ldots + \nu \left[ B(s) + \sum_{k \geq 1} \varepsilon^k e_k(s, \nu) \right], \\
\mathcal{F} &= C(s) + \varepsilon b_2^{(2)}(s)\nu^2 + \ldots + C(s) + \sum_{k \geq 1} \varepsilon^k b_{2k}^{(2)}\nu^{2k}, \\
\mathcal{G} &= 1 + \frac{\varepsilon \nu^2}{2} + \ldots + 1 + \sum_{k \geq 1} g_k \varepsilon^k \nu^{2k},
\end{align*}
\]
where the \( e_k(s, \nu) \) are polynomials in \( \nu \) of order \( \leq 2k \).

In order to construct the boundary layer type solution of equation (3) that satisfies the conditions
\[
v_{\nu\nu} = 0, \quad v_{\nu} \underset{\nu \rightarrow \infty}{\rightarrow} 1, \quad (4)
\]
we use, as an ansatz, the error integral
\[
v(s, \nu) = E(\Psi) = \sqrt{2/\pi} \int_0^\Psi e^{-s^2/2} ds, \quad (5)
\]
where \( \Psi(s, \nu, \varepsilon) \) is a series in powers of \( \varepsilon \) with polynomial in \( \nu \) coefficients,
\[
\Psi = \sum_{k \geq 0} \varepsilon^k \gamma_k(s, \nu).
\]
Note that such an ansatz automatically satisfies conditions (4) if \( \Psi \rightarrow \infty \) as \( \nu \rightarrow \infty \).

Following the standard scheme [4], we find equations for the polynomials \( \gamma_k(s, \nu) \). After substituting the function \( v \) defined by equality (5) in equation (3), we come to a recurrent system of differential equations for these polynomials.
**Lemma 1.** The leading term of the asymptotics of $\gamma_0(s, \nu)$ satisfies the nonlinear partial differential equation

$$\frac{\partial^2}{\partial \nu^2} \gamma_0 - \gamma_0 \left( \frac{\partial \gamma_0}{\partial \nu} \right)^2 + \nu B(s) \frac{\partial \gamma_0}{\partial s} + C(s) \frac{\partial \gamma_0}{\partial s} = 0. \quad (6)$$

The unique nontrivial polynomial solution of this equation, regular in a neighborhood of the saddle point $s = 0$, has the form

$$\gamma_0(s, \nu) = \nu \gamma(s),$$

where $\gamma(s)$ is a special solution of Bernoulli's equation

$$\frac{d \gamma}{ds} + \frac{B(s)}{C(s)} \gamma - \frac{\gamma^3}{C(s)} = 0, \quad (7)$$

$$\gamma(s) = F^{-1}(s) \left[ -2 \int_0^s dt \{C(t)F^2(t)\}^{-1} \right]^{-1/2}. \quad (8)$$

Here $F(s)$ is the nonzero solution of the homogeneous equation

$$F'(s) = B(s)C^{-1}(s)F(s). \quad (9)$$

**Proof.** The equation for $\gamma_0$ is obtained by straightforward calculations. Obviously, any polynomial solution of the equation for the leading term (6) is a polynomial in $\nu$ of order not higher than one,

$$\gamma_0(s, \nu) = \nu \gamma(s) + \beta(s). \quad (10)$$

Substituting expression (10) in the equation for the leading term and equating the coefficients at the first and at the zero power of $\nu$ to zero, we come to Bernoulli's equation (7) for $\gamma(s)$ and the following equation for $\beta(s)$, respectively:

$$\frac{d \beta(s)}{ds} = \frac{\gamma^2(s)}{C(s)} \beta(s). \quad (11)$$

Let us investigate in detail Bernoulli's equation (7). Substituting $u = \gamma^{-2}$, we reduce it to the linear equation

$$u' - \frac{2B(s)}{C(s)} u = -\frac{2}{C(s)},$$

whose general solution can be represented in the form

$$u(s) = \left[ A - 2 \int_0^s dt \{C(t)F^2(t)\}^{-1} \right] F^2(s), \quad (12)$$

