Euler Equations of Inviscid Fluids

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Abstract. Progresses on the dynamical system theory of 2D Euler equation are reported. An open problem is presented. The isospectral theory on Lax pairs of Euler equations is initiated.

1. Introduction

To understand the nature of turbulence, we select 2D Euler equation under periodic boundary condition as our primary example to study. 2D Navier-Stokes equation at high Reynolds number is regarded as a singularly perturbed 2D Euler equation. That is, we are interested in studying the zero viscosity limit problem.

To begin an infinite dimensional dynamical system study, we consider a simple fixed point and study the spectrum of the linear 2D Euler operator in \[1\]. The spectral theorem in \(\ell_2\) space is proved. The spectral theorem in Sobolev spaces is still open. Sobolev spaces are of more interest to us, since we are interested in understanding the invariant manifolds of 2D Euler equation at the fixed point. The main obstacle toward proving the invariant manifold theorem is twofold: 1. Spectral resolution for the linear Euler operator is not available, 2. The nonlinear term is non-Lipschitz.

Hyperbolic structures are the source of turbulence. To motivate a concrete understanding on the hyperbolic structure attached to the fixed point, in \[2\], a (dashed) line model is introduced. At a special parameter value, the explicit expression of the invariant manifolds of the dashed line model can be calculated. The stable and unstable manifolds are two dimensional ellipsoidal surfaces, and together they form a lip-shape hyperbolic structure. Such a structure appears to be robust with respect to the parameter.

Another more exciting development is the discovery of Lax pairs for Euler equations \[3\] \[4\]. From the Lax pair, we have obtained a Darboux transformation for the 2D Euler equation \[4\]. In principle, explicit expressions of the hyperbolic structures can be obtained from Darboux transformations \[5\].

Most importantly, the isospectral theory of the Lax pair for 3D Euler equations has the potential to provide useful invariants for solving the Clay problem on global...
well-posedness of 3D Navier-Stokes equations. In [6], a more complete isospectral theory is developed.

The article is organized as follows: In section 2, we will discuss the dynamical system theory of 2D Euler equation. In section 3, we will discuss the isospectral theory of Euler equations.

2. The Dynamical System Theory of 2D Euler Equation

This section is divided into four subsections. Subsection 2.1 is on the spectral problem of the linear 2D Euler operator in Sobolev spaces. Subsection 2.2 is on the open problem on proving the existence of invariant manifolds. In subsections 2.3 and 2.4, we will introduce two model problems, each of which carries only one of the two difficulties in the open problem.

2.1. The Spectral Theorem of the Linear 2D Euler Operator in Sobolev Spaces.

Consider the 2D Euler equation in vorticity form

$$\frac{\partial \Omega}{\partial t} + \{\Psi, \Omega\} = 0,$$

where \(\Omega\) is the vorticity, \(\Psi\) is the stream function, \(\Omega = \Delta \Psi\), \(\Delta\) is the 2D Laplacian, and the bracket \(\{ , \}\) is defined as

$$\{ f, g \} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g).$$

Expanding \(\Omega\) into Fourier series,

$$\Omega = \sum_{k \in \mathbb{Z}^2/\{0\}} \omega_k e^{ik \cdot X},$$

where \(\omega_{-k} = \overline{\omega_k}\), \(k = (k_1, k_2)^T\), and \(X = (x, y)^T\). The 2D Euler equation can be rewritten as

$$\dot{\omega}_k = \sum_{k=p+q} A(p, q) \omega_p \omega_q,$$

where \(A(p, q)\) is given by,

$$A(p, q) = \frac{1}{2} \begin{vmatrix} |q|^2 - |p|^2 & (p_1 q_2 - p_2 q_1) \\ \end{vmatrix},$$

where \(|q|^2 = q_1^2 + q_2^2\) for \(q = (q_1, q_2)^T\), similarly for \(p\). Denote \(\{\omega_k\}_{k \in \mathbb{Z}^2/\{0\}}\) by \(\omega\).

