2.1 Linear Equations

\[ \frac{dy}{dt} + p(t)y = g(t) \]

The general solution

\[ y(t) = \frac{\int \mu(s)g(s)ds + \text{Const}}{\mu(t)} \]

where

\[ \mu(t) = \exp \left( \int p(t)dt \right) \]
2.2 Separable Equations

\[ \frac{dy}{dx} = -\frac{M(x)}{N(y)} \]

\[ M(x)dx + N(y)dy = 0 \]

The general solution

\[ \int M(x)dx + \int N(y)dy = \text{Const} \]
2.6 Exact Equations

\[ M(x, y)dx + N(x, y)dy = 0 \]

where

\[ M = \Phi_x \]
\[ N = \Phi_y \]

for some \( \Phi = \Phi(x, y) \).

The general solution

\[ \Phi(x, y) = \text{Const} \]
Given smooth functions $M$ and $N$ on a nice domain with no holes, the equation

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if and only if

$$M_y = N_x$$

The general solution

$$\Phi(x, y) = \text{Const}$$

where

$$\Phi(x, y) = \int_{(a,b)}^{(x,y)} Mdx' + Ndy', \text{ the line integral over (any) path from (a, b) to (x, y)}$$
Integrating Factors

If $M_y \neq N_x$, but either

$$\frac{M_y - N_x}{N} = X(x) \neq 0$$

and

$$\frac{d\mu(x)}{dx} = X(x)\mu(x),$$

or

$$\frac{N_x - M_y}{M} = Y(y) \neq 0$$

and

$$\frac{d\mu(y)}{dy} = Y(y)\mu(y),$$

then the differential equation

$$\mu M \, dx + \mu N \, dy = 0$$

is exact.
3.1 Homogeneous Equations with Constant Coefficients

\[ ay'' + by' + cy = 0 \]

The characteristic equation

\[ ar^2 + br + c = 0 \quad (1) \]

The general solution

\[ y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \]

where \( r_1 \) and \( r_2 \) are real, different roots \( (r_1 \neq r_2) \) of the characteristic equation \( (1) \)
3.2 Solutions of Linear Homogeneous Equations; the Wronskian

The initial value problem

\[ y'' + py' + qy = 0 \quad (1) \]

\[ y(t_0) = y_0, \quad y'(t_0) = y'_0 \quad (2) \]

has a unique solution.

The general solution of the differential equation (1) is of the form

\[ y = C_1 y_1 + C_2 y_2 \]

where the solutions \( y_1(t) \) and \( y_2(t) \) of (1) form a **fundamental set** of solutions.
Linear Independence and the Wronskian

Two solutions \( y_1 \) and \( y_2 \) of the homogeneous differential equation
\[
y'' + py' + qy = 0
\]
\((p \text{ and } q \text{ are continuous functions on an open interval } I)\) form a fundamental set of solutions \textit{iff}

- \( y_1 \) and \( y_2 \) are linearly independent
- \( y_1 \) and \( y_2 \) are not proportional to each other
- \( W(t) \neq 0 \) for some point \( t \in I \)
- \( W(t) \neq 0 \) for all points \( t \in I \)

where \( W(t) \) is the Wronskian

\[
W(t) = W(y_1, y_2)(t) = y_1(t)y'_2(t) - y'_1(t)y_2(t)
\]
Abel’s Theorem If $y_1$ and $y_2$ are two solutions of the homogeneous differential equation

$$y'' + py' + qy = 0$$

($p$ and $q$ are continuous functions on an open interval $I$), then the Wronskian $W(y_1, y_2)$ is given by

$$W(y_1, y_2)(t) = \text{Const} \exp \left[ - \int p(t) dt \right]$$
3.3 Complex Roots of the Characteristic Equation

If the roots $r_1$ and $r_2$ of the characteristic equation

$$ar^2 + br + c = 0$$

are conjugate complex numbers

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu \quad (\mu > 0),$$

then the general solution is of the form

$$y(t) = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$$
3.4 Repeated Roots