where $F(s)$ satisfies equation (9), and $A$ and $d$ are some constants. The requirement that the function $u(s)$ must be regular as $s \to 0$ (consequently, the same holds for $\gamma(s)$) leads to the equalities $A = d = 0$, which proves representation (8). In order to verify this, first we shall prove that in a neighborhood of the point $s = 0$ the representation $F(s) \sim s^{-p}$ holds for
\[ p = -B(0)/C'(0). \] Indeed, equation (9) implies that up to a constant we have
\[ \log F(s) = \int_0^s (B(t)/C(t) + p/t) \, dt - p \log s \]
(the function \( B(t)/C(t) + p/t \) is regular at the point \( t = 0 \) and, thus, is integrable at zero). This implies that the function \( s^p F(s) \) has a finite limit as \( s \to 0 \), and thus the improper integral \( \int_0^s dt \{ C(t)F^2(t) \}^{-1} \) converges and is of order \( s^p \). In other words, it is proved that the function \( -2 \int_0^s (C(t)F^2(t))^{-1} \, dt F^2(s) \) is regular and, therefore, \( d \) can be set equal to zero in representation (12). Now, to eliminate the singularity of the function \( u(s) \) as \( s \to 0 \), it remains to set \( A = 0 \).

Further, we prove that
\[ \gamma(0) = \lim_{s \to 0} \gamma(s) = \sqrt{B(0)}, \]
\[ u(0) = -2 \lim_{s \to 0} \int_0^s \frac{dt C^{-1}(t)F^{-2}(t)}{F^{-2}(s)} = \lim_{s \to 0} \frac{F(s)}{F^2(s)C(s)} = \lim_{s \to 0} \frac{1}{B(s)} = \frac{1}{B(0)}. \]

We recall that \( \gamma = u^{-1/2} \). Now it is easily proved that the regular solution of equation (11) is identically zero. The general solution of equation (11) has the form
\[ \beta(s) = C\gamma(s)F(s). \]

Since \( F(s) \) has a singularity at \( s = 0 \), then, necessarily, we have \( C = 0 \) if the regularity of \( \beta(s) \) is required. The proof of Lemma 1 is completed.

The choice of the regular solution of (6) enables us to simplify the equations for the leading terms of the series \( \Psi(s, \nu, \varepsilon) \).

**Lemma 2.** The leading terms of the asymptotics of \( \gamma_k(s, \nu) \) are the solutions of the recurrent system of nonhomogeneous linear differential equations
\[ \frac{\partial \gamma_k}{\partial \nu} - \gamma_k \gamma^2 + \nu[B(s) - 2\gamma^2(s)]\frac{\partial \gamma_k}{\partial s} + C(s)\frac{\partial \gamma_k}{\partial s} = \Phi_k(s, \nu). \quad (13) \]

The right-hand sides \( \Phi_k \) depend on the solutions \( \gamma_l \), \( l \leq k - 1 \), already constructed, and, under the assumption that the latter are polynomials, the \( \Phi_k \) are polynomials in \( \nu \) themselves,
\[ \Phi_k = \sum_{l=0}^{M_k} \beta_{kl}(s)\nu^l, \]
and the order of these polynomials is no higher than \( 2k + 1 \), \( M_k \leq 2k + 1 \).

The coefficients of the polynomial solutions
\[ \gamma_k(s, \nu) = \sum_{l=0}^{M_k} \gamma_{kl}\nu^l \]
satisfy the recurrent system of ordinary singular differential equations

\[ C(s)\gamma'_{kl} + [KB(s) - (2k + 1)\gamma^2(s)]\gamma_{kl} = \beta_{kl} + \left\{ \begin{array}{ll}
0, & l = M_k, M_k - 1, \\
-(l + 2)(l + 1)\gamma_{kl+2}, & l = 0, 1, \ldots, M_k - 2.
\end{array} \right. \tag{14} \]