We consider the simple fixed point \(\omega^*:\)

$$\omega^*_p = \Gamma, \quad \omega^*_k = 0, \text{ if } k \neq p \text{ or } -p,$$

of the 2D Euler equation (2.2), where \(\Gamma\) is an arbitrary complex constant. The linearized two-dimensional Euler equation at \(\omega^*\) is given by,

$$\dot{\omega}_k = A(p, k - p) \Gamma \omega_{k-p} + A(-p, k + p) \Gamma \omega_{k+p}.$$

**Definition 2.1 (Classes).** For any \(\hat{k} \in \mathbb{Z}^2/\{0\}\), we define the class \(\Sigma_{\hat{k}}\) to be the subset of \(\mathbb{Z}^2/\{0\}\):

$$\Sigma_{\hat{k}} = \left\{ \hat{k} + np \in \mathbb{Z}^2/\{0\} \mid n \in \mathbb{Z}, \ p \text{ is specified in (2.4)} \right\}.$$
See Fig. 1 for an illustration of the classes. According to the classification defined in Definition 2.1, the linearized two-dimensional Euler equation (2.5) decouples into infinitely many invariant subsystems:

$$\dot{\tilde{\omega}}_{k+n} = A(p, \tilde{k} + (n-1)p) \Gamma \omega_{k+(n-1)p} + A(-p, \tilde{k} + (n+1)p) \bar{\Gamma} \omega_{k+(n+1)p}.$$  

(2.6)

Let $L_{\tilde{k}}$ be the linear operator defined by the right hand side of (2.6), and $H_s$ be the Sobolev space where $s \geq 0$ is an integer and $H^0 = \ell_2$.

**Theorem 2.2.** The eigenvalues of the linear operator $L_{\tilde{k}}$ in $H^s$ are of four types: real pairs $(c, -c)$, purely imaginary pairs $(id, -id)$, quadruples $(\pm c \pm id)$, and zero eigenvalues.

Proof. The same proof as in [1] works here. QED

The eigenvalues can be computed through continued fractions [1].

**Definition 2.3 (The Disk).** The disk of radius $|p|$ in $\mathbb{Z}^2/\{0\}$, denoted by $\bar{D}_{|p|}$, is defined as

$$\bar{D}_{|p|} = \left\{ k \in \mathbb{Z}^2/\{0\} \mid |k| \leq |p| \right\}.$$

See Fig. 1 for an illustration. For the Sobolev spaces $H^s$ where $s > 0$ is an integer, the following claims are still true.

1. Theorem VI.1 on page 747 of [1] is still true by simply replacing the $\ell_2$-norm by $H^s$-norm.
(2) Theorems VI.2 and VI.3 on page 750 of [1] are still true, where the inner product $(\cdot, \cdot)$ should still be an $\ell_2$ inner product.

(3) Theorem VI.4 on page 751 of [1] is still true, where $\ell_1$, $\ell_2$, and $\ell_\infty$ should be replaced by the Sobolev spaces $W^{s,1}$, $W^{s,2}$ ($= H^s$ in our notation), and $W^{s,\infty}$. In the expression (VI.60) of $f_n$ on page 753, one has

$$n^s f_n = \frac{2 [1 - w_4^s]}{W_0} \sum_{j=0}^{n-1} \left[ (n-j)^s w_{n-j}^s + (n-j)^s (-w_n)^{n-j} \right]$$

$$\left( \frac{n^s}{(n-j)^s (j+2)^s} \right) (j+2)^s y_{j+2} - \frac{2 [1 - w_4^s]}{W_0} \sum_{j=n}^{\infty}$$

$$\left[ w_j^{j-n} + (-w)^{j-n} \right] \left( \frac{n^s}{(j+2)^s} \right) (j+2)^s y_{j+2},$$

where both $\left( \frac{n^s}{(n-j)^s (j+2)^s} \right)$ and $\left( \frac{n^s}{(j+2)^s} \right)$ are bounded in $n$ and $j$, and the Riesz convexity theorem can be applied.

The main obstacle toward proving a spectral theorem for $L^\hat{k}$ is that for the Sobolev spaces here, $iL^\hat{H}$ in [1] is not self-adjoint anymore.

![Figure 2. The collocation of the modes in the line model.](image)

**2.2. The Open Problem.** In the neighborhood of the fixed point, the 2D Euler equation (2.2) can be rewritten as

$$\dot{\omega} = L\omega + N(\omega),$$
where $\omega = \{\omega_k\}_{k \in \mathbb{Z}^2/\{0\}}$, $L$ is the linear 2D Euler operator at the fixed point, and $N(\omega)$ is the remaining nonlinear term. Here we are dealing with the case that $L$ has eigenvalues with non-zero real parts.

• The Open Problem: Prove the existence of unstable, stable, and center manifolds.