If the two roots $r_1$ and $r_2$ of the characteristic equation are equal

$$r_1 = r_2 \equiv r,$$

then the general solution is of the form

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$
Reduction of Order

If \( y_1(t) \) is a nontrivial solution of the homogeneous equation

\[
y'' + py' + qy = 0
\]

then a general solution can be found in the form

\[
y(t) = v(t)y_1(t)
\]

where \( v \) is a general solution of

\[
v'' + \left( p + 2 \frac{y_1'}{y_1} \right) v' = 0
\]

This equation is the first order differential equation for

\[
w = \frac{d}{dt}v
\]

\[
w' + \left( p + 2 \frac{y_1'}{y_1} \right) w = 0
\]
Solve it!

Then the general solution $y(t)$ is given by

$$y(t) = \left( \int w(t) dt + \text{Const} \right) y_1(t)$$

**Remark** Compare with [Abel’s Theorem](#), which states that the general solution $y(t)$ can also be found by solving the first order differential equation

$$y_1y' - y_1'y = \text{Const} \exp \left[ -\int p(t) dt \right]$$
3.5 Method of undetermined Coefficients

A particular solution to the non-homogeneous differential equation

\[ ay'' + by' + cy = g(t) \]

with

\[ g(t) = \begin{cases} 
P_n(t) & I \\
e^{\alpha t} P_n(t) & II \\
e^{\alpha t} \left( P_n(t) \cos \beta t + \tilde{P}_n(t) \sin \beta t \right) & III 
\end{cases} \]

can be found in the form (Ansatz)

\[ y(t) = t^s \begin{cases} 
Q_n(t) & I \\
e^{\alpha t} Q_n(t) & II \\
e^{\alpha t} \left( Q_n(t) \cos \beta t + \tilde{Q}_n(t) \sin \beta t \right) & III 
\end{cases} \]
where $s$ is the multiplicity of the root $r_0$

\[
\begin{align*}
  r_0 &= \begin{cases}
    0 & \text{I} \\
    \alpha & \text{II} \\
    \alpha + i\beta & \text{III}
  \end{cases}
\end{align*}
\]

of the characteristic equation

\[
ar r_0^2 + br_0 + c = 0.
\]

Here $P_n(t)$ is a $n^{th}$-order polynomial

\[
P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n
\]

and $Q_n(t)$ and $\tilde{Q}_n(t)$ are polynomials

\[
Q_n(t) = A_0 t^n + A_1 t^{n-1} + \cdots + A_n
\]

\[
\tilde{Q}_n(t) = B_0 t^n + B_1 t^{n-1} + \cdots + B_n
\]

with unknown coefficients that are to be determined by substituting the Ansatz in the equation.
3.6 Variation of Parameters

Let \( y_1 \) and \( y_2 \) form a fundamental set of solutions to the **homogeneous** equation

\[
y'' + py' + qy = 0
\]

Then the general solution to the non-homogeneous

\[
y'' + py' + qy = g
\]
equation can be represented in the form

\[
y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)
\]

where

\[
u_1'y_1 + u_2'y_2 = 0
\]

and

\[
u_1'y_1' + u_2'y_2' = g
\]
Solving this system one gets the representation

\[ y(t) = -y_1(t) \int_0^t \frac{y_2(s)g(s)}{W(s)} \, ds \]
\[ + \, y_2(t) \int_0^t \frac{y_1(s)g(s)}{W(s)} \, ds \]

where

\[ W(s) = W(y_1, y_2)(s) \]

is the Wronskian associated with the pair \( \{y_1, y_2\} \).
6.1 Definition of the Laplace Transform

\[ \mathcal{L}\{f(\cdot)\}(s) = \int_0^\infty e^{-st} f(t) \, dt \]

\[ = \lim_{A \to \infty} \int_0^A e^{-st} f(t) \, dt \]