**Proof.** Let us write out the equation for the series $\Psi$:

\[
\frac{\partial^2 \psi}{\partial \nu^2} - \psi \left( \frac{\partial \psi}{\partial \nu} \right)^2 + \nu \left[ B(s) + \sum_{k \geq 1} e^k b_{2k}^{(2)}(s) \frac{\partial \psi}{\partial s} \right] + e \left[ 1 + \sum_{k \geq 1} e^k g_k \nu^{2k} \right] \left[ \frac{\partial^2 \psi}{\partial s^2} - \psi \left( \frac{\partial \psi}{\partial s} \right)^2 \right] = 0. \tag{15}\]

Equations (13) are obtained by setting to zero the terms at $e^k$ on the left-hand side of equation (15). The right-hand sides $\Phi_k(s, \nu)$ in equation (13) contain terms of the following form:

a) $\frac{\partial \gamma_m}{\partial \nu} \frac{\partial \gamma_l}{\partial \nu}$, $m + n + l = k$, $m, n, l \geq 0$, $m \neq k$, $n \neq k$, $l \neq k$,

b) $- \frac{\partial \gamma_m}{\partial \nu} e_{k-m}(s, \nu)$, $m = 0, 1, \ldots, k - 1$,

c) $- \frac{\partial \gamma_m}{\partial s} b_{2(k-m)}^{(2)}(s) \nu^{2(k-m)-2}$, $m = 0, 1, \ldots, k - 1$,

d) $- \frac{\partial \gamma_m}{\partial s} g_{k-m-1}(s) \nu^{2(k-m)-2}$, $m = 0, 1, \ldots, k - 1$,

e) $\frac{\partial \gamma_m}{\partial s} \frac{\partial \gamma_l}{\partial s} g_{k-(m+n+l)-1}(s) \nu^{2(k-(m+n+l)-1)}$, $m, n, l \geq 0$, $m \neq k$, $n \neq k$, $l \neq k$.

For definiteness, we write out $\Phi_1(s, \nu)$:

\[
\Phi_1(s, \nu) = - \frac{\partial \gamma_0}{\partial \nu} e_1(s, \nu) - \frac{\partial \nu_0}{\partial s} b_2^{(2)}(s) \nu^3 - \frac{\partial^2 \gamma_0}{\partial s^2} + \gamma_0 \left( \frac{\partial \gamma_0}{\partial s} \right)^2 = \nu^3(\gamma \gamma'^2 - \gamma b_2^{(1)}(s) - \gamma' b_2^{(2)}(s)) + \nu(\gamma - \gamma''(s)). \tag{17}\]

Here we have used the equality $e_1(s, \nu) = b_2^{(1)}(s) \nu^2 - 1$. Under the condition that the terms of the series $\gamma_l$ for $l \leq k - 1$ are polynomials, we obtain straightforwardly from representations a)–e) that the right-hand sides $\Phi_k$ are polynomials. To estimate the degrees of the polynomials $\Phi_k$ we use the method of complete induction. For $k = 1$, we have $\deg \Phi_1 = 3$ (the base),
which follows from representation (16). It remains to note that, under the induction assumption, every term in the representations a)–e) has degree in \( \nu \) no higher than \( 2k + 1 \) (during the verification it is sufficient to remember that \( \deg \gamma_0 = 1 \)).

The last statement of the lemma is verified by a direct substitution of the polynomial \( \gamma_k \) in equation (13). The recurrent system of equations is obtained by equating to zero all the coefficients at all powers of \( \epsilon \). The proof of Lemma 2 is completed.

We shall investigate the system of ordinary singular equations (14) obtained in more detail. Their unique solvability is guaranteed by the requirement that the solutions \( \gamma_{k,l} \) be regular at the saddle point. More precisely, the following statement is valid.

**Lemma 3** (the choice rule for regular solutions). The unique solution, regular in a neighborhood of the saddle \( s = 0 \), of the nonhomogeneous equation

\[
C(s)\alpha' + [kB(s) - (2k + 1)\gamma^2(s)]\alpha = \beta(s)
\]

(18)

has the form

\[
\alpha(s) = \gamma^{2k+1}(s)F^{k+1}(s) \int_0^s dt \beta(t)[\gamma^{2k+1}(t)F^{k+1}(t)C(t)]^{-1},
\]

(19)

and

\[
\lim_{s \to 0} \alpha(s) = -\frac{1}{k+1} \frac{\beta(0)}{B(0)}.
\]

In particular, the regular solution of the homogeneous equation is identically zero.