There are two difficulties: 1. The spectral resolution of $L$ is elusive. 2. The nonlinear term $N(\omega)$ is non-Lipschitz in $\omega$. It seems that a good approach is to solve the two difficulties one by one. In next two subsections, we will introduce two model problems, each of which has only one of the two difficulties. The second difficulty was also met when proving local well-posedness of quasi-linear equations. It is hopeful that Kato’s approach in [7] will also work here. A more hopeful approach is that of Bourgain [8].

2.3. The Line Model. To simplify our study, we study only the case when $\omega_k$ is real, $\forall k \in \mathbb{Z}^2/\{0\}$, i.e. we only study the cosine transform of the vorticity,

$$\Omega = \sum_{k \in \mathbb{Z}^2/\{0\}} \omega_k \cos(k \cdot X),$$

and the 2D Euler equation (2.1) preserves the cosine transform. To further simplify our study, we will study a concrete line model based upon the fixed point (2.4) with the mode $p = (1, 1)^T$ and parametrized by $\Gamma$. When $\Gamma \neq 0$, the fixed point has 4 eigenvalues which form a quadruple. These four eigenvalues appear in the invariant linear subsystem labeled by $\hat{k} = (-3, -2)^T$. We computed the eigenvalues through continued fractions, one of them is [1]:

$$(2.7) \tilde{\lambda} = 2\lambda/|\Gamma| = 0.24822302478255 + i 0.35172076526520.$$

In $\ell_2$, the essential spectrum (= continuous spectrum) of $L_{\hat{k}}$ with $\hat{k} = (-3, -2)^T$ is the segment on the imaginary axis. The essential spectrum (= continuous spectrum) of the linear 2D Euler operator at this fixed point is the entire imaginary axis. The line model is a Galerkin truncation with the modes on the line $\{\hat{k} + np, n \in \mathbb{Z}\}$ and $p$, where $\hat{k} = (-3, -2)^T$ and $p = (1, 1)^T$. See Figure 2 for an illustration of the modes in this model, which has the line nature leading to the name of the model. The line model is designed to model the hyperbolic structure in the neighborhood of the fixed point. For simplicity of presentation, we use the abbreviated notations, $\omega_n = \omega_{\hat{k} + np}$, $A_n = A(p, \hat{k} + np)$, $A_m,n = A(\hat{k} + mp, \hat{k} + np)$.

The line model is,

$$\dot{\omega}_n = A_{n-1}\omega_n - A_{n+1}\omega_{n+1},$$

$$\dot{\omega}_p = -\sum_{n \in \mathbb{Z}} A_{n-1,n}\omega_{n-1}\omega_n.$$

We also use $\omega^*$ to denote the fixed point of the line model $\{\omega_p = \Gamma; \omega_n = 0, \forall n \in \mathbb{Z}\}$. The linearized line model at the fixed point $\omega^*$ is the same with the invariant subsystem (2.6) with $\hat{k} = (-3, -2)^T$ and $p = (1, 1)^T$. In the neighborhood of the fixed point $\omega^*$, the line model can be rewritten as

$$(2.9) \dot{\omega} = L\omega + Q(\omega),$$
where
\[\omega = (\omega_p, \omega_n \ (n \in \mathbb{Z}))\],
\[[L\omega]_n = A_{n-1} \Gamma \omega_{n-1} - A_{n+1} \Gamma \omega_{n+1},\]
\[[L\omega]_p = 0,\]
\[[Q(\omega)]_n = A_{n-1} \omega_p \omega_{n-1} - A_{n+1} \omega_p \omega_{n+1},\]
\[[Q(\omega)]_p = - \sum_{n \in \mathbb{Z}} A_{n-1} \omega_n \omega_{n-1} \omega_n.\]

Of course, the spectral resolution of \(L\) is not available. But \(Q(\omega)\) is quadratic in \(\omega\), for \(\omega \in H^s (s \geq 1)\), a Banach algebra.