**Theorem**

\[ \mathcal{L}\{f'(\cdot)\}(s) = s \mathcal{L}\{f(\cdot)\}(s) - f(0) \]

\[ \mathcal{L}\{f''(\cdot)\}(s) = s^2 \mathcal{L}\{f(\cdot)\}(s) - sf(0) - f'(0) \]
6.2 Solution of Initial Value Problem

\[ ay'' + by' + cy = f(t) \]
\[ y(0) = y_0 \]
\[ y'(0) = y'_0 \]

is given by

\[ y(t) = \mathcal{L}^{-1}\{Y(\cdot)}\}(t) \]

where

\[ Y(s) = \mathcal{L}\{(y(t)}(s) \]

\[ = \frac{asy_0 + ay'_0 + by_0 + \mathcal{L}\{f(\cdot)}(s)}{as^2 + bs + d} \]
Useful formulas

\[ \mathcal{L}\{e^{at}\}(s) = \frac{1}{s - a} \]
\[ \mathcal{L}\{\sin at\}(s) = \frac{a}{s^2 + a^2} \]
\[ \mathcal{L}\{\cos at\}(s) = \frac{s}{s^2 + a^2} \]
\[ \mathcal{L}^{-1}\left\{\frac{1}{s - a}\right\}(t) = e^{at} \]
\[ \mathcal{L}^{-1}\left\{\frac{a}{s^2 + a^2}\right\}(t) = \sin at \]
\[ \mathcal{L}^{-1}\left\{\frac{s}{s^2 + a^2}\right\}(t) = \cos at \]
Theorem

\[ \mathcal{L} \left\{ t^n f(t) \right\}(s) = (-1)^n \frac{d^n}{ds^n} \mathcal{L} \{ f(t) \}(s) \]

\[ \mathcal{L}^{-1} \left\{ \frac{d^n}{ds^n} F(s) \right\}(t) = (-t)^n \mathcal{L}^{-1} \{ F(s) \}(t) \]
\[ \dot{X} = AX \]

The characteristic equation

\[ \lambda^2 - \text{tr } A \lambda + \det A = 0 \]

Eigenvectors
\[ Au = \lambda u \]

Root Vectors
\[ Av = \lambda v + u \]
Case I

The coefficient matrix $A$ has two distinct real eigenvalues $\lambda_1$ and $\lambda_2$
The general solution

$$X(t) = C_1 e^{\lambda_1 t} u_1 + C_2 e^{\lambda_2 t} u_2$$

where $u_1$ and $u_2$ are the corresponding eigenvectors

$$Au_1 = \lambda_1 u_1$$
$$Au_2 = \lambda_2 u_2$$
Case 2

The coefficient matrix $A$ has two distinct complex eigenvalues $\lambda$ and $\bar{\lambda}$ with

$$\lambda = \sigma + i\tau$$

The general solution

$$X(t) = C_1 e^{\sigma t} (\cos(\tau t) \text{Re}(u) - \sin(\tau t) \text{Im}(u))$$

$$+ C_2 e^{\sigma t} (\cos(\tau t) \text{Im}(u) + \sin(\tau t) \text{Re}(u))$$

where $u$ is the corresponding eigenvector

$$Au = \lambda u$$
Case 3

The coefficient matrix $A$ has only one eigenvalue $\lambda$ and only one linearly independent eigenvector $u$

$$Au = \lambda u$$

The general solution

$$X(t) = C_1 e^{\lambda t} u + C_2 e^{\lambda t} (tu + v)$$

where $v$ is the corresponding root vector

$$Av = \lambda v + u$$
The general solution of the non-homogeneous system
\[ \dot{X} = AX + F \]
is of the form
\[ X(t) = \Phi(t)C + \Phi(t) \int_{\$}^{t} \Phi^{-1}(\tau)F(\tau)d\tau \]
where \( \Phi \) is a fundamental matrix of the system
\[ \dot{\Phi} = A\Phi, \quad \det \Phi \neq 0 \]
and \( C \) is an arbitrary vector.