**Proof.** Transform the expression in square brackets using Bernoulli's equation (7):

\[
kB(s) - (2k + 1)\gamma^2(s) = kB(s)C(s) - (2k + 1) \left[ \frac{B(s)}{C(s)} + \frac{\gamma'(s)}{\gamma(s)} \right]
\]

\[= -(k + 1) \frac{B(s)}{C(s)} - (2k + 1)(\log \gamma(s))' = -(\log F^{k+1}(s)\gamma^{2k+1}(s))'.
\]

Thus the function \( F^{k+1}(s)\gamma^{2k+1}(s) \) is a solution of the corresponding homogeneous equation (18). Therefore the general solution of equation (18) has the form

\[
\alpha(s) = \gamma^{2k+1}(s)F^{k+1}(s) \int_0^s \frac{\beta(t) dt}{\gamma^{2k+1}(t)F^{k+1}(t)C(t)} + AF^{k+1}(s)\gamma^{2k+1}(s)
\]

where \( d \) and \( A \) are some constants. By the same argument as in Lemma 1, we come to the conclusion that \( \alpha(s) \) is a regular solution only in the case when \( A = d = 0 \).
In order to calculate the limit at zero of the regular function \( \alpha(s) \), we use the asymptotics \( F(s) \sim s^{-p} \). Then the following representations are valid:

\[
\alpha(s) \sim s^{-p(k+1)} \int_0^s \frac{\beta(0)}{C'(0)t} t^{p(k+1)} dt \sim \frac{\beta(0)}{p(k+1)C'(0)} = -\frac{1}{k+1} \frac{\beta(0)}{B(0)}. \]

The statements formulated above enable us to construct the complete asymptotic expansion of the boundary layer type in a tubular neighborhood of the saddle, \( s = O(1), \ n = O(e^{1/2-\delta}), \delta > 0 \), outside a neighborhood of the peaks.

We write out the explicit expression for the second term of the asymptotics of \( \gamma_1 \) using the representation (17) for \( \Phi_1 \):

\[
\Phi_1 = \nu^3 (\gamma \gamma'' - \gamma b_2^{(1)}(s) - b_2^{(2)}(s) \gamma') + \nu(\gamma - \gamma''). \]

The solution \( \gamma_1 \) must be sought in the form of a polynomial of degree three,

\[
\gamma_1(s, \nu) = \nu^3 \gamma_{13}(s) + \nu^2 \gamma_{12}(s) + \nu \gamma_{11}(s) + \gamma_{10}(s),
\]

whose coefficients satisfy the following recurrent system (Lemma 3):

\[
C(s)\gamma_1'' + [3B(s) - 7\gamma^2(s)]\gamma_1 = \gamma \gamma'' - b_2^{(2)}\gamma' - b_2^{(1)} \gamma,
\]

\[
C(s)\gamma_1' = [2B(s) - 3\gamma^2(s)]\gamma_1 = 0,
\]

\[
C(s)\gamma_1' + 3\gamma'' \gamma_1 = -\gamma'' - 6\gamma_1 + \gamma,
\]

\[
C(s)\gamma_1' + \gamma^2 \gamma_1 = -2\gamma_1. \]

The choice rule for regular solutions implies (Lemma 3) that \( \gamma_{12} = 0 \), and consequently \( \gamma_{10} = 0 \). Thus

\[
\gamma_1(s, \nu) = \nu^3 \gamma_{13}(s) + \nu \gamma_{11}(s), \quad (20)
\]

where

\[
\gamma_{13}(s) = \gamma^3(s)F^4(s) \int_0^s \frac{[\gamma \gamma'' - b_2^{(2)}\gamma' - b_2^{(1)} \gamma](t)}{\gamma^3(t)C(t)} dt, \quad (21)
\]

\[
\gamma_{11}(s) = \gamma^3(s)F^2(s) \int_0^s \frac{[\gamma \gamma'' - 6\gamma_1(t)]}{\gamma^3(t)C(t)} dt. \quad (22)
\]