A similar model can be designed for 2D Navier-Stokes equations. We consider the 2D Navier-Stokes equation with temporally periodic forcing, as a singular perturbation of the 2D Euler equation (2.1),

\[
(2.10) \quad \frac{\partial \Omega}{\partial t} + \{\Psi, \Omega\} = \epsilon [\Delta \Omega + f(t, x, y)],
\]
where \(\epsilon = 1/\text{Re}\) is the inverse of Reynolds number, and \(f(t, x, y)\) is periodic in \(t\), periodic in \(x\) and \(y\) of period \(2\pi\), and of spatial mean 0. The corresponding line model is

\[
\dot{\omega}_n = A_{n-1} \omega_p \omega_{n-1} - A_{n+1} \omega_p \omega_{n+1} + \epsilon [\kappa_n^2 \omega_n + f_n(t)],
\]
\[
(2.11) \quad \dot{\omega}_p = - \sum_{n \in \mathbb{Z}} A_{n-1} \omega_n - \epsilon [\kappa_p^2 \omega_p + f_p(t)],\]
where \(f_n\) and \(f_p\) are periodic functions of \(t\), and
\[
\kappa_n = |\hat{k} + np|, \quad \kappa_p = |p|, \quad \hat{k} = (-3, -2)^T, \quad p = (1, 1)^T.
\]

2.4. The Derivative Nonlinear Schrödinger Equation. Consider the derivative nonlinear Schrödinger equation

\[
iq_t = q_{xx} - i(|q|^2q)_x,
\]
under periodic boundary condition

\[q(t, x + 2\pi) = q(t, x).\]

It admits a monochromatic wave solution

\[q = ce^{i(\kappa x + \omega t)}\]

where \(\omega = k^2 - k|c|^2, k \in \mathbb{Z}\). Linearization around the monochromatic wave with

\[q = e^{i(\kappa x + \omega t)} (c + \hat{q}),\]

one gets

\[i\hat{q}_t = \hat{q}_{xx} + 2i k \hat{q}_x - k|c|^2 \hat{q} - (i \partial_x - k)[2|c|^2 \hat{q} + c^2 \hat{q}].\]

Setting

\[
\hat{q} = A e^{i\xi x + \Omega t} + B e^{-i\xi x + \Omega t},
\]

one obtains

\[\Omega = i[2k\xi - 2\xi|c|^2] \pm i\xi \sqrt{\xi^2 - 2k|c|^2 + |c|^4}.
\]
By choosing $k = 2$ and $2 - \sqrt{3} < |c|^2 < 2 + \sqrt{3}$, one obtains a quartet of eigenvalues given by $\xi = \pm 1$, with nonzero real parts. In the neighborhood of the monochromatic wave, the derivative nonlinear Schrödinger equation can be rewritten as

$$i\dot{q}_t = \dot{q}_{xx} + 2i k \dot{q}_x - k|c|^2 \dot{q} - (i\partial_x - k)(2|c|^2 \dot{q} + c^2 \ddot{q} + 2c|\dot{q}|^2 + \bar{c}\ddot{q} + |\dot{q}|^2 \dot{q}) .$$

Here of course the invariant subspaces exist. But the nonlinear terms are non-Lipschitz.

### 3. Isospectral Theory of Euler Equations

This section is divided into two subsections. Subsection 3.1 is on 3D Euler equation, and subsection 3.2 is on 2D Euler equation. A more complete isospectral theory is developed in [6].

#### 3.1. 3D Euler Equation

The 3D Euler equation can be written in vorticity form,

$$(3.1) \quad \partial_t \Omega + (u \cdot \nabla)\Omega - (\Omega \cdot \nabla)u = 0 ,$$

where $u = (u_1, u_2, u_3)$ is the velocity, $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the vorticity, $\nabla = (\partial_x, \partial_y, \partial_z)$, $\Omega = \nabla \times u$, and $\nabla \cdot u = 0$. $u$ can be represented by $\Omega$ for example through Biot-Savart law. The 3D Euler equation (3.1) has the Lax pair [4]

$$(3.2) \quad \begin{cases} L\phi = \lambda\phi , \\ \partial_t \phi + A\phi = 0 , \end{cases}$$

where

$L\phi = (\Omega \cdot \nabla)\phi$ , $A\phi = (u \cdot \nabla)\phi$ ,

$\lambda$ is a complex constant, and $\phi$ is a complex scalar-valued function. Another Lax pair of the 3D Euler equation (3.1) is given as [4]

$$(3.3) \quad \begin{cases} L\varphi = \lambda\varphi , \\ \partial_t \varphi + A\varphi = 0 , \end{cases}$$

where

$L\varphi = (\Omega \cdot \nabla)\varphi - (\varphi \cdot \nabla)\Omega$ , $A\varphi = (u \cdot \nabla)\varphi - (\varphi \cdot \nabla)u$ ,

$\lambda$ is a complex constant, and $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ is a complex 3-vector valued function.