3. Bernoulli's equation (7) and all the higher equations (14) are singular ordinary differential equations: the coefficient \( C(s) \) at the derivative vanishes at the saddle point of the potential and at the peak, i.e., at the point of maximum of the potential. The regular solution of Bernoulli's equation in a neighborhood of the saddle point \( s = 0 \) ceases to be regular in a neighborhood of the peak \( s = s^* \). This holds for all the higher equations. The reason for this phenomenon is that the derivative of the function \( C(s) \) has different signs at the saddle point and at the peak: \( C'(0) < 0 \) and \( C'(s^*) > 0 \). More precisely, the following statement holds.
Lemma 4. Under the assumption $B(s^*) = C'(s^*)$ (which automatically holds for the particular potential $\varphi$), the following asymptotic representations are valid:

$$
\gamma(s) = \sqrt{B(s^*)} \left( 1 - \frac{B''(s^*)}{2B(s^*)}(s - s^*)^2 \log(s^* - s) - \frac{KB(s^*)}{2}(s - s^*)^2 + o(s - s^*)^2 \right),
$$

$$
\gamma_{13}(s) = \frac{1}{4} \left( \frac{b_2^{(1)}(s^*)}{\sqrt{B(s^*)}} \right) + o(1),
$$

$$
\gamma_{11}(s) = -\frac{1}{2} B^{-3/2}(s^*) B''(s^*) \log(s^* - s) - \frac{1}{2} B^{1/2} \left[ B^{-1} + K + 2B^{-2}B'' - \frac{3}{2} B^{-2} b_2^{(1)} \right]_{s = s^*} + o(1),
$$

where $K$ is a constant determined by the functions $B(s)$ and $C(s)$.

Proof. The function $F(s)$ is regular in a neighborhood of the peak $s = s^*$: $F(s) \sim (s^* - s)^p$ for $p^* = B(s^*)/C'(s^*) = 1 > 0$. Indeed, the equality

$$
\log F(s) = \int_s^t \left( \frac{B(t)}{C(t)} - \frac{p^*}{t - s^*} \right) dt + p^* \log(s^* - s)
$$

holds up to a constant, and the function

$$
\frac{B(t)}{C(t)} - \frac{p^*}{t - s^*}
$$

is regular in a neighborhood of the peak $t = s^*$.

Let us obtain the asymptotics in a neighborhood of the peak of the regular at zero solution $u(s) = \gamma(s)$:

$$
u(s) = -2F^2(s) \int_s^t dt \left\{ C(t) F^2(t) \right\}^{-1}.
$$

Let us single out the singularities in the integrand:

$$
\frac{1}{C(s)F^2(s)} = \frac{a_{-3}}{(s - s^*)^3} + \frac{a_{-2}}{(s - s^*)^2} + \frac{a_{-1}}{(s - s^*)} + D(s).
$$

Here $D(s)$ is a function without singularities at the point $s = s^*$. Clearly, $a_{-3} = [C'(s^*)F^2(s^*)]^{-1}$.

Using the explicit representation

$$
F(s) = C \exp \int_{s_0}^s \frac{B(t)}{C(t)} \, dt
$$

and omitting the calculations, we obtain the following equalities:

$$
a_{-2} = 0, \quad a_{-1} = -\left. \frac{B''}{2B^2F^{1/2}} \right|_{s = s^*}.
$$
Integrating the leading terms of expansion (26) from zero to $s$,

$$
\int_0^s \left( \frac{a_{-3}}{(t-s)^2} + \frac{a_{-1}}{(t-s)^2} \right) \, dt = -\frac{1}{2} \left. \frac{a_{-3}}{(t-s)^2} \right|_0^s + a_{-1} \log|t-s^*| \bigg|_0^s \\
\sim -\frac{1}{2} \left. \frac{1}{C'F'\nu^2} (s-s^*)^2 - \frac{1}{2} \frac{B''}{B^2F^2} \right|_0^{s^*} \log(s^*-s) + o(s^*-s)^2,
$$

we come to the following asymptotics for the function $u$ in a neighborhood of the point $s = s^*$:

$$
u(s) = \frac{1}{B(s^*)} + \frac{B''}{B^2} \bigg|_0^{s^*} (s-s^*)^2 \log(s^*-s) + K(s-s^*)^2 + o(s-s^*)^2,
$$

where

$$
K = -2F'^2(s^*) \left[ \frac{a_{-3}}{2s^*} - a_{-1} \log s^* + \int_0^{s^*} D(t) \, dt \right]
$$

(the constant $K$, by means of the function $D(s)$, is defined by the global behavior of the functions $B(s)$ and $C(s)$ on the interval $(0, s^*)$). Using the relation $\gamma(s) = u^{-1/2}(s)$, we obtain the required asymptotics (23).

In order to obtain the asymptotics for the coefficients $\gamma_{13}$ and $\gamma_{11}$ of the first term of the expansion of $\gamma_1(s, \nu)$ in the series (5), we use the explicit formulas (21), (22). The expression $\gamma_1(s) = b_2(s)\gamma(s) - b_1(s)\gamma(s)$ that determines the function $\gamma_{13}(s)$ has, as $s \to s^*$, the finite limit:

$$
\gamma(s)\gamma^2(s) \sim (s-s^*)^2 \log^2(s^*-s),
\gamma(s)^2 \sim (s-s^*)^2 \log(s^*-s),
\gamma(s) \sim b_1(s)\gamma(s) \sim b_1(s)\gamma_{13}(s).
$$

Thus,

$$
\lim_{s \to s^*} (\gamma_2(s) - b_2(s)\gamma(s)) = -b_1(s)\gamma_{13}(s) = -b_1(s)/\sqrt{B(s^*)}.
$$

In order to obtain the asymptotics of the coefficients $\gamma_{11}$ and $\gamma_{13}$, we shall need the following simple statement. Let $\alpha_k$ be the regular at the point $s = 0$ solution of the equation

$$
C(s)\alpha_k + [kB(s) - (2k+1)\gamma^2(s)]\alpha_k = \beta(s).
$$

Then, if $\beta(s) = \beta(s^*) + o(1)$, we have

$$
\alpha_k(s) = -\frac{1}{k+1} \frac{\beta(s^*)}{B(s^*)} + o(1).\tag{28}
$$

If $\beta(s) = \log(s^*-s) + o(1)$, we have

$$
\alpha_2(s) = \frac{1}{2B(s^*)} \log(s^*-s) - \frac{1}{4B(s^*)} + o(1).\tag{29}
$$
If $\beta(s) \sim (s^* - s)^k$, we have

$$\alpha_m(s) \sim (s^* - s)^{-k}, \quad m = 1, 2, \ldots.$$  (30)

The proof is obtained by straightforward calculations of the asymptotics of integrals entering the explicit representation (19) for the regular at zero solution $\alpha_k$.

Thus, it is proved that integral (21) reproduces the asymptotics of the integrand

$$\lim_{s \to s^*} \gamma_{13}(s) = \frac{1}{4} \frac{b_1^{(1)}(s^*)}{\sqrt{B(s^*)}}.$$ 

The term $\gamma_{11}(s)$ is singular as $s \to s^*$. Indeed, let us calculate the asymptotics of the functions in the integrand in representation (22):

$$\gamma(s) = \sqrt{B(s^*)} + o(1),$$

$$-\gamma''(s) = \frac{B''}{\sqrt{B}} \bigg|_{s = s^*} \log(s^* - s) + KB^{3/2}(s^*) + \frac{3}{2} \frac{B}{\sqrt{B}} \bigg|_{s = s^*} + o(1),$$

$$-6\gamma_{13} = -\frac{3}{2} \frac{b_1^{(1)}(s^*)}{\sqrt{B(s^*)}} + o(1).$$

Now, representation (25) follows from relations (28), (29). The proof of the lemma is completed.