Denote by $H^s$ the Sobolev space $H^s(\mathbb{R}^3)$ or $H^s(\mathbb{T}^3)$, and $\| \|_s$ the $H^s$ norm.

**Theorem 3.1.** Let $\Omega$ be a solution to the 3D Euler equation (3.1), $\phi$ be a solution to the Lax pair (3.2) at $(\Omega, \lambda)$, then

$$I = \frac{\|(\Omega \cdot \nabla)\phi\|_s}{\|\phi\|_s}$$

is an invariant, i.e., $I$ is independent of $t$.

**Proof:** Take the $H^s$ norm on both side of the first equation in the Lax pair (3.3), then

$$I = \frac{\|(\Omega \cdot \nabla)\phi\|_s}{\|\phi\|_s} = |\lambda| .$$

By the isospectral property of the Lax pair (3.3), $I$ is independent of $t$. QED

Of course, the significance of the invariants comes from their potential in providing $a priori$ bounds and establishing the global well-posedness of 3D Euler equation, hence, of 3D Navier-Stokes equation (one of the Clay problems). The above theorem provides infinitely many invariants.
Theorem 3.2. If \( \{ \varphi_j \}_{j=1,2,...} \) is a complete base of \( H^s \), where \( \varphi_j \)'s solve the Lax pair (3.3) at different values of \( \lambda \); then
\[
\Omega = \sum_{j=1}^{\infty} a_j \varphi_j ,
\]
where \( a_j \)'s are complex constants.

Proof: The proof is completed by comparing the second equation in the Lax pair (3.3) and the 3D Euler equation. QED

3.2. 2D Euler Equation. The Lax pair of the 2D Euler equation (2.1) is given as
\[
\begin{cases}
L \varphi = \lambda \varphi , \\
\partial_t \varphi + A \varphi = 0 ,
\end{cases}
\]
where
\[
L \varphi = \{ \Omega, \varphi \} , \quad A \varphi = \{ \Psi, \varphi \} ,
\]
and \( \lambda \) is an imaginary constant, and \( \varphi \) is a complex-valued function.

Denote by \( H^s \) the Sobolev space \( H^s(\mathbb{R}^2) \) or \( H^s(\mathbb{T}^2) \), and \( \| \cdot \|_s \) the \( H^s \) norm.

Theorem 3.3. Let \( \Omega \) be a solution to the 2D Euler equation (2.1), \( \varphi \) be a solution to the Lax pair (3.4) at \( (\Omega, \lambda) \), then
\[
I = \frac{\| \{ \Omega, \varphi \} \|_s}{\| \varphi \|_s}
\]
is an invariant.

Theorem 3.4. If \( \varphi \) solves the Lax pair (3.4), then \( f(\varphi) \) solves
\[
\partial_t f(\varphi) + \{ \Psi, f(\varphi) \} = 0 ,
\]
for any \( f \) smooth enough.

Proof: The proof is completed by direct verification. QED

Theorem 3.5. If \( \{ \phi_j \}_{j=1,2,...} \) is a complete base of \( H^s \), where \( \phi_j \)'s are \( f(\varphi) \)'s at different values of \( \lambda \); then
\[
\Omega = \sum_{j=1}^{\infty} a_j \phi_j ,
\]
where \( a_j \)'s are complex constants.

Proof: Since Equation (3.5) is a linear equation, the claim of Theorem 3.4 implies the current theorem. QED

The Rossby wave equation is
\[
\partial_t \Omega + \{ \Psi, \Omega \} + \beta \partial_x \Psi = 0 ,
\]
where \( \Omega = \Omega(t,x,y) \) is the vorticity, \( \{ \Psi, \Omega \} = \Psi_x \Omega_y - \Psi_y \Omega_x \), and \( \Psi = \Delta^{-1} \Omega \) is the stream function. Its Lax pair can be obtained by formally conducting the transformation, \( \Omega = \tilde{\Omega} + \beta y \), to the 2D Euler equation,
\[
\begin{cases}
\{ \Omega, \varphi \} - \beta \partial_x \varphi = \lambda \varphi , \\
\partial_t \varphi + \{ \Psi, \varphi \} = 0 ,
\end{cases}
\]
where \( \varphi \) is a complex-valued function, and \( \lambda \) is a complex parameter.
References


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