4. The fact that there is a logarithmic term in the leading order of asymptotics (23) implies that the terms $\gamma_k$ with $k \geq 1$ become singular in a neighborhood of the point $s = s^*$, and the ratio $\gamma_{k+1}/\gamma_k$ for $k \geq 2$ has the order $(s^* - s)^{-2}$. Indeed, the main singularity of the polynomials $\Phi_k$ results from the terms $\partial^2 \gamma_k / \partial s^2$. But we have just verified that $\gamma_1 \sim \log(s^* - s)$ and, hence, $\Phi_2(s, \nu) \sim (s^* - s)^{-2}$. Since the solutions of equations of the type (27) reproduce the asymptotics of their right-hand sides (see (30)), we conclude that $\gamma_2 \sim (s^* - s)^{-2}$. Further, the main singularity of the polynomial $\Phi_3$ results from the term $\partial^2 \gamma / \partial s^2$; consequently $\Phi_3(s, \nu) \sim (s^* - s)^{-4}$, etc.

This shows that the series $\Psi$ ceases to be asymptotic in a $\sqrt{\varepsilon}$-neighborhood of the peak $s = s^*$, which makes it impossible to describe the boundary layer asymptotics by means of the ansatz (5).

The arguments given above justify the introduction of the new variables

$$\sigma = \left(\frac{B(s^*)}{\varepsilon}\right)^{1/2} (s^* - s), \quad \eta = \left(\frac{B(s^*)}{\varepsilon}\right)^{1/2} n, \quad n = \nu \sqrt{B(s^*)}.$$
Assuming $\sigma$ to be finite, $0 \leq C_1 \leq \sigma \leq C_2$, we calculate the asymptotics of the first two terms of the expansion of $\gamma_1$ and $\gamma_2$ in the new variables, omitting the terms of order $O(\epsilon)$ and higher. Using the results of Lemma 4, we obtain the following representations:

$$
\gamma_\nu = \eta \left( 1 - \frac{\epsilon B''}{2B^2} \sigma^2 \log \sigma \frac{\sqrt{\sigma}}{\sqrt{13}} - \frac{1}{2} Ke \sigma^2 \right) + o(\epsilon),
$$

$$
\epsilon \gamma_{13}^3 = \frac{\epsilon}{4} \eta^3 \frac{b_1^{(1)}(s^*)}{B^2(s^*)} + o(\epsilon),
$$

$$
\epsilon \gamma_{11}^3 = -\epsilon \eta \frac{B''}{2B^2} \log \sigma \frac{\sqrt{\sigma}}{\sqrt{B}} - \epsilon \eta \left[ B^{-1} + K + B^{-2} B'' - \frac{3}{2} B^{-2} b_2^{(1)} \right] + o(\epsilon).
$$

Gathering all the terms in (31) together, we obtain the following asymptotics:

$$
\nu \gamma(s) + \epsilon (\nu^3 \gamma_{13}(s) + \nu \gamma_{11}(s))
= \eta - \epsilon \log \epsilon \frac{B''}{B^2} \alpha(\eta, \sigma) + \epsilon \eta \frac{3}{4B^2} b_1^{(1)}(s^*) - \frac{\epsilon B''}{2B} \log \sigma \alpha(\eta, \sigma)
+ \epsilon \left( \frac{1}{4} \frac{B''}{B^2} \log B - \frac{K}{2} \right) \alpha(\eta, \sigma)
- \epsilon \eta \left[ \frac{1}{2B} + \frac{B''}{2B^2} - \frac{3}{4} B^{-2} b_2^{(1)} \right] + o(\epsilon),
$$

(32)

where $\alpha(\eta, \sigma) = \eta(\sigma^2 + 1)$.

This expansion holds even in a wider domain with respect to $\sigma$: $0 < C_1 \leq \sigma \leq C_2^\delta^{-1/2}$, $\delta > 0$. Moreover, it is clear that asymptotics (32) in this domain is the asymptotics of the whole series $\Psi$.

For the investigation of local asymptotic expansions in a neighborhood of the peak, it is necessary to know the asymptotics of the series $\Psi$. The presence of the logarithmic in $\epsilon$ term in (32) shows that these expansions must be carried out not only in powers of the small parameter, but also in powers of its logarithm [2]. This will enable us to match local expansions with the expansion (5). A detailed description of local expansions will be published elsewhere.

REFERENCES


Leningrad State University